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On the Gauss maps of space curves in characteristic p

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§0. Introduction

In this article, we discuss the extensions of function fields defined by Gauss maps of space curves in positive characteristic p . As is well-known, if $p = 0$, then the Gauss maps of space curves are always birational onto its image. However, if $p > 0$, then this is not true.

Precisely speaking, our purpose is to investigate the sets $\mathcal{K}'$, $\mathcal{K}'_{\text{un}}$ and $\mathcal{K}'_{\text{imm}}$ defined as follows: for a smooth connected complete curve C over an algebraically closed field of positive characteristic p , we define $\mathcal{K}'$ as the set of subfields K' of $K(C)$ such that there exists a morphism ι from C to some projective space $\mathbb{P}$, birational onto its image, such that $\iota(C)$ is not a line in $\mathbb{P}$ and the extension of function fields defined by the Gauss map coincides with $K(C)/K'$; replacing the word “morphism” above by “unramified morphism” and “closed immersion”, we moreover define $\mathcal{K}'_{\text{un}}$ and $\mathcal{K}'_{\text{imm}}$, respectively.

Our main results are

THEOREM 0.1:

- (a) *If C is an ordinary elliptic curve, then $\mathcal{K}'_{\text{imm}}$ contains any subfield K' of $K(C)$ such that $K(C)$ is finite, inseparable over K' and the separable closure of K' in $K(C)$ is a cyclic extension over K' with degree indivisible by p .*
- (b) *If C is a supersingular elliptic curve, then*

$$\mathcal{K}'_{\text{imm}} = \begin{cases} \{K(C)^p, K(C)^{p^2}\} & \text{if } p = 2 \\ \{K(C), K(C)^p\} & \text{otherwise.} \end{cases}$$

(see Section 5)

THEOREM 0.2:

- (a) *If C is a curve of genus $g \geq 2$, then an arbitrary element K' of $\mathcal{K}'_{\text{imm}}$ is of the form $K(C)^{p^l}$ for some integer $l \geq 0$.*
 (b) *Moreover, for an integer $l > 0$, let*

$$C \rightarrow C^{(p)} \rightarrow \cdots \rightarrow C^{(p^l)}$$

be a sequence of Frobenius morphisms of C . Then, the following conditions are equivalent:

- (1) $\mathcal{K}'_{\text{imm}}$ contains $K(C)^{p^l}$;
 (2) *there exists a rank 2 vector bundle $\mathcal{E}$ on $C^{(p^l)}$ such that $\mathcal{E}_{C^{(p)}}$ is stable and $\mathcal{E}_C$ is isomorphic to the bundle $\mathcal{P}_C^1(\mathcal{L})$ of principal parts of $\mathcal{L}$ of first order for some line bundle $\mathcal{L}$ on C , where $\mathcal{E}_{C^{(p)}}$ and $\mathcal{E}_C$ are the pull-backs of $\mathcal{E}$ to $C^{(p)}$ and C , respectively.*

(see Corollaries 4.4 and 6.2)

We shall show also the following results: $\mathcal{K}'$ (respectively, $\mathcal{K}'_{\text{imm}}$) contains $K(C)$ if and only if $p \neq 2$ (see Corollary 2.3); for a proper subfield K' of $K(C)$, $\mathcal{K}'$ contains K' if and only if $K(C)$ is finite, inseparable over K' (see Corollaries 2.2 and 3.4); if C is rational, then $\mathcal{K}' = \mathcal{K}'_{\text{imm}}$ (see Corollary 3.6); and, $\mathcal{K}'_{\text{un}} = \mathcal{K}'_{\text{imm}}$ for any C (see Corollary 4.3).

As a corollary to our results, one can show that a smooth plane curve has separable degree 1 over the dual curve via the Gauss map (see Corollary 4.5 and [16, p. 342]; compare with [24, Proposition 4.2]), and that if C is a Tango–Raynaud curve (see, for example, [2] or [27]), then $\mathcal{K}'_{\text{imm}}$ contains $K(C)^p$ (see Corollary 6.5).

The fundamental results in our study are Theorems 2.1, 3.1 and 4.1, from which we shall deduce all the results above.

§1. Gauss maps

Throughout this article, we shall work over an algebraically closed field k of positive characteristic, denoted by p .

Let C be a smooth, connected, complete curve defined over k . For a morphism ι from C to a projective space $\mathbb{P}$, birational onto its image, let V be a vector space $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$, and $\mathcal{P}_C^1(\iota^*\mathcal{O}_{\mathbb{P}}(1))$ the bundle of principal parts of $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ of first order on C (see, for example, [14, IV, A], [19, §1] or [25, §§2 and 6]). We have a natural map

$$a^1: V \otimes_k \mathcal{O}_C \rightarrow \mathcal{P}_C^1(\iota^*\mathcal{O}_{\mathbb{P}}(1)),$$

which is surjective if and only if ι is unramified. The image of a^1 is a quotient bundle of $V \otimes_k \mathcal{O}_C$ of rank 2.

Let $\mathbb{G}$ be a Grassmann manifold consisting of 2-dimensional quotient spaces of V with the universal quotient bundle

$$V \otimes_k \mathcal{O}_{\mathbb{G}} \rightarrow \mathcal{Q}.$$

DEFINITION: By virtue of the universality of $\mathbb{G}$, one obtains from a^1 a morphism

$$C \rightarrow \mathbb{G}.$$

We call this morphism the *Gauss map* of C in $\mathbb{P}$ via ι (see, for example, [3, 2, §4], [4, 1, (e)] or [14, IV, B]).

In particular, the image of a^1 is isomorphic to the pull-back of $\mathcal{Q}$ to C , denoted by $\mathcal{Q}_C$. We identify these bundles.

If one considers $\mathbb{G}$ to be consisting of lines in $\mathbb{P}$, then, for a general point x of C , the image of x in $\mathbb{G}$ under the Gauss map is corresponding to the tangent line to $\iota(C)$ at $\iota(x)$.

We always assume that $\iota(C)$ is not a line in $\mathbb{P}$, and denote by C^* the image of C in $\mathbb{G}$ under the Gauss map, which is called the *dual curve* of C if $\mathbb{P} = \mathbb{P}^2$ (see, for example, [5, I, Exercise 7.3] or [14, I, C]).

DEFINITION: For a curve C , we define $\mathcal{K}'$ as the set of subfields K' of $K(C)$ as follows: there exists a morphism ι from C to some projective space $\mathbb{P}$, birational onto its image, such that $\iota(C)$ is not a line in $\mathbb{P}$ and the extension $K(C)/K(C^*)$ defined by the Gauss map coincides with $K(C)/K'$. Replacing the word “morphism” above by “unramified morphism” and “closed immersion”, we moreover define $\mathcal{K}'_{\text{un}}$ and $\mathcal{K}'_{\text{imm}}$, respectively.

REMARK: It is well-known that if $p = 0$, then $\mathcal{K}' = \{K(C)\}$.

§2. Generic projections

Our first result is

THEOREM 2.1: Let $\iota: C \rightarrow \mathbb{P}$ be a morphism birational onto its image as before, let $\Pi: \mathbb{P} \rightarrow \mathbb{P}_1$ be a projection of $\mathbb{P}$ from a general point in $\mathbb{P}$, and let $\iota_1: C \rightarrow \mathbb{P}_1$ be a composition of Π with ι . Let C^* and C_1^* be the images of C

under the Gauss maps via ι and ι_1 , respectively, and let $K(C^*)_s$ and $K(C_1^*)_s$ be the separable closures of $K(C^*)$ and $K(C_1^*)$ in $K(C)$, respectively. If $\dim \mathbb{P} \geq 3$, then

$$K(C)/K(C^*)_s = K(C)/K(C_1^*)_s.$$

Proof: Denote by P the centre of Π , let $\mathbb{G}$ and $\mathbb{G}_1$ be Grassmann manifolds of lines in $\mathbb{P}$ and $\mathbb{P}_1$, respectively, and let $\pi: \mathbb{G} \rightarrow \mathbb{G}_1$ be a rational map naturally induced by Π . Then, we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{P} & \xleftarrow{\iota} & C & \longrightarrow & C^* \subset \mathbb{G} \\ \Pi \downarrow & & \parallel & & \downarrow \quad \downarrow \pi \\ \mathbb{P}_1 & \xleftarrow{\iota_1} & C & \longrightarrow & C_1^* \subset \mathbb{G}_1. \end{array}$$

We note that, via the Plücker embeddings of $\mathbb{G}$ and $\mathbb{G}_1$ into some projective spaces, π is compatible with a linear projection of the projective space, denoted by $\wedge \pi$. Writing $\sigma(P)$ for the subset of $\mathbb{G}$ consisting of lines in $\mathbb{P}$ passing through P , we see that the base locus of π is equal to $\sigma(P)$, which coincides with the centre of $\wedge \pi$. Choose a smooth point x of C^* , and consider the embedded tangent space $T_x C^*$ to C^* at x . Counting the dimensions, one proves that $T_x C^*$ and $\sigma(P)$ do not meet for a general P , which implies that C^* is separable over C_1^* (see, for example, [28, §3]). This completes our proof.

REMARK: Similarly, one can prove that if $\dim \mathbb{P} \geq 4$, then $K(C)/K(C^*) = K(C)/K(C_1^*)$. Moreover, it can be shown that when $\dim \mathbb{P} = 3$, this equality does not hold if and only if $\iota(C)$ is strange and not contained in any plane in $\mathbb{P}$, where $\iota(C)$ is called *strange* if there exists a point in $\mathbb{P}$ which lies on all the tangent lines at smooth points of $\iota(C)$ (see, for example, [5, IV, §3] or [29, II]).

COROLLARY 2.2: *For an element K' of $\mathcal{K}'$, if $K(C)$ is separable over K' , then $K(C) = K'$.*

Proof: For a morphism $\iota: C \rightarrow \mathbb{P}$ associated to K' , using Theorem 2.1, one can reduce the problem to the case $\dim \mathbb{P} = 2$. In this case, the result is known (see, for example, [9, §9.4] or [14, p. 310]).

COROLLARY 2.3: *For a curve C , the following conditions are equivalent:*

- (1) $\mathcal{K}'$ contains $K(C)$;
- (2) $\mathcal{K}'_{\text{un}}$ contains $K(C)$;
- (3) $\mathcal{K}'_{\text{imm}}$ contains $K(C)$;
- (4) $p \neq 2$.

Proof: The implications $(3) \Rightarrow (2) \Rightarrow (1)$ are obvious.

For a morphism $\iota: C \rightarrow \mathbb{P}$, birational onto its image, we denote by m the intersection multiplicity of $\iota(C)$ and a general tangent line to $\iota(C)$ at a general point. We note that $m + 1$ is equal to the third gap of the linear system defining ι . It follows from Theorem 2.1 and [6, Proposition 4.4] that $K(C)/K(C^*)$ is separable if and only if $p \neq 2$ and $m = 2$. This proves the implication $(1) \Rightarrow (4)$.

Moreover, it follows from the Riemann–Roch theorem that if ι is a closed immersion defined by a complete linear system with sufficiently large degree, then $m = 2$. Thus, the implication $(4) \Rightarrow (3)$ follows from Corollary 2.2.

REMARK: Combining Theorem 2.1 with [6, Proposition 4.4], one can prove that if $K(C)/K(C^*)$ is not separable, then its inseparable degree is equal to m .

REMARK: Using Theorem 2.1, one can prove that if the Gauss map of a curve C in $\mathbb{P}$ via ι gives a separable extension $K(C)/K(C^*)$, then $\iota(C)$ is not strange in $\mathbb{P}$.

§3. Gauss maps and ruled surfaces

The following plays a key role throughout the rest of this article.

THEOREM 3.1: *For a subfield K' of $K(C)$ such that $K(C)$ is finite, inseparable over K' , the following conditions are equivalent:*

- (1) $\mathcal{K}'$ contains K' ;
- (2) *there exist a rank 2 vector bundle $\mathcal{E}$ on a curve C' with $K(C') = K'$ and a morphism $h: C \rightarrow \mathbb{P}(\mathcal{E})$, birational onto its image, such that the extension $K(h(C))/K(C')$ defined by the projection of $\mathbb{P}(\mathcal{E})$ over C' coincides with $K(C)/K'$.*

This is true for $\mathcal{K}'_{\text{un}}$ (respectively, $\mathcal{K}'_{\text{imm}}$) if one assumes moreover that h is unramified (respectively, a closed immersion) in (2).

We first verify the implication $(1) \Rightarrow (2)$.

LEMMA 3.2: *Consider a trivial extension (ε_0) and a unique non-trivial extension (ε_1) of $\mathcal{O}_C$ by Ω_C^1 . Then, for a line bundle $\mathcal{L}$ on C , we have a natural extension*

$$0 \rightarrow \Omega_C^1 \otimes \mathcal{L} \rightarrow \mathcal{P}_C^1(\mathcal{L}) \rightarrow \mathcal{L} \rightarrow 0,$$

which coincides with $(\varepsilon_0) \otimes \mathcal{L}$ if the degree of $\mathcal{L}$ is divisible by the characteristic p . Otherwise, the extension coincides with $(\varepsilon_1) \otimes \mathcal{L}$.

Proof: See [11, §1].

For a morphism $\iota: C \rightarrow \mathbb{P}$, birational onto its image, combining the Euler sequence on $\mathbb{P}$ with the extension above, we have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & \iota^*(\Omega_{\mathbb{P}}^1 \otimes \mathcal{O}_{\mathbb{P}}(1)) & \rightarrow & V \otimes_k \mathcal{O}_C & \rightarrow & \iota^*\mathcal{O}_{\mathbb{P}}(1) \rightarrow 0 \\
 & & \downarrow & & a^! \downarrow & & \parallel \\
 0 & \rightarrow & \Omega_C^1 \otimes \iota^*\mathcal{O}_{\mathbb{P}}(1) & \rightarrow & \mathcal{P}_C^1(\iota^*\mathcal{O}_{\mathbb{P}}(1)) & \rightarrow & \iota^*\mathcal{O}_{\mathbb{P}}(1) \rightarrow 0
 \end{array}$$

with exact rows. Let C' be the normalization of the image C^* of C under the Gauss map, and let C_0 be a section of the ruled surface $\mathbb{P}(\mathcal{Q}_C)$ over C associated to the quotient $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ of $\mathcal{Q}_C$, where $\mathcal{Q}_C$ is the image of $a^!$. We have a commutative diagram (see [11, §1])

$$\begin{array}{ccccc}
 & & C \leftarrow \mathbb{P}(\mathcal{Q}_C) & & \hookrightarrow C_0 \\
 & \swarrow & \downarrow & \searrow & \\
 \mathbb{G} & \hookrightarrow C^* & & & X \hookrightarrow \mathbb{P} \\
 & \nwarrow & C' \leftarrow \mathbb{P}(\mathcal{Q}_{C'}) & \nearrow & \\
 & & f & &
 \end{array}$$

where X is an image of $\mathbb{P}(\mathcal{Q}_C)$ in $\mathbb{P}$, and f is a natural morphism induced by the Gauss map. Intuitively, C_0 is consisting of points of contact on $\mathbb{P}(\mathcal{Q}_C)$. We see that the induced morphism $C_0 \rightarrow \mathbb{P}$ above coincides with ι via $C_0 \simeq C$.

Now, let $\mathcal{E}$ be the pull-back $\mathcal{Q}_{C'}$ and h the composition of $f|_{C_0}$ with the isomorphism $C \rightarrow C_0$. Since ι is birational onto its image, so is h , and we have $K(h(C))/K(C') = K(C)/K(C^*)$. Moreover, if ι is unramified (respectively, a closed immersion), then so is h . This completes the proof of (1) $\Rightarrow$ (2).

REMARK: The curve $\iota(C)$ is strange if and only if the morphism $\mathbb{P}(\mathcal{Q}_{C'}) \rightarrow X$ above is not finite. It follows from the proof of [18, Proposition 3] that if ι is unramified, then $\iota(C)$ is not strange except for the case when $p = 2$ and $\iota(C)$ is a conic.

The converse (2) $\Rightarrow$ (1) follows from

LEMMA 3.3: *Let $\mathcal{E}$ be a rank 2 vector bundle on a curve C' , and let $h: C \rightarrow \mathbb{P}(\mathcal{E})$ be a morphism birational onto its image. If $h(C)$ is finite, inseparable over C' via the projection of $\mathbb{P}(\mathcal{E})$, then there exists a morphism $\iota: C \rightarrow \mathbb{P}$, birational onto its image, such that the extension $K(C)/K(C^*)$ defined by the Gauss map coincides with $K(C)/K(C')$. Moreover, if h is unramified (respectively, a closed immersion), then so is ι .*

Proof: Take a sufficiently very ample line bundle $\mathcal{M}$ on C' , and let ϱ be an embedding of $\mathbb{P}(\mathcal{E})$ into a projective space $\mathbb{P}$ as a scroll defined by a complete linear system associated to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{M}$, where π is the projection of $\mathbb{P}(\mathcal{E})$. We define ι to be a composition of ϱ with h . Then, the image of each fibre of $\mathbb{P}(\mathcal{E})$ under ϱ is a line in $\mathbb{P}$, tangent to $\iota(C)$ since $h(C)$ is inseparable over C' . Denoting by V the vector space $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{M})$, we have a commutative diagram

$$\begin{array}{ccc} V_{\mathbb{P}(\mathcal{E})} & \rightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{M} \\ \downarrow & & \downarrow \\ V_{h(C)} & \rightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{M}|_{h(C)}, \end{array}$$

where all the arrows are surjective, ϱ is determined by the upper arrow, and ι is determined by the lower arrow. Let $\mathcal{L}$ be a quotient line bundle of $(\pi \circ h)^*\mathcal{E}$ associated to h (see, for example, [5, II, Proposition 7.12]). Taking the inverse image by h and a morphism $\mathbb{P}((\pi \circ h)^*\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E})$, taking the direct image by the projection of $\mathbb{P}((\pi \circ h)^*\mathcal{E})$ over C , we get

$$\begin{array}{ccc} V_C & \rightarrow & (\pi \circ h)^*(\mathcal{E} \otimes \mathcal{M}) \\ \parallel & & \downarrow \\ V_C & \rightarrow & \mathcal{L} \otimes (\pi \circ h)^*\mathcal{M}, \end{array}$$

where all the arrows are surjective. We find that this diagram coincides with

$$\begin{array}{ccc} V_C & \rightarrow & \mathcal{Q}_C \\ \parallel & & \downarrow \\ V_C & \rightarrow & \iota^*\mathcal{O}_{\mathbb{P}}(1) \end{array}$$

which is obtained from the commutative diagram under Lemma 3.2 for our ι , and the extension of function fields defined by the Gauss map of C via ι is equal to $K(C)/K(C')$.

This completes the proof of Theorem 3.1.

COROLLARY 3.4: *The set $\mathcal{K}'$ contains any subfield K' of $K(C)$ such that $K(C)$ is finite, inseparable over K' .*

Proof: Let K' be a subfield of $K(C)$ as above, and let C' be a curve with $K(C') = K'$. According to Lemma 3.5 below, there is a primitive element ψ

of $K(C)$ over K' . From ψ and a morphism $C \rightarrow C'$ defined by $K(C)/K'$, we obtain a morphism

$$h: C \rightarrow \mathbb{P}^1 \times C'.$$

Then, we find that h is birational onto its image, and the result follows from Theorem 3.1.

LEMMA 3.5: *An arbitrary finite extension of function fields of dimension 1 over k is simple.*

Proof: Let K/K' be an extension as above, and let K'_s be the separable closure of K' in K . A finite, separable extension K'_s/K' is simple. On the other hand, an arbitrary intermediate field of the purely inseparable extension K/K'_s is of the form K'^l for some integer $l > 0$ (see, for example, [5, IV, Proposition 2.5]), whose number is finite. So, it follows from [10, Theorem 15, p. 55] that K/K'_s is simple. Thus, according to [10, Theorem 14, p. 54], the extension K/K' is simple.

From now on, we shall focus our attention on $\mathcal{K}'_{\text{un}}$ and $\mathcal{K}'_{\text{imm}}$.

COROLLARY 3.6: *If C is a rational curve, then $\mathcal{K}' = \mathcal{K}'_{\text{imm}}$.*

Proof: For a proper subfield K' in $\mathcal{K}'$, let C' be a curve with $K(C') = K'$, and let h be a graph morphism of $C \rightarrow C'$ defined by $K(C)/K'$:

$$h: C \rightarrow C \times C',$$

which is a closed immersion. The result follows from Theorem 3.1 and Corollary 2.3.

REMARK: A similar argument to the above can be found in [26, Example 2.13].

COROLLARY 3.7: *For an element K' of $\mathcal{K}'_{\text{un}}$ (respectively, $\mathcal{K}'_{\text{imm}}$), let K'_1 be an intermediate field of $K(C)/K'$. If $K(C)/K'_1$ is not separable, then $\mathcal{K}'_{\text{un}}$ (respectively, $\mathcal{K}'_{\text{imm}}$) contains K'_1 .*

Proof: Let C' be a curve with $K(C') = K'$. According to Theorem 3.1, we have a vector bundle $\mathcal{E}$ of rank 2 on C' and an unramified morphism $C \rightarrow \mathbb{P}(\mathcal{E})$, birational onto its image, such that $K(h(C))/K(C') = K(C)/K'$.

Let C'_1 be a curve with $K(C'_1) = K'_1$, and let $\mathcal{E}_1$ be a pull-back of $\mathcal{E}$ by a morphism $C'_1 \rightarrow C'$ defined by K'_1/K' . Using h and a morphism $C \rightarrow C'_1$ defined by $K(C)/K'_1$, we get an unramified morphism $C \rightarrow \mathbb{P}(\mathcal{E}_1)$, birational onto its image, such that $K(h_1(C))/K(C'_1) = K(C)/K'_1$. Then, the result for $\mathcal{K}'_{\text{un}}$ follows from Theorem 3.1. If h is a closed immersion, then so is h_1 . This completes the proof.

COROLLARY 3.8: *For an element K' of $\mathcal{K}'_{\text{un}}$ (respectively, $\mathcal{K}'_{\text{imm}}$), let K'_1 be a subfield of K' such that K'/K'_1 is finite, purely inseparable, let C' and C'_1 be curves with function fields K' and K'_1 , respectively, and let $f: C' \rightarrow C'_1$ be a morphism defined by K'/K'_1 . Then, the following conditions are equivalent:*

- (1) $\mathcal{K}'_{\text{un}}$ (respectively, $\mathcal{K}'_{\text{imm}}$) contains K'_1 ;
- (2) *there exists a rank 2 vector bundle $\mathcal{E}_1$ on C'_1 such that*

$$f^* \mathcal{E}_1 \simeq \mathcal{E}$$

for some vector bundle $\mathcal{E}$ on C' associated to K' as in Theorem 3.1.

Proof: (1) $\Rightarrow$ (2). This follows from Theorem 3.1.

(2) $\Rightarrow$ (1). Let h be the morphism $C \rightarrow \mathbb{P}(\mathcal{E})$ associated to K' as in Theorem 3.1, and let h_1 be the morphism h followed by a morphism $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E}_1)$ induced from f . By virtue of Theorem 3.1, it suffices to show that h_1 is unramified, birational onto its image (respectively, a closed immersion). Since f is purely inseparable, the morphism $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E}_1)$ is injective. So, it is sufficient to verify that a tangent vector to $h(C)$ in $\mathbb{P}(\mathcal{E})$ at each point of $h(C)$ is not vanished under $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E}_1)$. But, this is true since a fibre of $\mathbb{P}(\mathcal{E})$ over C' is tangent to $h(C)$, and mapped isomorphically into $\mathbb{P}(\mathcal{E}_1)$ by $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E}_1)$.

§4. Calculation on ruled surfaces

The third of our fundamental results is

THEOREM 4.1: *Let $\iota: C \rightarrow \mathbb{P}$ be a morphism birational onto its image, let g and g' be the genera of C and C' , respectively, and denote by $p_a(f(C_0))$ the arithmetic genus of $f(C_0)$, with the same notations as before. If ι is unramified, then we have*

$$p_a(f(C_0)) = g = g'.$$

We denote by d the degree of $\iota^*\mathcal{O}_{\mathbb{P}}(1)$, that is, the degree of $\iota(C)$ in $\mathbb{P}$, and by d^* the degree of C^* in $\mathbb{G}$ via the Plücker embedding, where C^* is the image of C in $\mathbb{G}$ under the Gauss map. We write n for the degree of the extension $K(C)/K(C^*)$, that is, the degree of the Gauss map of C . We denote by ν the morphism $\mathbb{P}(\mathcal{Q}_{C'}) \rightarrow X$, and by γ the degree of the cokernel of the map a^1 .

One can easily prove the following (see, for example, [11, §1], [19, §3] or [25, §3]).

LEMMA 4.2: *With the same notations as before, we have*

$$\begin{aligned} 2g - 2 + 2d - \gamma &= \deg \mathcal{Q}_C \\ &= nd^* \\ &= (\deg f)(\deg \nu)(\deg X). \end{aligned}$$

To prove Theorem 4.1, using a generic projection of $\mathbb{P}$, we may assume that $\mathbb{P} = \mathbb{P}^2$.

Writing D for the image $f(C_0)$, we first study the numerical class of D in the ruled surface $\mathbb{P}(\mathcal{Q}_{C'})$. For a general line L in $\mathbb{P}^2$, set

$$H := \nu^*L$$

in $\mathbb{P}(\mathcal{Q}_{C'})$. Then, H is a section of $\mathbb{P}(\mathcal{Q}_{C'})$ over C' . Since X coincides with $\mathbb{P}^2$, in particular, it has degree 1, it follows from Lemma 4.2 that ν has degree d^* . Thus, we have $(H^2) = d^*$. Since $\nu|_{f(C_0)}$ is birational onto its image, we have $(D \cdot H) = d$. On the other hand, denoting by F the numerical class of a fibre of the projection of $\mathbb{P}(\mathcal{Q}_{C'})$ over C' , we obtain $(D \cdot F) = n$ from the proof of (1) $\Rightarrow$ (2) in Theorem 3.1. Therefore, using Lemma 4.2, we find that

$$D \equiv nH - (2g - 2 + d - \gamma)F.$$

Denoting by K the numerical class of a canonical divisor of $\mathbb{P}(\mathcal{Q}_{C'})$, we see

$$K \equiv -2H + (2g' - 2 + d^*)F.$$

Now, it follows from the adjunction formula that

$$2p_a(D) - 2 = n\{(2g' - 2) - (2g - 2)\} + (nd^* - 2d + n\gamma).$$

So, we obtain by Lemma 4.2 that

$$\begin{aligned} & (2p_a(D) - 2) - (2g - 2) \\ &= n\{(2g' - 2) - (2g - 2)\} + (n - 1)\gamma. \end{aligned}$$

We here have $p_a(D) \geq g \geq g'$, $n \geq 1$, and $\gamma = 0$ since ι is unramified. Therefore, the result follows from the equality above. This completes the proof of Theorem 4.1.

COROLLARY 4.3: $\mathcal{K}'_{\text{un}} = \mathcal{K}'_{\text{imm}}$.

Proof: It follows from Theorem 4.1 that $f(C_0)$ is smooth, so that the unramified morphism h in Theorem 3.1 is a closed immersion.

COROLLARY 4.4: *If C has genus $g \geq 2$, then an arbitrary element K' of $\mathcal{K}'_{\text{un}}$ is of the form $K(C)^{p^l}$ for some integer $l \geq 0$.*

Proof: Use Hurwitz's formula (see, for example, [5, IV, §2]).

COROLLARY 4.5: *A smooth plane curve has separable degree 1 over the dual curve via the Gauss map.*

Proof: If $g \geq 2$, then the result follows directly from Corollary 4.4. If $g = 0$, then the result clearly follows. We hence assume that $g = 1$, so that $d = 3$. We obtain from Theorem 4.1 that $d^* \geq 3$, and the result follows from Corollary 2.2 and Lemma 4.2.

REMARK: At the last part of the proof above, we do not need Theorem 4.1 because we always have $n \leq d$.

REMARK: This result answers a question in [16, p. 342], and improves [24, Proposition 4.2].

§5. Elliptic curves

This section is devoted to elliptic curves.

THEOREM 5.1: *Let C be an ordinary elliptic curve in characteristic p , and let K' be a subfield of $K(C)$ such that $K(C)$ is finite, inseparable over K' . If the separable closure of K' in $K(C)$ is cyclic over K' and the separable degree of $K(C)/K'$ is not divisible by p , then $\mathcal{K}'_{\text{imm}}$ contains K' .*

Proof: Let C' be a curve with $K(C') = K'$. By virtue of Theorem 3.1, it is sufficient to show that there exist a rank 2 vector bundle $\mathcal{E}$ on C' and a closed immersion $h: C \rightarrow \mathbb{P}(\mathcal{E})$ such that $K(h(C))/K(C') = K(C)/K'$.

Writing s and q respectively for the separable and inseparable degree of $K(C)/K'$, we have an exact sequence of group schemes

$$0 \rightarrow \mathbb{Z}/s\mathbb{Z} \times \mu_q \rightarrow C \rightarrow C' \rightarrow 0.$$

Taking the dual, we get

$$0 \rightarrow \mathbb{Z}/s\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \rightarrow \hat{C}' \rightarrow \hat{C} \rightarrow 0.$$

Since s and q are coprime, one may choose a single element $\mathcal{L}$ of $\hat{C}'$ which generates the kernel above.

Now, we put

$$\mathcal{E} := \mathcal{O}_{C'} \oplus \mathcal{L}.$$

We note that there exist *only two* sections of $\mathbb{P}(\mathcal{E})$ over C' such that the self-intersection number is equal to zero. It follows

$$\mathcal{E}_C \simeq \mathcal{O}_C^{\oplus 2},$$

and we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \times C & \simeq & \mathbb{P}(\mathcal{E}_C) \xrightarrow{f} \mathbb{P}(\mathcal{E}) \\ & & \downarrow \qquad \downarrow \\ & & C \longrightarrow C', \end{array}$$

where f is a morphism induced by $K(C)/K'$.

We consider a constant section C_0 of $\mathbb{P}(\mathcal{E}_C)$ over C which comes from neither of the two sections of $\mathbb{P}(\mathcal{E})$ mentioned above, and let us prove that $f|_{C_0}$ is a closed immersion.

Let C'' be the normalization of the image $f(C_0)$. Using a base change by the naturally induced morphisms $C \rightarrow C'' \rightarrow C'$, we get a commutative diagram

$$\begin{array}{ccccc} C_0 & \rightarrow & C'' & \rightarrow & f(C_0) \\ \cap & & \cap & & \cap \\ \mathbb{P}(\mathcal{E}_C) & \rightarrow & \mathbb{P}(\mathcal{E}_{C''}) & \rightarrow & \mathbb{P}(\mathcal{E}) \\ \downarrow & & \downarrow & & \downarrow \\ C & \rightarrow & C'' & \rightarrow & C', \end{array}$$

where we denote by C''_0 a section of $\mathbb{P}(\mathcal{E}_{C''})$ over C'' naturally defined by the base change. Then, we find that C_0 is a pull-back of C''_0 and the self-intersection number of C''_0 is also equal to zero.

Now, suppose that $f|_{C_0}$ is not birational. It follows that $C \rightarrow C''$ should not be birational. Therefore, $\mathcal{E}_{C''}$ is not trivial and C''_0 comes from either of the two sections of $\mathbb{P}(\mathcal{E})$ specified above, so does C_0 . This is a contradiction, and it follows that $f|_{C_0}$ is birational.

It remains to show that $f(C_0)$ is smooth in $\mathbb{P}(\mathcal{E})$. Computing its arithmetic genus, one can easily deduce this result.

THEOREM 5.2: *If C is a supersingular elliptic curve in characteristic p , then*

$$\mathcal{K}'_{\text{imm}} = \begin{cases} \{K(C)^p, K(C)^{p^2}\} & \text{if } p = 2 \\ \{K(C), K(C)^p\} & \text{otherwise.} \end{cases}$$

Proof: We show the inclusion $\subseteq$. By Theorem 3.1, we see that the converse follows from the existence of suitable vector bundles, which can be verified simultaneously below (use, for example, [1]).

Take an element K' of $\mathcal{K}'_{\text{imm}}$, and let $\iota: C \rightarrow \mathbb{P}$ be a closed immersion associated to K' . We employ the same notations as before.

Case: $p \neq 2$

If $K(C) = K'$, then there is nothing to prove. We may assume that $K(C)$ is not separable over K' .

Let C_1 , C' and C'' be curves with function fields $K(C)^p$, K' and $K'^{1/p}$, respectively. We get a commutative diagram

$$\begin{array}{ccc} C & \rightarrow & C_1 \\ \downarrow & & \downarrow \\ C'' & \rightarrow & C'. \end{array}$$

Since ι is unramified, using Lemmas 3.2 and 4.2, we have

$$\mathcal{Q}_C \simeq \iota^* \mathcal{O}_{\mathbb{P}}(1)^{\oplus 2}$$

and C_0 is a constant section of $\mathbb{P}(\mathcal{Q}_C)$ over C via an isomorphism $\mathbb{P}(\mathcal{Q}_C) \simeq \mathbb{P}^1 \times C$.

I claim that $\mathcal{Q}_{C_1}$ is indecomposable. Suppose that $\mathcal{Q}_{C_1}$ is decomposed. Since C_1 is supersingular, a line bundle $\mathcal{L}$ on C_1 with $\mathcal{L}_C \simeq \iota^* \mathcal{O}_{\mathbb{P}}(1)$ is uniquely

determined. Thus, $\mathcal{Q}_{C_1}$ should be of the form $\mathcal{L}^{\oplus 2}$ for some line bundle $\mathcal{L}$. This implies that the map

$$C_0 \hookrightarrow \mathbb{P}(\mathcal{Q}_C) \rightarrow \mathbb{P}(\mathcal{Q}_{C_1})$$

is not birational onto its image. Then, we find a contradiction because $f|_{C_0}$ is birational onto its image.

It follows that $\mathcal{Q}_{C'}$ is indecomposable. I claim moreover that $\mathcal{Q}_{C'}$ has even degree. Suppose that $\mathcal{Q}_{C'}$ has odd degree. It clearly follows that $\mathcal{Q}_{C''}$ should have also odd degree. Since $\mathcal{Q}_C$ is a direct sum of two line bundles with the same degree, $\mathcal{Q}_{C''}$ must be indecomposable. Therefore, $\mathcal{Q}_{C'}$ and $\mathcal{Q}_{C''}$ are both indecomposable if they have odd degree. Since $C \rightarrow C''$ is isomorphic to $C_1 \rightarrow C'$ as a finite cover of abstract curves, $\mathcal{Q}_C$ is indecomposable if and only if so is $\mathcal{Q}_{C_1}$ (see, for example, [1, II]). This is a contradiction.

Since C'' is supersingular, we have

$$\mathcal{Q}_{C''} \simeq \mathcal{M}^{\oplus 2}$$

for some line bundle $\mathcal{M}$ on C'' .

If $K(C)/K'$ is not purely inseparable of degree p , again this contradicts the birationality of $f|_{C_0}$ because the map

$$\begin{array}{c} C_0 \hookrightarrow \mathbb{P}(\mathcal{Q}_C) \\ \downarrow \\ \mathbb{P}(\mathcal{Q}_{C''}) \end{array}$$

should not be birational onto its image. This completes the proof.

Case: $p = 2$

According to Corollaries 2.2 and 2.3, $K(C)$ is always inseparable over K' . Since ι is unramified, we have an exact sequence

$$(\varepsilon) \quad 0 \rightarrow \iota^* \mathcal{O}_{\mathbb{P}}(1) \rightarrow \mathcal{Q}_C \rightarrow \iota^* \mathcal{O}_{\mathbb{P}}(1) \rightarrow 0.$$

It follows from Lemma 3.2 that (ε) splits if and only if d is even.

By the similar way to the former case, let us consider the curves C_1 , C' , C'' and the commutative diagram above.

Subcase: (ε) splits

I first claim that $\mathcal{Q}_{C_1}$ is indecomposable and has even degree. The indecomposability of $\mathcal{Q}_{C_1}$ follows by the similar way to the former case. If $\mathcal{Q}_{C_1}$ has odd

degree, then, according to [23, Theorem 2.16], $\mathcal{Q}_C$ should be indecomposable. This is a contradiction.

Next, I claim that $K(C_1)/K(C')$ is either trivial or not separable. Suppose the contrary. Since C_1 is supersingular, the degree of $C_1 \rightarrow C'$ should not be divisible by p , that is, odd. Since $\mathcal{Q}_{C_1}$ is indecomposable with even degree, so is $\mathcal{Q}_{C'}$. Thus, $\mathcal{Q}_{C''}$ should be of the form $\mathcal{M}^{\oplus 2}$ for some line bundle $\mathcal{M}$ on C'' because C'' is supersingular, and this contradicts the birationality of $f|_{C_0}$ as above.

Now, if $K(C_1) = K(C')$, then $K(C)$ is purely inseparable of degree p over K' . We hence consider the case when $C_1 \rightarrow C'$ is not separable. Then, one can take a curve C_2 between C_1 and C' with $K(C_2) = K(C)^{p^2}$, and C''' between C and C'' with $K(C''') = K'^{1/p^2}$. We obtain a commutative diagram

$$\begin{array}{ccccc} C & \rightarrow & C_1 & \rightarrow & C_2 \\ \downarrow & & & & \downarrow \\ C''' & \rightarrow & C'' & \rightarrow & C'. \end{array}$$

I claim that $\mathcal{Q}_{C_2}$ has odd degree. Suppose that $\mathcal{Q}_{C_2}$ has even degree. Then, $\mathcal{Q}_{C_1}$ should be decomposed since C_1 is supersingular. This is a contradiction.

Therefore, $\mathcal{Q}_{C'}$ is indecomposable with odd degree and $\mathcal{Q}_{C''}$ is of the form $\mathcal{N}^{\oplus 2}$ for some line bundle $\mathcal{N}$ on C'' .

If $K(C)/K'$ is not purely inseparable of degree p^2 , then we find a contradiction by the similar way to the last part of the former case. This completes our proof.

Subcase: (ε) does not split

We see that C_0 is a *unique* section of $\mathbb{P}(\mathcal{Q}_C)$ over C such that the self-intersection number is equal to zero, and $\mathcal{Q}_C$ is indecomposable with even degree.

I claim that $\mathcal{Q}_{C_1}$ has odd degree. Suppose that $\mathcal{Q}_{C_1}$ has even degree. Then, $\mathcal{Q}_{C_1}$ should have a non-trivial extension

$$(\varepsilon_1) \quad 0 \rightarrow \mathcal{L} \rightarrow \mathcal{Q}_{C_1} \rightarrow \mathcal{L} \rightarrow 0$$

with some line bundle $\mathcal{L}$ on C_1 because $\mathcal{Q}_{C_1}$ is indecomposable. We find that the pull-back of (ε_1) to C must coincide with (ε) . Therefore, similarly as above, this contradicts the birationality of $f|_{C_0}$.

Hence, $\mathcal{Q}_{C'}$ and $\mathcal{Q}_{C''}$ are indecomposable with odd and even degree, respectively, and we have $K(C_1) = K(C')$ since $f|_{C_0}$ is birational onto its image. Thus, $K(C)/K'$ is purely inseparable of degree p .

REMARK: In the latter subcase above, we have not used the assumption that C is supersingular.

§6. Curves of higher genus

The main purpose of this section is to prove

THEOREM 6.1: *Let C be a curve in characteristic p with genus $g \geq 2$, let*

$$f: C \rightarrow C'$$

be a Frobenius morphism of C , and let $\mathcal{L}$ be a line bundle on C such that the degree of $\mathcal{P}_C^1(\mathcal{L})$ is divisible by p . Then, the following conditions are equivalent:

- (1) $\mathcal{K}'_{\text{imm}}$ contains $K(C)^p$;
- (2) *there exists a stable vector bundle $\mathcal{E}$ of rank 2 on C' such that*

$$\mathcal{P}_C^1(\mathcal{L}) \simeq f^*\mathcal{E}.$$

Combining Theorem 6.1 with Corollary 3.8, we get

COROLLARY 6.2: *Let C be as above. For an integer $l > 0$, let*

$$C \rightarrow C^{(p)} \rightarrow \cdots \rightarrow C^{(p^l)}$$

be a sequence of Frobenius morphisms of C . Then, the following conditions are equivalent:

- (1) $\mathcal{K}'_{\text{imm}}$ contains $K(C)^{p^l}$;
- (2) *there exists a vector bundle $\mathcal{E}$ of rank 2 on $C^{(p^l)}$ such that $\mathcal{E}_{C^{(p)}}$ is stable and $\mathcal{E}_C \simeq \mathcal{P}_C^1(\mathcal{L})$ for some line bundle $\mathcal{L}$ on C .*

To prove Theorem 6.1, let us use Theorem 3.1. For a rank 2 vector bundle $\mathcal{E}$ on C' , to give a morphism $h: C \rightarrow \mathbb{P}(\mathcal{E})$, it is equivalent to give a quotient line bundle $\mathcal{L}$ of $f^*\mathcal{E}$, and there is a commutative diagram

$$\begin{array}{ccccccc} C & \longleftarrow & \mathbb{P}(f^*\mathcal{E}) & \supset & \mathbb{P}(\mathcal{L}) & & \\ f \downarrow & & \downarrow & \swarrow & \searrow & & \\ C' & \longleftarrow & \mathbb{P}(\mathcal{E}) & & \xleftarrow{h} & C, & \end{array}$$

where $\mathbb{P}(\mathcal{L})$ is a section of $\mathbb{P}(f^*\mathcal{E})$ over C . Then, we have

LEMMA 6.3: *The following conditions are equivalent:*

- (1) *h is birational onto its image;*
- (2) *the quotient line bundle $\mathcal{L}$ of $f^*\mathcal{E}$ does not come from any quotient line bundle of $\mathcal{E}$.*

Proof: Since $\mathbb{P}(\mathcal{L})$ is purely inseparable of degree p over C' , one of the two cases below occurs: $\mathbb{P}(\mathcal{L})$ is purely inseparable of degree p over $h(C)$, and $K(h(C)) = K(C')$, so $h(C)$ is a section of $\mathbb{P}(\mathcal{E})$ over C' ; h is birational onto its image, and $K(h(C))/K(C')$ is purely inseparable of degree p . This completes the proof.

PROPOSITION 6.4: *Let C be a curve in characteristic p , let $f: C \rightarrow C'$ be a Frobenius morphism of C , and let $\mathcal{L}$ be a line bundle on C such that the degree of $\mathcal{P}_C^1(\mathcal{L})$ is divisible by p . Then, the following conditions are equivalent:*

- (1) *$\mathcal{K}'_{\text{imm}}$ contains $K(C)^p$;*
- (2) *there exists a vector bundle $\mathcal{E}$ of rank 2 on C' such that*

$$\mathcal{P}_C^1(\mathcal{L}) \simeq f^*\mathcal{E}$$

and the natural quotient line bundle $\mathcal{L}$ of $\mathcal{P}_C^1(\mathcal{L})$ does not come from any quotient line bundle of $\mathcal{E}$.

Proof: We first show the implication (1) $\Rightarrow$ (2) in case of $p \neq 2$. For a closed immersion $\iota: C \rightarrow \mathbb{P}$ associated to $K(C)^p$ in $\mathcal{K}'_{\text{imm}}$, we have

$$\mathcal{P}_C^1(\iota^*\mathcal{O}_{\mathbb{P}}(1)) \simeq f^*\mathcal{Q}_{C'}.$$

In particular, $\mathcal{L}$ and $\iota^*\mathcal{O}_{\mathbb{P}}(1)$ have the same degree modulo p since $p \neq 2$. It follows from Lemma 3.2 that

$$\mathcal{P}_C^1(\mathcal{L}) \simeq \mathcal{P}_C^1(\iota^*\mathcal{O}_{\mathbb{P}}(1)) \otimes (\mathcal{L} \otimes \iota^*\mathcal{O}_{\mathbb{P}}(1)^\vee),$$

where $\iota^*\mathcal{O}_{\mathbb{P}}(1)^\vee$ is the dual of $\iota^*\mathcal{O}_{\mathbb{P}}(1)$. Thus, there exists a vector bundle $\mathcal{E}$ on C' such that $\mathcal{P}_C^1(\mathcal{L}) \simeq f^*\mathcal{E}$, and it follows from Lemma 6.3 (1) $\Rightarrow$ (2) that $\mathcal{E}$ has the required property.

Next, we consider the case $p = 2$. For a closed immersion $\iota: C \rightarrow \mathbb{P}$ such that the degree of $\iota(C)$ is even (respectively, odd), it follows from Corollaries 2.2 and 2.3 that the extension $K(C)/K(C^*)$ defined by the Gauss map via ι is not separable. Using Corollary 3.7, we obtain a closed immersion $\iota_1: C \rightarrow \mathbb{P}_1$ such that the degree of $\iota_1(C)$ is even (respectively, odd) and the

extension defined by the Gauss map via ι_1 is purely inseparable of degree p . In particular, this implies the condition (1) in case of $p = 2$. Now, apply the argument in case of $p \neq 2$ above to this ι_1 . We obtain the required $\mathcal{E}$ in (2) when the degree of $\mathcal{L}$ is even (respectively, odd). Thus, we have proved that both (1) and (2) are always satisfied when $p = 2$.

Finally, we show the implication (2) $\Rightarrow$ (1). For the natural quotient line bundle $\mathcal{L}$ of $f^*\mathcal{E}$ via $f^*\mathcal{E} \simeq \mathcal{P}_C^1(\mathcal{L})$, by virtue of Lemma 6.3 (2) $\Rightarrow$ (1), we have a morphism $h: C \rightarrow \mathbb{P}(\mathcal{E})$ such that $K(h(C))/K(C')$ is purely inseparable of degree p . Computing the arithmetic genus of $h(C)$, we see that $h(C)$ is smooth, so that h is a closed immersion. Then, the result follows from Theorem 3.1.

We here consider a special case of this situation.

DEFINITION: For a curve C with a Frobenius morphism $f: C \rightarrow C'$, if C' has a line bundle $\mathcal{N}$ such that:

- (1) $f^*\mathcal{N} \simeq \Omega_C^1$, and
 - (2) $f^*: H^1(C', \mathcal{N}^\vee) \rightarrow H^1(C, f^*\mathcal{N}^\vee)$ is not injective,
- then C is called a *Tango-Raynaud curve*.

COROLLARY 6.5: *If C is a Tango-Raynaud curve in characteristic p , then $\mathcal{K}'_{\text{imm}}$ contains $K(C)^p$.*

Proof: It follows from the assumption that there is a non-zero element ξ of $H^1(C', \mathcal{N}^\vee)$ for some line bundle $\mathcal{N}$ on C' such that $f^*(\xi) = 0$ in $H^1(C, f^*\mathcal{N}^\vee)$. Take $\mathcal{E}$ to be the extension of $\mathcal{N}$ by $\mathcal{O}_{C'}$ determined by ξ , and $\mathcal{L}$ to be $f^*\mathcal{O}_{C'}$ in the situation above. The result follows from Proposition 6.4.

REMARK: If a line bundle $\mathcal{N}$ on C' with $\deg f^*\mathcal{N} = 2g - 2$ satisfies the condition (2), then we have $f^*\mathcal{N} \simeq \Omega_C^1$.

REMARK: Consider a curve C of genus $g \geq 2$, and denote by $n(C)$ Tango's invariant of C in [32, Definition 11]. Then, C is a Tango-Raynaud curve in our sense if and only if

$$n(C) = (2g - 2)/p,$$

and our definition above is equivalent to the ordinary one (see, for example, [2] or [27]).

REMARK: Neither rational nor ordinary elliptic curve C is a Tango–Raynaud curve. On the other hand, a supersingular elliptic curve C is a Tango–Raynaud curve, because $\mathcal{O}_C$ enjoys the required properties.

REMARK: Any curve C in characteristic 2 is a Tango–Raynaud curve if C is neither rational nor ordinary elliptic, because the cokernel of a natural map $\mathcal{O}_C \rightarrow f_*\mathcal{O}_C$ is a line bundle having the required properties (see, for example, [2, Exemple i), p. 81] or [20, 2.2]).

Now, we go back to the proof of Theorem 6.1. For a rank 2 vector bundle $\mathcal{F}$ on a curve C and a quotient line bundle $\mathcal{L}$ of $\mathcal{F}$, we put

$$s(\mathcal{F}, \mathcal{L}) := 2 \deg \mathcal{L} - \deg \mathcal{F}.$$

Moreover, we put

$$s(\mathcal{F}) := \inf_{\mathcal{L}} s(\mathcal{F}, \mathcal{L}),$$

where $\mathcal{L}$'s are quotient line bundles of $\mathcal{F}$. The value $s(\mathcal{F})$ is called the *stability degree* of $\mathcal{F}$ because of the fact that $\mathcal{F}$ is stable if and only if $s(\mathcal{F}) > 0$.

LEMMA 6.6: *For a rank 2 vector bundle $\mathcal{F}$ on a curve, if $s(\mathcal{F}) < 0$, then a quotient line bundle $\mathcal{L}$ of $\mathcal{F}$ with $s(\mathcal{F}, \mathcal{L}) \leq 0$ is uniquely determined.*

Proof: Taking a quotient line bundle $\mathcal{L}$ of $\mathcal{F}$ with $s(\mathcal{F}, \mathcal{L}) < 0$, we denote by $\mathcal{M}$ the kernel of $\mathcal{F} \rightarrow \mathcal{L}$. Let $\mathcal{L}'$ be an arbitrary quotient line bundle of $\mathcal{F}$ with $s(\mathcal{F}, \mathcal{L}') \leq 0$, and denote by $\mathcal{M}'$ the kernel of $\mathcal{F} \rightarrow \mathcal{L}'$. Then, we have that $\deg \mathcal{M} > \deg \mathcal{L}$, and $\deg \mathcal{M}' \geq \deg \mathcal{L}'$. It follows that $\deg \mathcal{M} > \deg \mathcal{L}'$, and $\deg \mathcal{M}' > \deg \mathcal{L}$. Thus, we see that $\mathcal{L} = \mathcal{L}'$ as a quotient line bundle of $\mathcal{F}$.

For a line bundle $\mathcal{L}$ on a curve C with genus g , we have $s(\mathcal{P}_C^1(\mathcal{L}), \mathcal{L}) = -(2g - 2)$. Moreover, if $g \geq 2$, then it follows from Lemma 6.6 that

$$s(\mathcal{P}_C^1(\mathcal{L})) = -(2g - 2).$$

REMARK: One can say that $\mathcal{P}_C^1(\mathcal{L})$ is the farthest from a stable bundle on C even if $\mathcal{P}_C^1(\mathcal{L})$ is indecomposable (see, for example, [5, V, Theorem 2.12]).

PROPOSITION 6.7: *Let $f: C \rightarrow C'$ be a finite morphism of curves, and let $\mathcal{F}$ be a rank 2 vector bundle on C' . If the pull-back $f^*\mathcal{F}$ has a quotient line bundle $\mathcal{L}$ with $s(f^*\mathcal{F}, \mathcal{L}) < 0$, then the following conditions are equivalent:*

- (1) *the quotient line bundle $\mathcal{L}$ of $f^*\mathcal{F}$ does not come from any quotient line bundle of $\mathcal{F}$;*
- (2) *$\mathcal{F}$ is stable.*

Proof: It suffices to show the implication (1) $\Rightarrow$ (2) since the converse is obvious. Assume that there exists a quotient line bundle $\mathcal{L}'$ of $\mathcal{F}$ such that $s(\mathcal{F}, \mathcal{L}') \leq 0$. It clearly follows that $s(f^*\mathcal{F}, f^*\mathcal{L}') \leq 0$. According to Lemma 6.6, the quotient line bundle $\mathcal{L}$ must coincide with the pull-back $f^*\mathcal{L}'$. This completes the proof.

Finally, Theorem 6.1 follows from Propositions 6.4 and 6.7.

REMARK: In Theorem 6.1, if the degree of $\mathcal{L}$ is not divisible by p and there exists a vector bundle $\mathcal{E}$ on C' such that $f^*\mathcal{E} \simeq \mathcal{P}_C^1(\mathcal{L})$, then $\mathcal{E}$ is stable.

REMARK: In characteristic 2, the line bundle $\iota^*\mathcal{O}_p(1)$ is not necessarily related to $\mathcal{L}$ in Theorem 6.1 and the condition (1) is always satisfied, as we have seen in the proof of Proposition 6.4.

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