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## A universal construction for projective Hjelmslev planes of level $n$

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### 1. Introduction

In [1], B. Artmann constructs for any projective plane  $\mathcal{H}_1$  and for any positive integer  $n > 0$ , a projective Hjelmslev plane  $\mathcal{H}_n$  of level  $n$  having  $\mathcal{H}_1$  as epimorphic image. This way, one gets an infinite sequence

$$\cdots \rightarrow \mathcal{H}_n \rightarrow \cdots \rightarrow \mathcal{H}_2 \rightarrow \mathcal{H}_1$$

of epimorphic projective Hjelmslev planes and the inverse limit of such a sequence exists and is a projective plane.

In [2], D.A. Drake constructs a finite projective Hjelmslev plane of level  $n$  (for arbitrary  $n$ ) in which all point-neighbourhood-structures (which are affine Hjelmslev planes of level  $n - 1$ ) may be chosen freely up to their order. All line-neighbourhood-structures though are isomorphic. Our goal is to give a construction where one has as much freedom as possible. Moreover, our construction will be universal and will give precise information about what can be chosen arbitrarily and what is determined by other substructures.

The motivation for this work is twofold. First, by [3], projective Hjelmslev planes of level  $n$  appear naturally in affine buildings of type  $\tilde{A}_2$ . As a consequence of our construction, all such planes arise this way. Secondly, a beautiful theorem by M.A. Ronan [4] yields a construction for projective Hjelmslev planes of level  $n$  (see [3]), but in this construction, it is not completely clear how the various choices one has, act upon the geometrical structure of the planes. Our construction will answer that question.

### 2. Projective Hjelmslev planes of level $n$

The following definition is justified by [3] and [5], 3.5.5.

Suppose  $\mathcal{H}_n = (P(\mathcal{H}_n), L(\mathcal{H}_n), I)$  is a rank 2 incidence geometry with point-set  $P(\mathcal{H}_n)$ , line-set  $L(\mathcal{H}_n)$  and symmetric incidence relation  $I$ , which will be our

standard notation for any incidence relation throughout this paper. We call  $\mathcal{H}_n$  a projective Hjelmslev plane of level  $n$  (definition by means of induction on  $n$ ), or briefly a level  $n$  PH-plane, when

- (1)  $\mathcal{H}_1$  is a customary projective plane,
- (2) for every  $j \in \{0, 1, \dots, n\}$ , there is a given partition  $P_j(\mathcal{H}_n)$  of  $P(\mathcal{H}_n)$  such that  $P_j(\mathcal{H}_n)$  is finer than  $P_{j+1}(\mathcal{H}_n)$ ,  $P_0(\mathcal{H}_n) = \{\{P\} \mid P \in P(\mathcal{H}_n)\}$ ,  $P_n(\mathcal{H}_n) = \{P(\mathcal{H}_n)\}$  and every partition class of  $P_1(\mathcal{H}_n)$  contains at least two elements. Dually, there are given partitions  $L_j(\mathcal{H}_n)$  of  $L(\mathcal{H}_n)$  with similar properties. These partitions must satisfy (N.1), (N.2) and (N.2') below.

For every  $j \in \{0, 1, \dots, n\}$ , we define the incidence geometry  $\mathcal{H}_j = (P(\mathcal{H}_j), L(\mathcal{H}_j), I)$  as follows.  $P(\mathcal{H}_j) = P_{n-j}(\mathcal{H}_n)$ ,  $L(\mathcal{H}_j) = L_{n-j}(\mathcal{H}_n)$  and if  $C \in P(\mathcal{H}_j)$ ,  $D \in L(\mathcal{H}_j)$ , then  $C I D$  if there exist  $P \in C$  and  $L \in D$  such that  $P I L$  in  $\mathcal{H}_n$ . This clearly induces a canonical epimorphism  $\pi_j^!: \mathcal{H}_n \rightarrow \mathcal{H}_j$ . We now state (N.1).

- (N.1) The geometry  $\mathcal{H}_j$  as defined above is a level  $j$  PH-plane, for all  $j, 0 < j < n$ .

We call  $\mathcal{H}_j$  the underlying level  $j$  PH-plane of  $\mathcal{H}_n$ . Note that for  $j = n$ ,  $\mathcal{H}_j$  coincides with  $\mathcal{H}_n$ , justifying the notation. Now suppose  $P, Q \in P(\mathcal{H}_n)$ . If  $\pi_j^!(P) = \pi_j^!(Q)$ , then we write  $P(\sim j)Q$ . If moreover  $\pi_{j+1}^!(P) \neq \pi_{j+1}^!(Q)$ , then we denote  $P(\simeq j)Q$ . The negation of  $(\sim j)$  is  $(\not\sim j)$ . Similar definitions hold for lines. For  $P \in P(\mathcal{H}_n)$  (resp.  $L \in L(\mathcal{H}_n)$ ), we denote by  $\sigma(P)$  (resp.  $\sigma(L)$ ) the set of lines (resp. points) incident with  $P$  (resp.  $L$ ). We can now state (N.2) and (N.2')

- (N.2) If  $L, M \in L(\mathcal{H}_n)$  and  $L(\sim j)M$ , then there is a unique  $C \in P_j(\mathcal{H}_n)$  such that  $C \cap \sigma(L) = \sigma(L) \cap \sigma(M) = \sigma(M) \cap C \neq \emptyset$ .
- (N.2') Dual to (N.2).

The order  $q$  of  $\mathcal{H}_1$  ( $q$  possibly infinite) is by definition also the order of  $\mathcal{H}_n$ . If  $P \in P(\mathcal{H}_n)$ , then we denote  $S_j(P) = \{Q \in P(\mathcal{H}_n) \mid Q(\sim n-j)P\} \in P_j(\mathcal{H}_n)$  and  $LS_j(P) = \{L \in L(\mathcal{H}_n) \mid \text{there exists } Q I L \text{ such that } P(\sim n-j)Q\}$ . Dually, one defines  $S_j(L)$  and  $PS_j(L)$ ,  $L \in L(\mathcal{H}_n)$ . Observe that  $LS_j(P)$  is the union of all  $S_j(L)$  with  $L I P$ . In particular,  $S_j(P) \subseteq PS_j(L)$  if  $P I L$ . Note that, strictly speaking,  $\pi_j^!(P) = S_{n-j}(P)$ , but we use  $\pi_j^!(P)$  to express that we conceive this as a point of the geometry  $\mathcal{H}_j$ , and we use  $S_{n-j}(P)$  to express that we conceive this as a set of points of  $\mathcal{H}_n$ .

Note also that  $\mathcal{H}_j$  is canonically isomorphic to the underlying level  $j$  PH-plane of  $\mathcal{H}_k$ ,  $0 < j \leq k \leq n$ . We denote the corresponding epimorphism by  $\pi_j^k$ . So we have  $\pi_j^! = \pi_j^k \circ \pi_k^!$ .

Throughout, we keep the notation of this section, if not explicitly mentioned otherwise.

### 3. H-planes of level $n$

We define subgeometries of a given level  $n$  PH-plane  $\mathcal{H}_n$ .

Suppose  $L \in L(\mathcal{H}_n)$ , then the incidence geometry  $\mathcal{A}_n = (P(\mathcal{A}_n), L(\mathcal{A}_n), I) = (P(\mathcal{H}_n) - PS_{n-1}(L), L(\mathcal{H}_n) - S_{n-1}(L), I)$  is called an affine Hjelmslev plane of level  $n$  (or briefly a level  $n$  AH-plane). The order  $q$  of  $\mathcal{H}_n$  is also called the order of  $\mathcal{A}_n$ . Dually, for every point  $P \in P(\mathcal{H}_n)$ , the incidence geometry  $\mathcal{A}_n^* = (P(\mathcal{A}_n^*), L(\mathcal{A}_n^*), I) = (P(\mathcal{H}_n) - S_{n-1}(P), L(\mathcal{H}_n) - LS_{n-1}(P), I)$  is called a dual affine Hjelmslev plane of level  $n$  (or briefly a level  $n$  DH-plane), with order  $q$ . If moreover,  $P \perp L$ , then we call  $\mathcal{V}_n = (P(\mathcal{V}_n), L(\mathcal{V}_n), I) = (P(\mathcal{H}_n) - PS_{n-1}(L), L(\mathcal{H}_n) - LS_{n-1}(P), I)$  a helicopter Hjelmslev plane of level  $n$  (briefly a level  $n$  HH-plane), also with order  $q$ . A level  $n$  H-plane is a common name for level  $n$  PH-, AH-, DH- and HH-planes. The reader should be aware of the fact that these definitions are not completely the same as in [2], but in the context of this paper, they are the most plausible. In the usual sense of duality, one can see easily that the dual of a level  $n$  PH-plane (resp. AH-plane, DH-plane, HH-plane, H-plane) is a level  $n$  PH-plane (resp. DH-plane, AH-plane, HH-plane, H-plane).

It is also easy to see, in view of (N.2) and (N.2'), that the partitions  $P_j(\mathcal{H}_n)$  and  $L_j(\mathcal{H}_n)$ ,  $0 \leq j \leq n$ , are completely determined by  $P(\mathcal{H}_n)$ ,  $L(\mathcal{H}_n)$  and the incidence relation  $I$ . Hence, we do not need to include these partitions in the notation for  $\mathcal{H}_n$  and we can speak about *the* underlying level  $j$  PH-plane of  $\mathcal{H}_n$ . One can also define in the obvious way the underlying level  $j$  H-plane of a given level  $n$  H-plane,  $0 < j \leq n$ .

### 4. H-planes of level $j < n$ inside H-planes of level $n$

If  $P \in P(\mathcal{H}_n)$ , then we call  $S_j(P)$  a  $j$ -neighbourhood of  $P$ , or a  $j$ -point-neighbourhood or a point-neighbourhood or simply a neighbourhood of  $P$ . Dually for lines. Suppose now  $P \perp L$ ,  $L \in L(\mathcal{H}_n)$ . Then the sets  $b_p = S_j(P) \cap \sigma(L)$  and  $b_l = S_{n-j}(L) \cap \sigma(P)$  are non-empty. The pair  $(b_p, b_l)$  has the property that the set of lines incident with all elements of  $b_p$  is exactly  $b_l$  and the set of points incident with all elements of  $b_l$  is exactly  $b_p$  (see [5], 5.1.9). We call the pair  $(b_p, b_l)$  a  $j$ -short line. If  $b$  is a  $j$ -short line, then we denote  $b = (b_p, b_l)$ . The set of all  $j$ -short lines of  $\mathcal{H}_n$  is denoted by  $B_n^j$ . Similarly, the set of all  $j$ -short lines of  $\mathcal{H}_m$ ,  $m < n$ , is denoted by  $B_m^j$ . By [4], 5.1.10, the projection  $\pi_{n-1}^n(b_p)$  is the point-set corresponding to a  $(j - 1)$ -short line of  $\mathcal{H}_{n-1}$  and  $\pi_{n-1}^n(b_l)$  is the line-set corresponding to a  $j$ -short line of  $\mathcal{H}_{n-1}$ , for all  $b \in B_n^j$ ,  $0 < j < n$ . Hence  $(\pi_{n-1}^n(b_p), \pi_{n-1}^n(b_l))$  is again a short line in  $\mathcal{H}_{n-1}$  if and only if  $j = 0$  or  $j = n$  (a short line is an  $i$ -short line for some  $i$ ,  $0 \leq i \leq n$ ).

Suppose now  $C \in P_j(\mathcal{H}_n)$ . We define the incidence geometry  $\mathcal{H}_j(C) = (P(C), L(C), I)$  as follows:  $P(C) = C$ ,  $L(C) = \{b \mid b \in B_n^j \text{ and } b_p \subseteq C\}$  and for

$P \in P(C), b \in L(C), P I b$  if  $P \in b_p$ . Then it is well-known (see e.g. [2]), that  $\mathcal{H}_j(C)$  is a level  $j$  AH-plane. If  $0 < k < j$ , then  $\pi_{n-k}^n$  defines a bijection from  $P_j(\mathcal{H}_n)$  to  $P_{j-k}(\mathcal{H}_{n-k})$ . Suppose  $C' = \pi_{n-k}^n(C)$ , then we can consider similarly as above  $\mathcal{H}_{j-k}(C')$ . This is canonically isomorphic to the underlying level  $j - k$  AH-plane of  $\mathcal{H}_j(C)$ . We can identify these two structures and denote it by  $\mathcal{H}_{j-k}(C)$ . Dually, one can define for every  $D \in L_j(\mathcal{H}_n)$  and every  $k < j$  a unique level  $(j - k)$  DH-plane  $\mathcal{H}_{j-k}(D)$ , where  $\mathcal{H}_j(D) = (\{b \mid b \in B_n^{n-j} \text{ and } b_l \subseteq D\}, D, I)$ . Suppose now  $b \in B_k, 0 < j < k < n$ . Then  $b_p$  is a subset of some  $C \in P_j(\mathcal{H}_k)$ . Let  $(\pi_k^n)^{-1}(C) = C' \in P_{n-k+j}(\mathcal{H}_n)$ . So  $b$  is a line in  $\mathcal{H}_j(C')$  and hence, if  $k < m \leq n$ , then  $b$  can be viewed as an  $(m - k)$ -line neighbourhood in  $\mathcal{H}_{m-k+j}(C')$ . Similarly as above, this defines a level  $m - k$  HH-plane  $\mathcal{H}_{m-k}(b)$  in  $\mathcal{H}_{m-k+j}(C')$ . So the points of  $\mathcal{H}_{m-k}(b)$  are all  $j$ -short lines of  $\mathcal{H}_{m-k+j}(C')$  which line-set is a subset of the  $(m - k)$ -line neighbourhood  $b$  and the lines of  $\mathcal{H}_{m-k}(b)$  are all elements of the latter line-neighbourhood. Hence, the points of  $\mathcal{H}_{m-k}(b)$  are all  $j$ -short lines  $b'$  of  $\mathcal{H}_m$  such that  $\pi_k^n(b_i) = b_i$ , and since lines in  $\mathcal{H}_{m-k+j}(C')$  are  $(m - k + j)$ -short lines of  $\mathcal{H}_m$ , the lines of  $\mathcal{H}_{m-k}(b)$  are all  $(m - k + j)$ -short lines  $b''$  of  $\mathcal{H}_m$  such that  $\pi_k^n(b_p) = b_p$ . With this notation,  $b' I b''$  if and only if  $b'_p \subseteq b''_p$  if and only if  $b'_l \subseteq b''_l$ . We can also obtain  $\mathcal{H}_{m-k}(b)$  in a dual way, starting from  $b_l$ ; then  $\mathcal{H}_{m-k}(b)$  arises as a point-neighbourhood in a DH-plane which is a line-neighbourhood in  $\mathcal{H}_m$ . Both ways are equivalent and the above description of the points and lines in  $\mathcal{H}_{m-k}(b)$  is self-dual.

We can give similar definitions in level  $n$  AH-planes (resp. DH-planes, HH-planes). This way, point-neighbourhoods are structured to AH-planes (resp. HH-planes, HH-planes), line-neighbourhoods are structured to HH-planes (resp. DH-planes, HH-planes) and the H-planes related to short lines are all HH-planes.

### 5. The local data of a level $n$ H-plane

By definition, the local data of  $\mathcal{H}_n$  are given by the 7-tuple  $(P(\mathcal{H}_n), L(\mathcal{H}_n), \mathcal{H}_{n-1}, \{\mathcal{H}_{n-1}(P^1) \mid P^1 \in P(\mathcal{H}_1)\}, \{\mathcal{H}_{n-1}(L^1) \mid L^1 \in L(\mathcal{H}_1)\}, \{(P, \pi_{n-1}^n(P)) \mid P \in P(\mathcal{H}_n)\}, \{(L, \pi_{n-1}^n(L)) \mid L \in L(\mathcal{H}_n)\})$ . Now, what information about  $\mathcal{H}_n$  can not be obtained a priori by the local data? Well, one knows the line structure of all  $(n - 1)$ -point-neighbourhoods, but it is not known which lines of  $\mathcal{H}_n$  intersect a given  $(n - 1)$ -point-neighbourhood in a given line of that neighbourhood. At least, this is not explicitly mentioned in the local data. The following example shows that this information cannot be obtained by the local data. For a suitable notation of points and lines, the local data of the classical level 2 PH-planes over the rings  $Z(\text{mod } 4)$  and  $Z(\text{mod } 2)[t]/t^2$  (the dual numbers over  $Z(\text{mod } 2) = \text{GF}(2)$ ) are identical and yet, the corresponding PH-planes are not isomorphic. But for  $n > 2$ , it will follow from our universal construction that the local data

determine  $\mathcal{H}_n$  completely and that is why we have to treat the case  $n = 2$  separately. Similary, one defines the local data for any H-plane.

We are now ready to give the universal construction, first for the case  $n = 2$ .

**6. A universal construction for level 2 PH-planes**

In every dual affine plane, the relation “being non-collinear with” is an equivalence relation in the set of points. We call the corresponding equivalence classes “non-collinearity classes”. We denote by  $\cup$  the disjoint union.

**THEOREM 1.** *Suppose  $\mathcal{H}_1 = (P(\mathcal{H}_1), L(\mathcal{H}_1), I)$  is any projective plane of order  $q$ ,  $q$  possibly infinite. We assign to every point  $P \in P(\mathcal{H}_1)$  an arbitrary affine plane  $\mathcal{A}(P)$  of order  $q$  and to every line  $L \in L(\mathcal{H}_1)$  an arbitrary dual affine plane  $\mathcal{A}^*(L)$  of order  $q$ . Then there exists a level 2 PH-plane  $\mathcal{H}_2$  with local data  $(P(\mathcal{H}_2) = \cup\{Q \mid Q \text{ is a point of } \mathcal{A}(P)\} \mid P \in P(\mathcal{H}_1)\}, L(\mathcal{H}_2) = \cup\{M \mid M \text{ is a line of } \mathcal{A}^*(L)\} \mid L \in L(\mathcal{H}_1)\}, \mathcal{H}_1, \{\mathcal{A}(P) \mid P \in P(\mathcal{H}_1)\}, \{\mathcal{A}^*(L) \mid L \in L(\mathcal{H}_1)\}, \{(Q, P) \mid Q \text{ is a point of } \mathcal{A}(P), P \in P(\mathcal{H}_1)\}, \{(M, L) \mid M \text{ is a line of } \mathcal{A}^*(L), L \in L(\mathcal{H}_1)\})$ .*

*Proof.* Suppose  $P \in P(\mathcal{H}_1)$ . Choose an arbitrary bijection  $\beta_P$  from the set of lines of  $\mathcal{H}_1$  incident with  $P$  to the set of parallel classes of lines in  $\mathcal{A}(P)$ . Suppose  $L \in L(\mathcal{H}_1)$ . Choose an arbitrary bijection  $\beta_L$  from the set of points of  $\mathcal{H}_1$  incident with  $L$  to the set of non-collinearity classes of points in  $\mathcal{A}^*(L)$ . Suppose  $P \perp L$  in  $\mathcal{H}_1$ . Choose an arbitrary bijection  $\beta_{(P,L)}$  from the set  $\beta_P(L)$  of parallel lines of  $\mathcal{A}(P)$  to the set  $\beta_L(P)$  of non-collinear points of  $\mathcal{A}^*(L)$ . Denote  $\beta_{(L,P)} = \beta_{(P,L)}^{-1}$ . We now define incidence in  $\mathcal{H}_2$ . Let  $Q \in P(\mathcal{H}_2)$  and  $M \in L(\mathcal{H}_2)$ . Suppose  $Q$  is a point of  $\mathcal{A}(P)$ ,  $P \in P(\mathcal{H}_1)$  and  $M$  is a line of  $\mathcal{A}^*(L)$ ,  $L \in L(\mathcal{H}_1)$ . Then  $Q \perp M$  if and only if  $P \perp L$  in  $\mathcal{H}_1$  and the unique line  $L^*$  of  $\mathcal{A}(P)$  of the parallel class  $\beta_P(L)$  incident with  $Q$  is the image under  $\beta_{(L,P)}$  of the unique point  $P^*$  of  $\mathcal{A}^*(L)$  of the non-collinearity class  $\beta_L(P)$  incident with  $M$ , i.e.  $\beta_{(L,P)}(P^*) = L^*$ . We show that  $\mathcal{H}_2$  is a level 2 PH-plane.

(1) Suppose  $M_1, M_2 \in L(\mathcal{H}_2)$  and  $M_1 (\simeq 0) M_2$ . We show that there is a unique point  $Q \in L(\mathcal{H}_2)$  incident with both  $M_1$  and  $M_2$ . Throughout, let  $i = 1, 2$ . Suppose  $M_i$  is a line of  $\mathcal{A}^*(L_i)$ ,  $L_i \in L(\mathcal{H}_1)$ , then  $L_1 \neq L_2$ . Let  $P = \sigma(L_1) \cap \sigma(L_2)$  in  $\mathcal{H}_1$ . Clearly  $\sigma(M_1) \cap \sigma(M_2)$  must be a subset of the set of points of  $\mathcal{A}(P)$ . Let  $C_i = \beta_P(L_i)$ , then  $C_1 \neq C_2$ . Let  $D_i = \beta_{L_i}(P)$  and  $P_i$  be the unique point of  $\mathcal{A}^*(L_i)$  in  $D_i$ , incident with  $M_i$ . Finally, let  $K_i = \beta_{(L,P)}(P_i)$ . Clearly, every point  $Q_i$  incident with  $M_i$  and lying in  $\mathcal{A}(P)$  is incident with  $K_i$  in  $\mathcal{A}(P)$ . Hence  $\sigma(M_1) \cap \sigma(M_2) = \sigma(K_1) \cap \sigma(K_2)$  in  $\mathcal{A}(P)$ . Since  $K_i \in C_i$  and  $C_1 \neq C_2$ ,  $\sigma(K_1) \cap \sigma(K_2)$  is a singleton, which proves the assertion.

(2) Dually, one can show that, if  $Q_1, Q_2 \in P(\mathcal{H}_2)$  and  $Q_1 (\simeq 0) Q_2$ , there is a unique line  $M \in L(\mathcal{H}_2)$  incident with both  $Q_1$  and  $Q_2$ .

(3) Suppose now  $M_1, M_2 \in L(\mathcal{H}_2)$  and  $M_1 (\simeq 1) M_2$ . Suppose again that  $M_i$  is

a line in  $\mathcal{A}^*(L_i), i = 1, 2$ , then  $L_1 = L_2 =: L$ . So  $M_1$  and  $M_2$  are two distinct lines in  $\mathcal{A}^*(L)$  and hence they have a unique point  $P^*$  of  $\mathcal{A}^*(L)$  in common. Denote by  $D$  the unique non-collinearity class of points of  $\mathcal{A}^*(L)$  containing  $P^*$  and let  $P = \beta_L^{-1}(D), L^* = \beta_{(L,P)}(P^*)$ . Note that  $L^*$  is a line of  $\mathcal{A}(P)$  and  $L^* \in \beta_P(L)$ . By definition, all points of  $\mathcal{A}(P)$  incident with  $L^*$  are incident with both  $M_1$  and  $M_2$  in  $\mathcal{H}_2$  and no other points of  $\mathcal{A}(P)$  are incident with either  $M_1$  or  $M_2$ . If  $Q \in P(\mathcal{H}_2)$  is incident with both  $M_1$  and  $M_2$  in  $\mathcal{H}_2$ , with  $Q$  not a point of  $\mathcal{A}(P)$ , then by the second part (2) of this proof,  $M_1 = M_2$ , a contradiction. This shows (N.2) completely.

(4) Dually, one shows (N.2'). The other axioms are trivial to verify. Q.E.D.

Clearly, every level 2 PH-plane must be constructed as in the above proof. Hence, this yields a universal construction for all level 2 PH-planes.

REMARK 1. This universal construction of level 2 PH-planes implies also a universal construction of level 2 H-planes by deleting the suitable subsets of points and lines.

We now turn to the general case.

### 7. The general case

DEFINITION. Suppose  $\mathcal{A}_n$  is a level  $n$  AH-plane (with underlying level 1 AH-plane  $\mathcal{A}_1$ ) and let  $L^1, M^1 \in L_{n-1}(\mathcal{A}_n) = L(\mathcal{A}_1)$  (notation of section 2). The level  $n - 1$  HH-planes  $\mathcal{H}_{n-1}(L^1)$  and  $\mathcal{H}_{n-1}(M^1)$  are called parallel if  $L^1$  and  $M^1$  are parallel in the affine plane  $\mathcal{A}_1$ .

EXAMPLE. With the notation of the previous sections, let  $\mathcal{H}_n$  be a level  $n$  PH-plane with underlying level  $j$  PH-plane  $\mathcal{H}_j, 0 < j \leq n$  and canonical epimorphisms  $\pi_j^n$ . Suppose  $P^1 \in P(\mathcal{H}_1)$  and  $L_1 \in L(\mathcal{H}_1)$  and  $P^1 \perp L^1$  in  $\mathcal{H}_1$ . Then  $L^1$  defines in the level  $n - 1$  AH-plane  $\mathcal{H}_{n-1}(P^1)$  a class of mutually parallel level  $n - 2$  HH-planes as follows. Take any  $L^2 \in L(\mathcal{H}_2)$  such that  $\pi_1^2(L^2) = L^1$ . We now view  $P^1$  as an element of  $P_1(\mathcal{H}_2)$ . So  $\sigma(L^2) \cap P^1$  (in  $\mathcal{H}_2$ ) defines a line  $b$  in  $\mathcal{H}_1(P^1)$ . Hence  $\mathcal{H}_{n-2}(b)$  is a level  $n - 2$  HH-plane inside  $\mathcal{H}_{n-1}(P^1)$ . If  $M^2 \in L(\mathcal{H}_2)$  is such that  $\pi_1^2(M^2) = L^1$ , then  $\sigma(M^2) \cap P^1$  defines a line  $b'$  in  $\mathcal{H}_1(P^1)$  parallel to  $b$ . This shows our assertion. Note that this set is a maximal set of mutually parallel  $n - 2$  HH-planes in  $\mathcal{H}_{n-1}(P^1)$ .

REMARK 2. Suppose  $\mathcal{A}_n$  is a level  $n$  AH-plane (with underlying level 1 AH-plane  $\mathcal{A}_1$ ). Suppose  $\{\mathcal{H}_{n-1}(L_j^1), j \in J\}$  is a maximal set of mutually parallel level  $n - 1$  HH-planes. We can view  $\{L_j^1, j \in J\}$  as a class of parallel lines in  $\mathcal{A}_1$ . For every  $P \in P(\mathcal{A}_n)$ , there is a unique  $j \in J$  and a unique 1-short line  $b$  of  $\mathcal{A}_n$  such that  $P \in b$ ,

and  $b$  is a point of  $\mathcal{H}_{n-1}(L_j^1)$ . Also, in that case,  $L_j^1$  is the unique line of  $\mathcal{A}_1$  for which  $\pi_1^n(P) \cap L_j^1$  in  $\mathcal{A}_1$  ( $\pi_1^n$  is a canonical epimorphism).

**LEMMA 1.** *Suppose  $\mathcal{A}_n$  is a level  $n$  AH-plane with underlying level  $n - 1$  AH-plane  $\mathcal{A}_{n-1}$  and corresponding canonical epimorphism  $\pi_{n-1}^n$ ,  $n \geq 2$ . Suppose  $\{\mathcal{H}_{n-1}(L_j^1), j \in J\}$  is a maximal set of mutually parallel level  $n - 1$  HH-planes of  $\mathcal{A}_n$  and let  $P^n \in P(\mathcal{A}_n)$ . Let  $b$  be the unique 1-short line of  $\mathcal{A}_n$  such that  $P^n \in b_p$  and  $b$  is a point of  $\mathcal{H}_{n-1}(L_j^1)$  for some (unique)  $j \in J$ . Then  $\{\mathcal{H}_{n-2}(L_i^1), i \in J\}$  is a maximal set of mutually parallel level  $n - 2$  HH-planes in  $\mathcal{A}_{n-1}$ , and if  $b'$  is the unique 1-short line of  $\mathcal{A}_{n-1}$  such that  $\pi_{n-1}^n(P^n) \in b'_p$  and  $b'$  is a point of  $\mathcal{H}_{n-2}(L_i^1)$ ,  $i \in J$ , then  $b'_i = \pi_{n-1}^n(b_i)$  and consequently  $i = j$ .*

*Proof.* Clearly  $\{\mathcal{H}_{n-2}(L_i^1), i \in J\}$  is a maximal set of mutually parallel level  $n - 2$  HH-planes in  $\mathcal{A}_{n-1}$ . We now show, under the given assumptions,  $b'_i = \pi_{n-1}^n(b_i)$ . Note that  $\pi_{n-1}^n(b_i)$  is indeed a set of lines corresponding to a 1-short line in  $\mathcal{A}_{n-1}$ . Since this set of lines completely determines the 1-short line, it suffices to show  $\pi_{n-1}^n(b_i) \subseteq b'_i$ . So suppose  $M^n \in b_i$ , then  $P^n \cap M^n$  and  $M^n$  is a line of  $\mathcal{H}_{n-1}(L_j^1)$  (by (N.2)). But  $\pi_{n-1}^n(M^n)$  is a line of  $\mathcal{H}_{n-2}(L_i^1)$  and is moreover incident with  $\pi_{n-1}^n(P)$ , hence  $\pi_{n-1}^n(M^n) \in b'_i$ . Q.E.D.

**LEMMA 2.** *Suppose  $\mathcal{H}_n$  is a level  $n$  PH-plane ( $n \geq 2$ ) and  $b \in B_2^1$  (cp. section 4). If  $c$  is a point of  $\mathcal{H}_{n-2}(b)$  ( $c$  is a 2-short line of  $\mathcal{H}_n$ ) and  $d$  is a line of  $\mathcal{H}_{n-2}(b)$  ( $d$  is an  $(n - 2)$ -short line) incident with  $c$  in  $\mathcal{H}_{n-2}(b)$  (for  $n = 2$ , this means  $c = d = b$ ), then every point  $P^n \in c_p$  is incident with every line  $L^n \in d_i$  in  $\mathcal{H}_n$ .*

*Proof.* Let  $\{P^1\} = \pi_1^n(b_p)$ , then  $c$  is a 1-short line in  $\mathcal{H}_{n-1}(P^1)$  and  $d$  is a line in  $\mathcal{H}_{n-1}(P^1)$ . Since  $\mathcal{H}_{n-2}(b)$  is an HH-plane inside  $\mathcal{H}_{n-1}(P^1)$ ,  $c_p \subseteq d_p$  (by definition of incidence in  $\mathcal{H}_{n-2}(b)$ ) and hence  $P^n \in d_p$ . Consequently  $P^n \cap L^n$  in  $\mathcal{H}_n$ . Q.E.D.

**THEOREM 2.** *Suppose  $\mathcal{H}_{n-1}$  is a level  $n - 1$  PH-plane of order  $q$ ,  $q$  possibly infinite. There exists a level  $n$  PH-plane  $\mathcal{H}_n$  with underlying level  $n - 1$  PH-plane  $\mathcal{H}_{n-1}$  such that (UC.1) through (UC.n) holds.*

(UC.1) for every  $b \in B_{n-1}^j$ ,  $0 \leq j \leq n - 1$ ,  $\mathcal{H}_1(b)$  is any desired level 1 H-plane (AH-plane ( $j = 0$ ), DH-plane ( $j = n - 1$ ) or HH-plane (all other cases)) of order  $q$ .

Suppose  $b \in B_k^j$ ,  $0 \leq j \leq k \leq n - 1$ . We denote  $P(b) = \cup\{Q \mid Q \text{ is a point of } \mathcal{H}_1(c)\} \mid c \text{ is a point of } \mathcal{H}_{n-k-1}(b)\}$ ,  $L(b) = \cup\{M \mid M \text{ is a line of } \mathcal{H}_1(d)\} \mid d \text{ is a line of } \mathcal{H}_{n-k-1}(b)\}$ .

(UC.k) for every  $b \in B_{n-k}^j$ ,  $0 \leq j \leq n - k < n - 1$ ,  $k \neq n$ ,  $\mathcal{H}_k(b)$  is any desired level  $k$  H-plane (AH-plane ( $j = 0$ ), DH-plane ( $j = n - k$ ) or HH-plane (all other cases)) with local data  $(P(b), L(b), \mathcal{H}_{k-1}(b), \{\mathcal{H}_{k-1}(C) \mid C \text{ is a point of } \mathcal{H}_1(b)\}, \{\mathcal{H}_{k-1}(D) \mid D \text{ is a line of } \mathcal{H}_1(b)\}, \{(P, Q) \mid Q \in B_{n-1}^0 = P(\mathcal{H}_{n-1}) \text{ and } P \text{ is a point of } \mathcal{H}_1(b)\})$ .

$\mathcal{H}_1(Q)$ ,  $P \in P(b)$ ,  $\{(L, M) \mid M \in B_{n-1}^{-1} = L(\mathcal{H}_{n-1}) \text{ and } L \text{ is a line of } \mathcal{H}_1(M), L \in L(b)\}$ .

(UC.n)  $\mathcal{H}_n$  has local data  $(\cup\{Q \mid Q \text{ is a point of } \mathcal{H}_1(P^{n-1}) \mid P^{n-1} \in P(\mathcal{H}_{n-1})\}, \cup\{M \mid M \text{ is a line of } \mathcal{H}_1(L^{n-1}) \mid L^{n-1} \in L(\mathcal{H}_{n-1})\}, \mathcal{H}_{n-1}, \{\mathcal{H}_{n-1}(c) \mid c \in B_1^0\}, \{\mathcal{H}_{n-1}(d) \mid d \in B_1^1\}, \{(P, Q) \mid Q \in B_{n-1}^0 \text{ and } P \text{ is a point of } \mathcal{H}_1(Q)\}, \{(L, M) \mid M \in B_{n-1}^{-1} \text{ and } L \text{ is a line of } \mathcal{H}_1(M)\})$ .

*Proof.* First note that e.g. in (UC.k), expressions as  $\mathcal{H}_{k-1}(b)$  live in  $\mathcal{H}_{n-1}$  and hence the local data of  $\mathcal{H}_k(b)$  only depend on  $\mathcal{H}_{n-1}$  and the choices we made in (UC.1) up to (UC.k - 1). Hence, the condition (UC.k) is equivalent to Theorem 2 applied for  $k = n$  and for  $\mathcal{H}_n$  a general HH-plane. So it is obvious that we will proceed by induction on  $n > 0$ . The theorem is trivial for  $n = 1$ , and for  $n = 2$ , it is equivalent to Theorem 1. So suppose  $n > 2$ . So we assume that the theorem holds for  $n - 1$ . But clearly, the theorem then also holds for level  $n - 1$  HH-planes, AH-planes and DH-planes (cp. Remark 1). Hence, as remarked above, the conditions (UC.1) up to (UC.n - 1) will follow from the induction hypothesis. So, it suffices to show that the local data in (UC.n) determine  $\mathcal{H}_n$  completely.

(1) Note that the local data determine completely all partitions and hence also the canonical epimorphisms. So we can use the notation  $\pi_j^k$  for these epimorphisms (as in Section 2),  $0 < j \leq k \leq n$ .

(2) Lemma 2 shows that, if  $\mathcal{H}_n$  exists, incidence in  $\mathcal{H}_n$  is determined by the local data. Indeed, let  $P^n \in P(\mathcal{H}_n)$ ,  $L^n \in L(\mathcal{H}_n)$  and denote  $P^j = \pi_j^n(P^n)$ ,  $L^j = \pi_j^n(L^n)$ ,  $0 < j \leq n$ . If  $P^2 \text{ I } L^2$  in  $\mathcal{H}_2$ , then we must define  $P^n \text{ I } L^n$ . Suppose now  $P^2 \text{ I } L^2$  in  $\mathcal{H}_2$ . Then  $L^2$  and  $P^1$  define a unique element  $b \in B_2^1$  such that  $b_p = \sigma(L^2) \cap S_1(P^2)$  (in  $\mathcal{H}_2$ ) and  $b_l = \sigma(P^2) \cap S_1(L^2)$ . By the example preceding remark 2,  $L^1$  defines a class of parallel lines in  $\mathcal{H}_1(P^1)$  (one of these lines is  $b$ ) and this defines a maximal set of parallel level  $n - 2$  HH-planes in  $\mathcal{H}_{n-1}(P^1)$  (amongst them  $\mathcal{H}_{n-2}(b)$ ). Denote by  $c$  the unique 1-short line in  $\mathcal{H}_{n-1}(P^1)$  such that  $P^n \in c_p$  and  $c$  is a point of  $\mathcal{H}_{n-2}(b)$  (cp. Remark 2). Dually, we have an  $(n - 2)$ -short line  $d$  in  $\mathcal{H}_{n-1}(L^1)$  such that  $L^n \in d_l$  and  $d$  is a line of  $\mathcal{H}_{n-2}(b)$ . By Lemma 2, we must define  $P^n \text{ I } L^n$  if and only if  $c \text{ I } d$  in  $\mathcal{H}_{n-2}(b)$ .

We now show in four steps that  $\mathcal{H}_n = (P(\mathcal{H}_n), L(\mathcal{H}_n), \text{I})$ , I as defines above, is a level  $n$  PH-plane.

(3) We determine the set  $T$  of points of  $\mathcal{H}_{n-1}(P^1)$  (for given  $P^1$  as above) incident with  $L^n$  ( $L^n$  also as above). Let also  $d$  be as above, then  $T$  is the union of all sets of points corresponding to all 1-short lines  $c'$  of  $\mathcal{H}_{n-1}(P^1)$  incident with  $d$  in  $\mathcal{H}_{n-2}(b)$ . Since  $d$  is a line of  $\mathcal{H}_{n-1}(P^1)$ ,  $T$  is the set of points incident with  $d$  in  $\mathcal{H}_{n-1}(P^1)$ . Hence  $T = d_p$ . Now, a  $j$ -short line  $b'$  in  $\mathcal{H}_{n-2}(b)$  corresponds to a  $(j + 1)$ -short line  $b^*$  in  $\mathcal{H}_{n-1}(P^1)$  by  $\cup\{c_p^* \mid c^* \in b'_p\} = b_p^*$  and this is the point set of a  $(j + 1)$ -short line  $b''$  in  $\mathcal{H}_n$ ,  $0 \leq j \leq n - 1$ .

(4)(a) Let, with the notation of (2),  $P^n \text{ I } L^n$  in  $\mathcal{H}_n$ . Then  $c \text{ I } d$  in  $\mathcal{H}_{n-2}(b)$ . By

projecting  $c$  and  $d$  into  $\mathcal{H}_{n-3}(b)$  (for  $n = 3$ , this means: map  $c$  and  $d$  down onto  $b$ ), we see, by Lemma 1 and Lemma 2, that  $P^{n-1} \text{ I } L^{n-1}$ .

(b) Suppose now  $P^{n-1} \text{ I } L^{n-1}$  in  $\mathcal{H}_{n-1}$  with  $P^{n-1} \in P(\mathcal{H}_{n-1})$  and  $L^{n-1} \in L(\mathcal{H}_{n-1})$  and choose an arbitrary  $L^n \in L(\mathcal{H}_n)$  such that  $\pi_{n-1}^n(L^n) = L^{n-1}$ . Denote  $P^1 = \pi_1^{n-1}(P^{n-1})$ . The set of points of  $\mathcal{H}_n$  incident with  $L^n$  in  $\mathcal{H}_{n-1}(P^1)$  is the set of points incident with a line  $d$  of  $\mathcal{H}_{n-1}(P^1)$ , by (3). But by (4)(a), the projection  $d'$  of  $d$  onto  $\mathcal{H}_{n-2}(P^1)$  is incident in  $\mathcal{H}_{n-2}(P^1)$  with all points of  $\mathcal{H}_{n-1}$  incident with  $L^{n-1}$  and lying in  $S_{n-2}(P^{n-1})$ . Hence  $P^{n-1} \text{ I } d'$  in  $\mathcal{H}_{n-2}(P^1)$ . By a general "lifting property" for H-planes, there exists a point  $P^n$  incident with  $d$  in  $\mathcal{H}_{n-1}(P^1)$  with  $\pi_{n-1}^n(P^n) = P^{n-1}$  and hence  $P^n \text{ I } L^n$  in  $\mathcal{H}_n$ . This shows the axiom (N.1).

(5) Suppose  $L^n, M^n \in L(\mathcal{H}_n)$  with  $L^n (\simeq 0) M^n$ . Let  $L^1 = \pi_1^n(L^n)$ ,  $M^1 = \pi_1^n(M^n)$ . Then  $M^1 \neq L^1$  and hence, in  $\mathcal{H}_1$ , they have a unique intersection point  $P^1$ . So the points of intersection of  $L^n$  and  $M^n$  are to be found in  $\mathcal{H}_{n-1}(P^1)$ . Denote by  $d$  (resp.  $e$ ) the line of  $\mathcal{H}_{n-1}(P^1)$  such that  $\sigma(d) \subseteq \sigma(L^n)$  (resp.  $\sigma(e) \subseteq \sigma(M^n)$ ) (cp. (3)). Since  $L^1$  and  $M^1$  define non-parallel lines in  $\mathcal{H}_1(P^1)$ ,  $d (\simeq 0) e$  in  $\mathcal{H}_{n-1}(P^1)$  and hence  $d$  and  $e$  meet in a unique point  $P^n$ . Hence  $P^n$  is the unique point of  $\mathcal{H}_n$  incident with both  $L^n$  and  $M^n$ . This shows (N.2) for  $j = 0$ . Dually, one shows (N.2') for  $j = 0$ .

(6) Suppose  $L^n, M^n \in L(\mathcal{H}_n)$  with  $L^n (\simeq j) M^n, n > j > 0$ . Again let  $L^1 = \pi_1^n(L^n) = \pi_1^n(M^n)$ . Then  $L^n (\simeq j - 1) M^n$  in  $\mathcal{H}_{n-1}(L^1)$ . Hence  $L^n$  and  $M^n$  meet in a  $(j - 1)$ -short line  $c$  in  $\mathcal{H}_{n-1}(L^1)$  by (N.2) applied in  $\mathcal{H}_{n-1}(L^1)$ . Now,  $c_p$  is a subset of the point set of  $\mathcal{H}_{n-2}(b)$ , for some  $b \in B_2^1$  ( $b$  is a point of  $\mathcal{H}_1(L^1)$ ) and exists since  $j - 1 < n - 1$ . So  $c_p$  is the set of points corresponding to  $(j - 1)$ -short line  $c'$  of  $\mathcal{H}_{n-2}(b)$ . Let  $c''$  be the  $j$ -short line in  $\mathcal{H}_n$  corresponding to  $c'$  as in (3), then  $c_p'' \subseteq \sigma(L^n) \cap \sigma(M^n)$  in  $\mathcal{H}_n$ . Since  $L^n$  and  $M^n$  determine the same  $(n - 2)$ -line neighbourhood in  $\mathcal{H}_{n-1}(S_1(P^n))$ , for every  $P^n \in \sigma(L^n) \cap \sigma(M^n)$ ,  $P^n$  belongs to the set of points corresponding to a 1-short line  $c^*$  for which  $c_p^* \subseteq \sigma(L^n) \cap \sigma(M^n)$ , and hence  $c_p'' = \sigma(L^n) \cap \sigma(M^n)$ . This shows (N.2) completely, remarking that the case  $j = n$  is trivial in view of (N.2') for  $j = 0$ . Dually, (N.2') also holds in  $\mathcal{H}_n$ .

This completes the proof of the theorem. Q.E.D.

Clearly, every level  $n$  PH-plane is constructed as in the above proof. So Theorem 2 yields a universal construction for all level  $n$  PH-planes. It also shows that the local data of a level  $n$  H-plane,  $n > 2$ , determine the incidence relation completely. Since the third component of the local data of  $\mathcal{H}_n$  was arbitrary in Theorem 2, this also implies that for any given level  $n - 1$  PH-plane  $\mathcal{H}_{n-1}$ , there exists a level  $n$  PH-plane  $\mathcal{H}_n$  with underlying level  $n - 1$  PH-plane  $\mathcal{H}_{n-1}$ . Hence the following corollaries.

**COROLLARY 1.** *Every level  $n$  PH-plane  $\mathcal{H}_n$  occurs in an infinite sequence  $\dots \rightarrow \mathcal{H}_{n+1} \rightarrow \mathcal{H}_n \rightarrow \dots \rightarrow \mathcal{H}_1$  of PH-planes (where  $\mathcal{H}_k$  is of level  $k$  and where  $\mathcal{H}_j$  is isomorphic to the underlying level  $j$  PH-plane of  $\mathcal{H}_k, j < k$ ) with as inverse limit*

