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## Some properties of positive superharmonic functions

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In this paper we present some results similar to the well-known Cartan lemma (cf. [5]) which estimates the set where a potential is large. These results were inspired by a Hall type lemma proved in [2] and they have useful applications regarding the behaviour of a subharmonic function at a point or regarding the boundary behaviour of Green potentials. We further prove a reversed Hölder inequality for positive superharmonic functions and in Section 4 we construct a simple counterexample to an assertion of Tolsted in [10] according to which a Green potential in the unit disk has boundary limit zero almost everywhere on the unit circle along rotations of any fixed normal curve.

### 1. A Cartan-type result

In [2] Davis and Lewis have proved the estimate  $\sigma(\{u > s\}^*) \leq Cu(0)/s$ . Here  $\sigma$  is surface measure on  $|x| = 1$ ,  $u$  is a positive superharmonic function in  $|x| < 1$ ,  $\{u > s\}^*$  denotes the radial projection of the open set where  $u$  is larger than  $s$ , and  $C$  is a constant only depending on the dimension. Our aim is to estimate  $\{u > s\}$  more closely. We first prove a Cartan type result for potentials which by means of Riesz decomposition implies a better estimate than the above one for the projection of the part of  $\{u > s\}$  in  $|x| \leq \frac{1}{2}$ .

**THEOREM 1.** *Let  $\mu$  be a nonnegative measure in  $\mathbb{R}^n$ ,  $n \geq 2$ , and let  $u(x)$  be its Newtonian potential (or logarithmic potential, if  $n = 2$ ):  $u(x) = \int |x - y|^{2-n} d\mu(y)$  (or  $u(x) = \int \log|x - y|^{-1} d\mu(y)$ , respectively); if  $n = 2$  we assume in addition that  $\mu$  is finite with support in  $|y| \leq \rho$ , where  $\rho < 1$ . Then the open set  $\{u > s\}$  can for each sufficiently large  $s$  (depending on  $u(0)$ ) be covered by balls  $B(x_j, t_j)$  such that*

$$\sum \left( \frac{t_j}{|x_j|} \right)^{n-1} \leq \begin{cases} C(u(0)/s)^{(n-1)/(n-2)} & \text{for } n > 2 \\ Ce^{-cs/u(0)} & \text{for } n = 2. \end{cases}$$

In particular,  $\sigma(\{u > s\}^*) \leq C(u(0)/s)^{(n-1)/(n-2)} (\leq C \exp(-cs/u(0)))$ , respectively). The constants  $C$  and  $c = c_\rho > 0$  do not depend on  $u$ .

REMARK. From the theorem one can obtain (by means of Riesz decomposition) some classical facts about the behaviour of a general superharmonic function at a point; e.g. that  $u(x) \rightarrow u(0)$  along almost every radius through the origin (cf. Deny [3] who proves an even stronger result). Indeed, a standard argument shows that  $u(x) \rightarrow u(0)$  as  $x \rightarrow 0$  through the complement of a collection of balls  $B(x_j, t_j)$  such that  $t_j < |x_j|$  and  $\sum(t_j/|x_j|)^{n-1} < \infty$ . (Here as well as in the theorem the exponent  $n - 1$  could be replaced by any  $\alpha > n - 2$ ). Compare also with the work of Essén-Jackson [3a] on thin sets.

*Proof of Theorem 1.* We modify the proof in [5] of the Cartan lemma. Suppose first that  $n > 2$  and, without restriction, that  $u(0) = 1$ . Let  $m$  be the measure defined by  $m(E) := E|y|^{2-n} d\mu(y)$  so that  $m(\mathbb{R}^n) = u(0) = 1$ . For fixed  $x \in \mathbb{R}^n$  we put  $\tilde{m}(t) := \tilde{m}_x(t) := m(B(x, t))$ ,  $t \geq 0$ .

Let  $A \geq 2$  be some constant and assume that  $\tilde{m}(t) \leq (At/r)^{n-1}$  for  $0 \leq t < t_0 := r/A$  where  $r := |x|$ . For  $|x - y| < t_0$  we have  $|y| \leq 2r$ , hence

$$\begin{aligned} \int_{|x-y| < t_0} |x - y|^{2-n} d\mu(y) &\leq (2r)^{n-2} \int_{|x-y| < t_0} |x - y|^{2-n} dm(y) \\ &= (2r)^{n-2} \int_{[0, t_0]} t^{2-n} d\tilde{m}(t) \leq (n - 1)(2A)^{n-2}, \end{aligned}$$

where the last inequality has been obtained by means of an integration by parts and our assumption on  $\tilde{m}$ . Further, for  $|y - x| \geq t_0$  we have  $|y| \leq |x| + |y - x| \leq (A + 1)|x - y|$ , hence

$$\int_{|x-y| \geq t_0} |x - y|^{2-n} d\mu(y) \leq (A + 1)^{n-2} u(0) \leq (2A)^{n-2}.$$

Combining the above estimates we conclude that  $u(x) \leq s$  for all  $x$  such that  $\tilde{m}_x(t) \leq (At/r)^{n-1}$ , where  $A$  is defined by  $s = n(2A)^{n-2}$ , and where  $s \geq n4^{n-2}$ , say. Thus the set  $\{u > s\}$  can be covered by balls  $B(x, t_x)$  such that  $0 < t_x < t_0 \leq \frac{1}{2}|x|$  and such that  $m(B(x, t_x)) > (At_x/r)^{n-1}$ . The conclusion of the theorem follows now readily from a suitable version of the Besicovitch covering lemma which yields a countable subcovering with bounded overlaps (cf. [4]).

For  $n = 2$  a similar proof can be given. □

## 2. Another covering result

The full content of the Davis-Lewis result (see § 1) can be obtained if Theorem 1 is combined with the following related covering result. We formulate it in the rather

general setting of  $C^{1+\alpha}$  and Dini domains (see [11] for the definition of a Dini domain).

**THEOREM 2.** *Let  $u$  be a positive superharmonic function in a  $C^{1+\alpha}$  or Dini domain  $D \in \mathbb{R}^n$ . Then for each  $s \geq 0$  the set  $\{u > s\}$  can be covered by balls  $B(x_j, t_j)$  such that*

$$\sum t_j^{n-1} \leq Cs^{-1} \|u\|_1,$$

where  $\|u\|_1 = \int_D u(x) dx < \infty$  and  $C = C_D$ .

Theorem 2 can be obtained as a consequence of some recent results of Wu [12] on harmonic measures. However, since the techniques of [12] are rather complicated we outline an easier direct proof of Theorem 2 in the spirit of the proof of Theorem 1 (cf. also Kudina [4a]). It is based upon the following classical estimates due to Widman [11] for the Green function  $G(x, y)$  and the Poisson kernel  $P(x, y)$  of a Dini domain  $D$ .

**LEMMA** (cf. [11]). *The following uniform estimates hold for  $G$  and  $P$  ( $d(\cdot)$  denotes distance to  $\partial D$ ):*

- (1)  $G(x, y) \leq \begin{cases} \log (Cd(x)/|x - y|) & \text{for } n = 2 \text{ and } |x - y| \leq \frac{1}{2}d(x) \\ |x - y|^{2-n} & \text{for } n > 2 \end{cases}$
- (2)  $G(x, y) \leq Cd(x)|x - y|^{1-n}$
- (3)  $G(x, y) \leq Cd(x)d(y) |x - y|^{-n}$
- (4)  $P(x, y) \leq Cd(x) |x - y|^{-n}$  for  $y \in \partial D$
- (5)  $G(x, y) \geq cd(y)$  for  $x \in K \in D, y \in D$  (where  $c > 0$ )
- (6)  $P(x, y) \geq c$  for  $x \in K \in D, y \in \partial D$ .

*Proof of Theorem 2.* Since  $u$  is positive it has a global Riesz decomposition  $u = G\mu + Pv$ , where  $G\mu(x) = \int_D G(x, y)d\mu(y)$ ,  $Pv(x) = \int_{\partial D} P(x, y)dv(y)$ . ( $\mu$  is the Riesz measure and  $\nu$  is the “boundary” measure of  $u$ .) We define the measure  $m$  by  $m(E) = \int_{E \cap D} d(y)d\mu(y) + \nu(E \cap \partial D)$ . Then  $m(\bar{D}) < \infty$ , cf. (5) of the lemma. Moreover, with the aid of the Fubini theorem it follows readily from the estimates (2), (4), (5) and (6) of the lemma that  $m(\bar{D}) \approx \|u\|_1$ . Hence it suffices (cf. the proof of Theorem 1) to show that  $u(x) \leq s$  for all  $x \in D$  satisfying  $m(B(x, t)) \leq Cst^{n-1}$ ,  $t \geq 0$ , where  $C = C_D$ . An integration by parts in the integrals defining  $G\mu$  and  $Pv$  shows that this is indeed the case: for  $G\mu$  we use (1) if  $|x - y| < \frac{1}{2}d(x)$ , (3) if  $|x - y| \geq \frac{1}{2}d(x)$ , while for  $Pv$  we use (4). □

Next we obtain a reversed Hölder inequality for positive superharmonic functions which seems to be new:

**THEOREM 3.** *Let  $D$  and  $u$  be as in Theorem 2. Then  $\|u\|_p$  is finite for  $0 < p < n/(n - 1)$ . Moreover, given  $0 < p < q < n/(n - 1)$ , there is a constant*

$C = C(p, q, D)$  such that

$$\|u\|_q \leq C\|u\|_p.$$

*Proof.* By Hölder’s inequality it suffices to show that  $\|u\|_q \leq C_q\|u\|_1$ ,  $1 < q < n/(n - 1)$ , and  $\|u\|_1 \leq C_p\|u\|_p$ ,  $0 < p < 1$ . We may assume that  $\|u\|_1 = 1$ . Let  $\omega(s)$  denote the Lebesgue measure of the set  $\{u > s\}$ . Then Theorem 2 implies the weak type estimate

$$(7) \quad \omega(s) \leq Cs^{-n/(n-1)}$$

The inequality  $\|u\|_q \leq C_q$  for  $1 < q < n/(n - 1)$  follows now from (7) (for  $s \geq 1$ ) and the identity  $\int_D u^q dx = q \int_0^\infty s^{q-1} \omega(s) ds$ . The second desired inequality may be obtained by repeatedly using the first one together with the superharmonicity of  $u^p$  when  $0 < p < 1$ . □

### 3. Some applications

(i) Rippon [8] and Wu [13] have obtained the following extension of Littlewood’s radial limit theorem in the unit disk: suppose to each  $\xi \in \partial D$  (where  $D$  may be any Dini domain) there corresponds a curve  $\gamma_\xi$  in  $D$  which tends to  $\xi$  nontangentially; suppose also that the family  $\Gamma := (\gamma_\xi)$  satisfies the separation condition:

$$(8) \quad \text{distance}(\gamma_\xi, \gamma_{\xi'}) \geq c|\xi - \xi'| \quad \text{for } \xi, \xi' \in \partial D,$$

where  $c$  is a positive constant. Then every Green potential  $G\mu(x) := \int_D G(x, y)d\mu(y)$  satisfies  $G\mu(x) \rightarrow 0$  for  $x \rightarrow \xi$  along  $\gamma_\xi$ , for almost all  $\xi \in \partial D$ . This result can be obtained directly from Theorem 2. Indeed, a standard argument shows that  $G\mu(x) \rightarrow 0$  for  $x \rightarrow \partial D$  outside a family of balls  $B(x_j, t_j)$  with  $\Sigma t_j^{n-1} < \varepsilon$ , where  $\varepsilon > 0$  may be chosen arbitrarily small; further (8) implies that the “ $\Gamma$ -projection” of those balls onto  $\partial D$  has surface measure bounded by  $C\varepsilon$ .

(ii) A minor modification in the proof of Theorem 2 yields the following extension of the theorem: if  $w(x)$  is any positive function in  $D$  then the set  $\{uw > s\}$  can be covered by balls  $B(x_j, t_j)$  such that  $\Sigma t_j^{n-1}/w(x_j) \leq C\|u\|_1/s$ . Choosing in particular  $w(x) = d(x)^{n-1}$ , where as before  $d(x)$  denotes the distance of  $x$  to  $\partial D$ , we obtain the following analogue of the statement above: namely,  $d(x)^{n-1}G\mu(x) \rightarrow 0$  for  $x \rightarrow \partial D$  outside a family of balls  $B(x_j, t_j)$  such that  $\Sigma(t_j/d(x_j))^{n-1} < \varepsilon$ . This is an improvement and generalization of a result of Stoll [9] for the disk and may be compared to results obtained in [6] and [7].

#### 4. A counterexample

Dahlberg [1] has shown that the separation condition (8) is rather essential for the above-mentioned Rippon-Wu result by constructing a  $C^{1+\alpha}$  domain in  $\mathbb{R}^2$  with the property that Littlewood's theorem fails for the family of interior normals. Here we present a similar but more explicit construction directly in a halfplane for the special case where the family  $\Gamma$  consists of translations of a fixed curve  $\gamma$ . This construction was motivated by an incorrect statement of Tolsted ([10], Corollary 3.23) according to which the mere existence of a tangent at the endpoint of  $\gamma$  would imply the validity of Littlewood's theorem for the family of rotations  $e^{i\theta}\gamma$ . (Tolsted worked in the unit disk rather than in the halfplane.)

**THEOREM 4.** *There exist a Green potential  $u(z)$  in the halfplane  $\operatorname{Re} z > 0$  and a rectifiable curve  $\gamma$  in  $\operatorname{Re} z \geq 0$ , with parametrization  $t \rightarrow t + i\phi(t)$ ,  $0 \leq t \leq 1$ , where  $\phi(0) = \phi'(0) = 0$ ,  $\phi$  is piecewise linear for  $t > 0$  and Hölder continuous of order  $\alpha < 1$  on  $[0, 1]$ , such that  $\limsup_{z \rightarrow iy, z \in \gamma_y} u(z) = \infty$  for all  $y \in \mathbb{R}$ . Here  $\gamma_y := \gamma + iy$  denotes the vertical translation of  $\gamma$  over  $y$ .*

**REMARK.** According to the Rippon-Wu result of Section 3 the family  $(\gamma_y)$  of the theorem can not satisfy (8). On the other hand, it is not hard to prove that the family of vertical translations of a given curve  $t \rightarrow t + i\phi(t)$  satisfies (8) if and only if  $\phi$  is Lipschitz continuous. The theorem shows that the analogue of Littlewood's theorem for such a family does in general not hold under any weaker smoothness condition.

*Proof of Theorem 4.* Fix  $\alpha \in (0, 1)$  and put  $a_k := 2^{-k^2}$ ,  $\varepsilon_k := a_k/k$ ,  $\delta_k := \varepsilon_k^{1/\alpha}$  ( $k \geq 2$ ). Then one can verify that the function  $\phi(t)$  which is 0 at the points  $t = 0, t = a_k$ , equals the value  $-\varepsilon_k$  at the points  $t = a_k - \delta_k$ , and is linear on each of the intervals  $[a_{k+1}, a_k - \delta_k]$  and  $[a_k - \delta_k, a_k]$ , satisfies the requirements. Also  $\gamma: t \rightarrow t + i\phi(t)$  is rectifiable.

We now define a Green potential  $u(z)$  by

$$u(z) := \sum_{k=1}^{\infty} k^{-3/2} \sum_{j=1}^{2^{k^2}} \log |z + \overline{c_{jk}}|,$$

where  $c_{jk} = a_k + ib_{jk}$  and where the real numbers  $b_{jk}$  ( $1 \leq j \leq 2^{k^2}$ ,  $k \geq 2$ ) are so chosen that the intervals  $[b_{jk}, b_{jk} + \varepsilon_k]$  cover the real axis infinitely often. This is possible since  $\sum 2^{k^2} \varepsilon_k = \sum 1/k = \infty$ . Also  $u \not\equiv \infty$  since

$$\sum_{k,j} k^{-3/2} \operatorname{Re} c_{jk} = \sum k^{-3/2} 2^{k^2} a_k = \sum k^{-3/2} < \infty.$$

As the translations  $\gamma_y := \gamma + iy$  intersect the disk  $|z - c_{jk}| \leq \delta_k$  for  $y \in [b_{jk}, b_{jk} + \varepsilon_k]$  (because this is true for  $y = b_{jk}$  and for  $y = b_{jk} + \varepsilon_k$ , by definition of  $\phi$ ), it follows that each  $\gamma_y$  intersects infinitely many disks  $|z - c_{jk}| \leq \delta_k$ . However, in each disk  $|z - c_{jk}| \leq \delta_k$  we have

$$\begin{aligned} u(z) &\geq k^{-3/2} \log \left| \frac{a_k}{z - c_{jk}} \right| \\ &\geq k^{-3/2} \log \frac{a_k}{\delta_k} \geq \left( \frac{1}{\alpha} - 1 \right) k^{1/2} \log 2 \rightarrow \infty. \end{aligned}$$

This completes the proof of Theorem 4. □

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