COMPOSITIO MATHEMATICA

REIN L. ZEINSTRA

Some properties of positive superharmonic functions

Compositio Mathematica, tome 72, n° 1 (1989), p. 115-120
<http://www.numdam.org/item?id=CM_1989__72_1_115_0>

Compositio Mathematica **72**: 115–120, 1989.

Some properties of positive superharmonic functions

REIN L. ZEINSTRA
Universität Osnabrück, Abteilung Vechta, Fachbereich 13, Driverstr. 22, D-2848 Vechta

Received 18 October 1988; accepted 9 March 1989

In this paper we present some results similar to the well-known Cartan lemma (cf. [5]) which estimates the set where a potential is large. These results were inspired by a Hall type lemma proved in [2] and they have useful applications regarding the behaviour of a subharmonic function at a point or regarding the boundary behaviour of Green potentials. We further prove a reversed Hölder inequality for positive superharmonic functions and in Section 4 we construct a simple counterexample to an assertion of Tolsted in [10] according to which a Green potential in the unit disk has boundary limit zero almost everywhere on the unit circle along rotations of any fixed normal curve.

1. A Cartan-type result

In [2] Davis and Lewis have proved the estimate $\sigma(\{u > s\}^*) \leqslant Cu(0)/s$. Here σ is surface measure on $|x| = 1$, u is a positive superharmonic function in $|x| < 1$, $\{u > s\}^*$ denotes the radial projection of the open set where u is larger than s, and C is a constant only depending on the dimension. Our aim is to estimate $\{u > s\}$ more closely. We first prove a Cartan type result for potentials which by means of Riesz decomposition implies a better estimate than the above one for the projection of the part of $\{u > s\}$ in $|x| \leqslant \frac{1}{2}$.

THEOREM 1. *Let μ be a nonnegative measure in $\mathbb{R}^n, n \geqslant 2$, and let $u(x)$ be its Newtonian potential (or logarithmic potential, if $n = 2$): $u(x) = \int |x - y|^{2-n}\, d\mu(y)$ (or $u(x) = \int \log|x - y|^{-1}\, d\mu(y)$, respectively); if $n = 2$ we assume in addition that μ is finite with support in $|y| \leqslant \rho$, where $\rho < 1$. Then the open set $\{u > s\}$ can for each sufficiently large s (depending on $u(0)$) be covered by balls $B(x_j, t_j)$ such that*

$$\sum \left(\frac{t_j}{|x_j|}\right)^{n-1} \leqslant \begin{cases} C(u(0)/s)^{(n-1)/(n-2)} & \text{for } n > 2 \\ Ce^{-cs/u(0)} & \text{for } n = 2. \end{cases}$$

In particular, $\sigma(\{u > s\}^*) \leqslant C(u(0)/s)^{(n-1)/(n-2)}$ ($\leqslant C\exp(-cs/u(0))$, respectively). The constants C and $c = c_\rho > 0$ do not depend on u.

REMARK. From the theorem one can obtain (by means of Riesz decomposition) some classical facts about the behaviour of a general superharmonic function at a point; e.g. that $u(x) \to u(0)$ along almost every radius through the origin (cf. Deny [3] who proves an even stronger result). Indeed, a standard argument shows that $u(x) \to u(0)$ as $x \to 0$ through the complement of a collection of balls $B(x_j, t_j)$ such that $t_j < |x_j|$ and $\Sigma(t_j/|x_j|)^{n-1} < \infty$. (Here as well as in the theorem the exponent $n - 1$ could be replaced by any $\alpha > n - 2$). Compare also with the work of Essén-Jackson [3a] on thin sets.

Proof of Theorem 1. We modify the proof in [5] of the Cartan lemma. Suppose first that $n > 2$ and, without restriction, that $u(0) = 1$. Let m be the measure defined by $m(E) := E|y|^{2-n}\, d\mu(y)$ so that $m(\mathbb{R}^n) = u(0) = 1$. For fixed $x \in \mathbb{R}^n$ we put $\tilde{m}(t) := \tilde{m}_x(t) := m(B(x, t)), t \geqslant 0$.

Let $A \geqslant 2$ be some constant and assume that $\tilde{m}(t) \leqslant (At/r)^{n-1}$ for $0 \leqslant t < t_0 := r/A$ where $r := |x|$. For $|x - y| < t_0$ we have $|y| \leqslant 2r$, hence

$$\int_{|x-y|<t_0} |x - y|^{2-n}\, d\mu(y) \leqslant (2r)^{n-2} \int_{|x-y|<t_0} |x - y|^{2-n}\, dm(y)$$

$$= (2r)^{n-2} \int_{[0,t_0)} t^{2-n}\, d\tilde{m}(t) \leqslant (n - 1)(2A)^{n-2},$$

where the last inequality has been obtained by means of an integration by parts and our assumption on $\tilde{m}$. Further, for $|y - x| \geqslant t_0$ we have $|y| \leqslant |x| + |y - x| \leqslant (A + 1)|x - y|$, hence

$$\int_{|x-y|\geqslant t_0} |x - y|^{2-n}\, d\mu(y) \leqslant (A + 1)^{n-2} u(0) \leqslant (2A)^{n-2}.$$

Combining the above estimates we conclude that $u(x) \leqslant s$ for all x such that $\tilde{m}_x(t) \leqslant (At/r)^{n-1}$, where A is defined by $s = n(2A)^{n-2}$, and where $s \geqslant n4^{n-2}$, say. Thus the set $\{u > s\}$ can be covered by balls $B(x, t_x)$ such that $0 < t_x < t_0 \leqslant \frac{1}{2}|x|$ and such that $m(B(x, t_x)) > (At_x/r)^{n-1}$. The conclusion of the theorem follows now readily from a suitable version of the Besicovitch covering lemma which yields a countable subcovering with bounded overlaps (cf. [4]).

For $n = 2$ a similar proof can be given. □

2. Another covering result

The full content of the Davis-Lewis result (see § 1) can be obtained if Theorem 1 is combined with the following related covering result. We formulate it in the rather general setting of $C^{1+\alpha}$ and Dini domains (see [11] for the definition of a Dini domain).

THEOREM 2. *Let u be a positive superharmonic function in a $C^{1+\alpha}$ or Dini domain $D \Subset \mathbb{R}^n$. Then for each $s \geqslant 0$ the set $\{u > s\}$ can be covered by balls $B(x_j, t_j)$ such that*

$$\sum t_j^{n-1} \leqslant Cs^{-1} \|u\|_1,$$

where $\|u\|_1 = \int_D u(x)\,dx < \infty$ and $C = C_D$.

Theorem 2 can be obtained as a consequence of some recent results of Wu [12] on harmonic measures. However, since the techniques of [12] are rather complicated we outline an easier direct proof of Theorem 2 in the spirit of the proof of Theorem 1 (cf. also Kudina [4a]). It is based upon the following classical estimates due to Widman [11] for the Green function $G(x, y)$ and the Poisson kernel $P(x, y)$ of a Dini domain D.

LEMMA (cf. [11]). *The following uniform estimates hold for G and P ($d(.)$ denotes distance to ∂D):*

(1) $G(x, y) \leqslant \begin{cases} \log\,(Cd(x)/|x-y|) & \text{for } n = 2 \text{ and } |x-y| \leqslant \tfrac{1}{2}d(x) \\ |x-y|^{2-n} & \text{for } n > 2 \end{cases}$

(2) $G(x, y) \leqslant Cd(x)|x-y|^{1-n}$

(3) $G(x, y) \leqslant Cd(x)d(y)\,|x-y|^{-n}$

(4) $P(x, y) \leqslant Cd(x)\,|x-y|^{-n}$ for $y \in \partial D$

(5) $G(x, y) \geqslant cd(y)$ for $x \in K \Subset D, y \in D$ (where $c > 0$)

(6) $P(x, y) \geqslant c$ for $x \in K \Subset D, y \in \partial D$.

Proof of Theorem 2. Since u is positive it has a global Riesz decomposition $u = G\mu + Pv$, where $G\mu(x) = \int_D G(x, y)d\mu(y)$, $Pv(x) = \int_{\partial D} P(x, y)dv(y)$. ($\mu$ is the Riesz measure and v is the "boundary" measure of u.) We define the measure m by $m(E) = \int_{E\cap D} d(y)d\mu(y) + v(E \cap \partial D)$. Then $m(\bar{D}) < \infty$, cf. (5) of the lemma. Moreover, with the aid of the Fubini theorem it follows readily from the estimates (2), (4), (5) and (6) of the lemma that $m(\bar{D}) \approx \|u\|_1$. Hence it suffices (cf. the proof of Theorem 1) to show that $u(x) \leqslant s$ for all $x \in D$ satisfying $m(B(x, t)) \leqslant Cst^{n-1}$, $t \geqslant 0$, where $C = C_D$. An integration by parts in the integrals defining $G\mu$ and Pv shows that this is indeed the case: for $G\mu$ we use (1) if $|x-y| < \tfrac{1}{2}d(x)$, (3) if $|x-y| \geqslant \tfrac{1}{2}d(x)$, while for Pv we use (4). □

Next we obtain a reversed Hölder inequality for positive superharmonic functions which seems to be new:

THEOREM 3. *Let D and u be as in Theorem 2. Then $\|u\|_p$ is finite for $0 < p < n/(n-1)$. Moreover, given $0 < p < q < n/(n-1)$, there is a constant*

$C = C(p, q, D)$ *such that*

$$\|u\|_q \leqslant C\|u\|_p.$$

Proof. By Hölder's inequality it suffices to show that $\|u\|_q \leqslant C_q\|u\|_1$, $1 < q < n/(n-1)$, and $\|u\|_1 \leqslant C_p\|u\|_p$, $0 < p < 1$. We may assume that $\|u\|_1 = 1$. Let $\omega(s)$ denote the Lebesgue measure of the set $\{u > s\}$. Then Theorem 2 implies the weak type estimate

$$(7)\quad \omega(s) \leqslant Cs^{-n/(n-1)}$$

The inequality $\|u\|_q \leqslant C_q$ for $1 < q < n/(n-1)$ follows now from (7) (for $s \geqslant 1$) and the identity $\int_D u^q\,dx = q\int_0^\infty s^{q-1}\omega(s)ds$. The second desired inequality may be obtained by repeatedly using the first one together with the superharmonicity of u^p when $0 < p < 1$. □

3. Some applications

(i) Rippon [8] and Wu [13] have obtained the following extension of Littlewood's radial limit theorem in the unit disk: suppose to each $\xi \in \partial D$ (where D may be any Dini domain) there corresponds a curve γ_ξ in D which tends to ξ nontangentially; suppose also that the family $\Gamma := (\gamma_\xi)$ satisfies the separation condition:

$$(8)\quad \text{distance}\,(\gamma_\xi, \gamma_{\xi'}) \geqslant c|\xi - \xi'| \quad \text{for } \xi, \xi' \in \partial D,$$

where c is a positive constant. Then every Green potential $G\mu(x) := \int_D G(x, y)d\mu(y)$ satisfies $G\mu(x) \to 0$ for $x \to \xi$ along γ_ξ, for almost all $\xi \in \partial D$. This result can be obtained directly from Theorem 2. Indeed, a standard argument shows that $G\mu(x) \to 0$ for $x \to \partial D$ outside a family of balls $B(x_j, t_j)$ with $\Sigma t_j^{n-1} < \varepsilon$, where $\varepsilon > 0$ may be chosen arbitrarily small; further (8) implies that the "Γ-projection" of those balls onto ∂D has surface measure bounded by $C\varepsilon$.

(ii) A minor modification in the proof of Theorem 2 yields the following extension of the theorem: if $w(x)$ is any positive function in D then the set $\{uw > s\}$ can be covered by balls $B(x_j, t_j)$ such that $\Sigma t_j^{n-1}/w(x_j) \leqslant C\|u\|_1/s$. Choosing in particular $w(x) = d(x)^{n-1}$, where as before $d(x)$ denotes the distance of x to ∂D, we obtain the following analogue of the statement above: namely, $d(x)^{n-1}G\mu(x) \to 0$ for $x \to \partial D$ outside a family of balls $B(x_j, t_j)$ such that $\Sigma(t_j/d(x_j))^{n-1} < \varepsilon$. This is an improvement and generalization of a result of Stoll [9] for the disk and may be compared to results obtained in [6] and [7].

4. A counterexample

Dahlberg [1] has shown that the separation condition (8) is rather essential for the above-mentioned Rippon-Wu result by constructing a $C^{1+\alpha}$ domain in $\mathbb{R}^2$ with the property that Littlewood's theorem fails for the family of interior normals. Here we present a similar but more explicit construction directly in a halfplane for the special case where the family Γ consists of translations of a fixed curve γ. This construction was motivated by an incorrect statement of Tolsted ([10], Corollary 3.23) according to which the mere existence of a tangent at the endpoint of γ would imply the validity of Littlewood's theorem for the family of rotations $e^{i\theta}\gamma$. (Tolsted worked in the unit disk rather than in the halfplane.)

THEOREM 4. *There exist a Green potential $u(z)$ in the halfplane $\mathbb{R}ez > 0$ and a rectifiable curve γ in $\mathbb{R}ez \geqslant 0$, with parametrization $t \to t + i\phi(t), 0 \leqslant t \leqslant 1$, where $\phi(0) = \phi'(0) = 0$, ϕ is piecewise linear for $t > 0$ and Hölder continuous of order $\alpha < 1$ on $[0, 1]$, such that $\limsup_{z \to iy,\, z \in \gamma_y} u(z) = \infty$ for all $y \in \mathbb{R}$. Here $\gamma_y := \gamma + iy$ denotes the vertical translation of γ over y.*

REMARK. According to the Rippon-Wu result of Section 3 the family (γ_y) of the theorem can not satisfy (8). On the other hand, it is not hard to prove that the family of vertical translations of a given curve $t \to t + i\phi(t)$ satisfies (8) if and only if ϕ is Lipschitz continuous. The theorem shows that the analogue of Littlewood's theorem for such a family does in general not hold under any weaker smoothness condition.

Proof of Theorem 4. Fix $\alpha \in (0, 1)$ and put $a_k := 2^{-k^2}$, $\varepsilon_k := a_k/k$, $\delta_k := \varepsilon_k^{1/\alpha} (k \geqslant 2)$. Then one can verify that the function $\phi(t)$ which is 0 at the points $t = 0, t = a_k$, equals the value $-\varepsilon_k$ at the points $t = a_k - \delta_k$, and is linear on each of the intervals $[a_{k+1}, a_k - \delta_k]$ and $[a_k - \delta_k, a_k]$, satisfies the requirements. Also $\gamma: t \to t + i\phi(t)$ is rectifiable.

We now define a Green potential $u(z)$ by

$$u(z) := \sum_{k=1}^{\infty} k^{-3/2} \sum_{j=1}^{2^{k^2}} \log |z + \overline{c_{jk}}|,$$

where $c_{jk} = a_k + ib_{jk}$ and where the real numbers $b_{jk} (1 \leqslant j \leqslant 2^{k^2}, k \geqslant 2)$ are so chosen that the intervals $[b_{jk}, b_{jk} + \varepsilon_k]$ cover the real axis infinitely often. This is possible since $\Sigma 2^{k^2} \varepsilon_k = \Sigma 1/k = \infty$. Also $u \not\equiv \infty$ since

$$\sum_{k,j} k^{-3/2} \,\mathbb{R}e c_{jk} = \sum k^{-3/2} 2^{k^2} a_k = \sum k^{-3/2} < \infty.$$

As the translations $\gamma_y := \gamma + iy$ intersect the disk $|z - c_{jk}| \leqslant \delta_k$ for $y \in [b_{jk}, b_{jk} + \varepsilon_k]$ (because this is true for $y = b_{jk}$ and for $y = b_{jk} + \varepsilon_k$, by definition of ϕ), it follows that each γ_y intersects infinitely many disks $|z - c_{jk}| \leqslant \delta_k$. However, in each disk $|z - c_{jk}| \leqslant \delta_k$ we have

$$u(z) \geqslant k^{-3/2} \log\left|\frac{a_k}{z - c_{jk}}\right|$$

$$\geqslant k^{-3/2} \log \frac{a_k}{\delta_k} \geqslant \left(\frac{1}{\alpha} - 1\right)k^{1/2} \log 2 \to \infty.$$

This completes the proof of Theorem 4. □

References

1. B. Dahlberg, On the existence of radial boundary values for functions subharmonic in a Lipschitz domain. *Indiana Univ. Math. J.* 27 (1978) 515–526.
2. B. Davis, and J. Lewis, Paths for subharmonic functions. *Proc. London Math. Soc.* 48 (1984) 401–427.
3. J. Deny, Un théorème sur les ensembles effilés. *Ann. Univ.* Grenoble 23 (1948) 139–142.

3a. M. Essén, and H. L. Jackson, A comparision between thin sets and generalized Azarin sets. Canad. Math. Bull. 18 (1975) 335–346.

4. M. de Guzman, Différentiation of integrals in $\mathbb{R}^n$. *Lecture Notes in Maths.* 481. Springer-Verl., Berlin 1975.

4.a L.S. Kudina, Estimates for functions that can be represented as a difference of subharmonic functions in a ball (Russian). Teorija Funkcii, *Funkcionalnij Analiz i Prilozjenija* 14 (Charkov 1971).

5. N.S. Landkof, Foundations of modern potential theory. Springer-Verl., Berlin 1972.
6. D.H. Luecking, Boundary behavior of Green potentials. *Proc. Am. Math. Soc.* 96 (1986) 481–488.
7. Y. Mizuta, Boundary limits of Green potentials of general order. *Proc. Am. Math. Soc.* 101 (1987) 131–135.
8. P.J. Rippon, On the boundary behaviour of Green potentials. *Proc. London Math. Soc.* 38 (1979) 461–480.
9. M. Stoll, Boundary limits of Green potentials in the unit disc. *Arch. Math.* 44 (1985) 451–455.
10. E. Tolsted, Limiting values of subharmonic functions. *Proc. Am. Math. Soc.* 1 (1950) 636–647.
11. K.O. Widman, Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations. *Math. Scand.* 21 (1967) 17–37.
12. J.M. Wu, Content and harmonic measure – an extension of Hall's lemma. *Indiana Univ. Math. J.* 36 (1987) 403–420.
13. J.M. Wu, Boundary limits of Green's potentials along curves. *Studia Math.* 60 (1977) 137–144.