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Invariant theory for S_5 and the rationality of M_6

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Introduction

The main purpose of this note is to show that if V is any finite-dimensional complex representation of the symmetric group S_5 of degree five, then the quotient varieties $\mathbf{P}(V)/S_5$, and so V/S_5, are rational. In particular, it follows that the moduli space M_6 for curves of genus six is rational over $\mathscr{C}$; this follows from the well known fact that the canonical model of such a curve lies as a quadric section on a unique quintic Del Pezzo surface Σ, so that if $U_2 = H^0(\mathscr{O}_\Sigma(2))$, then $M_6 \sim \mathbf{P}(U_2)/S_5$, since $S_5 = \mathrm{Aut}\,\Sigma$. (We let the symbol $\sim$ denote birational equivalence.) This is essentially equivalent to the classical fact that a generic curve C of genus six has five g_6^2's, and each g_6^2 maps C to a plane sextic with four nodes in general position.

In the final section, we shall extend this result to an arbitrary base field. It turns out that the geometry of Σ is the key to other actions of S_5; see Proposition 9.

Preliminaries

We gather various well known facts and set up some notation.

The irreducible representations of S_5 will be denoted by $\mathbf{1}$, ϕ, χ, ψ, χ', ϕ', σ, of degrees 1, 4, 5, 6, 4, 1 respectively. $\mathbf{1}$ is the trivial representation, σ is the signature, ϕ is the representation of S_5 as the Weyl group $W(A_4)$, $\phi' = \phi \otimes \sigma$ and $\chi' = \chi \otimes \sigma$. For the convenience of the reader, the complete character table of S_5 is reproduced at the end of the paper.

The quintic Del Pezzo surface Σ is obtained by blowing up four distinct points $P_1, \ldots, P_4$ in $\mathbf{P}^2$, no three of which are collinear, and is anticanonically embedded in $\mathbf{P}^5$ via the system of cubics through $P_1, \ldots, P_4$. The group of

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Cremona transformations of $\mathbf{P}^2$ based at $P_1, \ldots, P_4$ is isomorphic to S_5 and acts biregularly on Σ; S_5 is the whole of Aut Σ, and the action on $K_\Sigma^\perp \subseteq H^2(\Sigma, \mathbf{Z}) \cong \mathrm{Pic}\,\Sigma$ is the representation of S_5 as $W(A_4)$. Σ contains five pencils of conics and ten lines; both sets are permuted transitively by Aut Σ. Finally, the action of S_5 on Σ extends to a linear action on $\mathbf{P}^5$, and since $\mathcal{O}_\Sigma(1)$ is the determinant of the tangent bundle T_Σ, it is S_5-linearized; i.e., the action of S_5 on $\mathbf{P}^5$ is induced from a representation of S_5 on the six-dimensional vector space $H^0(\mathcal{O}_\Sigma(1))$.

Throughout, we shall denote the vector space $H^0(\mathcal{O}_\Sigma(n))$ by U_n.

Representations of S_5 associated to Σ

LEMMA 1. $U_1 \cong \psi$.

Proof: It is enough to show that U_1 contains no one-dimensional representation of S_5, or equivalently that there is no Aut Σ-invariant hyperplane in $\mathbf{P}^5$. So suppose that there is such a hyperplane, say H. Since Aut Σ acts transitively on the lines in Σ, $H \cap \Sigma$ cannot contain any line, since $\deg(\Sigma \cap H) = 5$. Similarly $H \cap \Sigma$ cannot contain any conic, and so $H \cap \Sigma$ is a reduced and irreducible quintic curve of arithmetic genus one. Then the normalization of $H \cap \Sigma$ is either $\mathbf{P}^1$ or elliptic; however, S_5 cannot act effectively on such a curve. Q.E.D.

We shall let W denote the space of quadrics in $\mathbf{P}^5$ through Σ. Because Σ is projectively normal (since a hyperplane section, a quintic elliptic curve, is so), W is five-dimensional, and in fact Σ is cut out by the elements of W.

PROPOSITION 2 (Mukai): *W is irreducible.*

Proof: It is well-known (and easy to see) that each line in Σ lies in six pentagons contained in Σ. Hence there are 12 pentagons in Σ, each of which is a hyperplane section of Σ, and if $\Pi_1 = \{l_1, \ldots, l_5\}$ is a pentagon, then the remaining lines $\{m_1, \ldots, m_5\}$ on Σ also form a pentagon Π_1'. Hence the twelve pentagons fall into six pairs $\{\Pi_1, \Pi_1'\}, \ldots, \{\Pi_6, \Pi_6'\}$. Choose linear forms L_i, L_i' cutting out Π_i, Π_i' respectively on Σ. Then the quadrics $Q_i = L_i L_i'$ form a six-dimensional space W' upon which S_5 acts (up to twisting by a character ℓ or σ) as a transitive permutation group. So W' has irreducible five-and one-dimensional components. The five-dimensional component W'' is generated by the differences $Q_{ij} = Q_i - Q_j$, all of which vanish along Σ, and so contain Σ. Hence $W'' = W$, and we know that $W'' = \chi$ or χ'. Q.E.D.

COROLLARY 3. *Either* $U_2 \cong \mathbf{1} \oplus \phi \oplus 2\chi \oplus \sigma$ or $U_2 \cong \mathbf{1} \oplus \phi \oplus \chi \oplus \chi' \oplus \sigma$.

Proof: By the projective normality of Σ, we have $U_2 \cong \mathrm{Symm}^2(U_1)/W$. A brief computation involving the character table of S_5 shows that $\mathrm{Symm}^2(U_1) \cong \mathbf{1} \oplus \phi \oplus 2x \oplus x' \oplus \sigma$, and now the Corollary follows from Proposition 2.

Curves of genus six

Our aim is to give a proof of the classical fact mentioned in the introduction, that the canonical model of a general curve of genus six lies on a unique quintic Del Pezzo surface.

PROPOSITION 4. *Suppose that* $\Gamma \subset \mathbf{P}^5$ *is the canonical model of a non-hyperelliptic smooth curve of genus six, and that* Γ *lies on a smooth quintic Del Pezzo surface* Σ. *Then* Γ *is a quadric section of* Σ, Γ *has exactly five* g_4^1's (*cut out by the pencils of conics on* Σ) *and* Σ *is the only quintic Del Pezzo surface on which* Γ *lies.*

Proof: By the Hodge Index Theorem, Γ is numerically, and so linearly, equivalent to a quadric section of Σ. Since Σ is projectively normal, it follows that Γ is a quadric section of Σ. Note that since Σ is an intersection of quadrics, so is Γ, and so Γ is not trigonal. Suppose that $|D|$ is a g_4^1 on Γ; then $|D|$ has no base points, and so $|K_\Gamma - D|$ is a g_6^2. By the geometric version of Riemann–Roch, every divisor $E \in |K_\Gamma - D|$ lies in a 3-plane L in $\mathbf{P}^5$; then L meets Σ in at least six points, and so L meets Σ in a curve C. Since L moves in a net, so does C, and so $|C|$ is a net of twisted cubics on Σ.

It follows that if $|H|$ is the system of hyperplane sections of Σ, then $|H{-}C|$ cuts out $|D|$, the given g_4^1, on Γ, and $|H{-}C|$ is a pencil of conics. So every g_4^1 on Γ is cut out by a pencil of conics on Σ.

If there were two pencils $|A|$ and $|B|$ of conics on Σ that cut out the same g_4^1 on Γ, then a member of $|A|$ would meet a member of $|B|$ in at least four points, and so either $|A| = |B|$ or $A \cdot B \geqslant 4$; the latter is impossible, since $A \cdot B = 1$ if $|A| \neq |B|$, and so $|A| = |B|$. Hence Γ has exactly five g_4^1's, and they are all cut out by pencils of conics on Σ. Also Γ has just five g_6^2's, residual to the g_4^1's, and they are cut out by the nets of twisted cubics on Σ.

Finally, suppose that Γ lies on two smooth quintic Del Pezzo surfaces, Σ and Σ'. Choose a g_6^2 on Γ, say $|D|$; then $|D|$ is cut out by a net $|A|$ of twisted cubics on Σ and another such net $|A'|$ on Σ'. Then every member of $|A|$ meets some member of $|A'|$ in at least six points; however, distinct twisted cubics can meet in at most five points, and so every member of $|A|$ lies on Σ'. Since $|A|$ sweeps out Σ and $|A'|$ sweeps out Σ', it follows that $\Sigma = \Sigma'$. Q.E.D.

COROLLARY 5: $M_6 \sim \mathbf{P}(U_2)/\mathrm{Aut}\,\Sigma$.

Proof: By Proposition 4, if $X \subset \mathbf{P}(U_2)$ is the locus of smooth quadric sections of Σ, then the natural map $X/\mathrm{Aut}\,\Sigma \to M_6$ is injective. Since both are of dimension fifteen, the corollary follows:

THEOREM 6: *M_6 is rational.*

Proof: By Corollary 5, it is enough to show that $\mathbf{P}(U_2)/S_5$ is rational. By Corollary 3, U_2 contains a copy of ϕ. Let $\alpha: \tilde{\mathbf{P}} \to \mathbf{P}\,(U_2)$ be the blow-up of the base locus of the projection $\mathbf{P}(U_2) \to \mathbf{P}(\phi)$ and $\pi: \tilde{\mathbf{P}} \to \mathbf{P}(\phi)$ the induced morphism. Put $\mathscr{L} = \alpha^*\mathcal{O}(1)$; then S_5 acts freely on an open subvariety $\mathbf{P}_0$ of $\mathbf{P}(\phi)$ and the sheaf $\mathscr{L}$ is S_5-linearized. Hence by [4, Prop. 7.1] the quotient $\tilde{\mathbf{P}}/S_5$ is generically a Severi–Brauer scheme over $\mathbf{P}(\phi)/S_5$; moreover the sheaf $\mathscr{L}$ descends to $\tilde{\mathbf{P}}/S_5$ and cuts out $\mathcal{O}(1)$ on the fibres of the map $\tilde{\mathbf{P}}/S_5 \to \mathbf{P}(\phi)/S_5$. Hence $\tilde{\mathbf{P}}/S_5 \sim \mathbf{P}(\phi)/S_5 \times \mathbf{P}^{11}$; since $\mathbf{P}(\phi)/S_5$ is rational, by the theorem on symmetric functions, it follows that $\tilde{\mathbf{P}}/S_5$, and so $\mathbf{P}(U_2)/S_5$, is also rational. Q.E.D.

REMARK 7. [1, Lemma 1.3]. One key point in the preceding proof is that if a reductive algebraic group G acts generically freely on $\mathbf{P}(U)$, where U is a representation of G, and if $\mathbf{P}(U)/G$ is rational, then $\mathbf{P}(U \oplus V)/G$ is rational for any representation V of G. In particular, for $G = S_5$, to prove that $\mathbf{P}(U)/G$ is rational for all U, we have only to consider irreducible representations U of S_5.

Other representations of S_5

LEMMA 8. *If V is a representation of the reductive group G and σ is a character of G, then the quotients $\mathbf{P}(V)/G$ and $\mathbf{P}(V \otimes \sigma)/G$ are birationally equivalent.*

Proof: Obvious.

In view of Remark 7 and Lemma 8, to prove that $\mathbf{P}(Y)/S_5$ is rational for every representation Y of S_5, it is enough to prove the result for the cases $Y = \chi$ and $Y = \psi$.

REMARK. Recall that $\mathbf{1} \oplus \chi$ is the restriction to S_5 of the permutation representation of S_6, where S_5 is embedded in S_6 as a transitive subgroup. If S_6 permutes the variables $v_1, \ldots, v_6$ and σ_i is the i'th elementary symmetric function of $v_1, \ldots, v_6$, then the field of invariants $\mathbf{C}(\mathbf{1} \oplus \chi)^{S_5}$ is $\mathbf{C}(\sigma_1, \ldots, \sigma_6, W)$, where W is the expression givlen on p. 679 of [7]. From this description, however, it is not clear that the field is rational.

PROPOSITION 9. There are birational equivalences $\mathbf{P}(\psi)/S_5 \sim \mathbf{P}(\chi)/S_5 \times \mathbf{P}^1$ and $\mathbf{P}(\chi)/S_5 \sim \Sigma^{(2)}/S_5$, where $\Sigma^{(2)}$ denotes the symmetric square of Σ.

Proof: Recall that W is the space of quadrics through Σ and that $W \cong \chi$ or χ', so that $\mathbf{P}(W) \cong \mathbf{P}(\chi)$ as S_5-spaces. Let $\beta\colon \mathbf{P}^5 \to \mathbf{P}(\chi)$ denote the rational map defined by the linear system $\mathbf{P}(W)$. Let $H \subset \mathbf{P}^5$ be a generic hyperplane; then the induced map $\beta|_H\colon H \to \mathbf{P}(\chi)$ is defined by the linear system of quadrics through the quintic elliptic curve $\Sigma \cap H$. By [5, VIII 5.2, pp. 181–2] (for a proof, use [2, Ex. 9. 1.12]) this map is a birational equivalence, and so β is generically a $\mathbf{P}^1$-bundle. Hence, as in the proof of Theorem 6, $\mathbf{P}^5/S_5 \sim \mathbf{P}(\chi)/S_5 \times \mathbf{P}^1$, which is the first part of the Proposition. Moreover, we see that the generic fibres of β are just the secant lines to Σ, so that a generic point in $\mathbf{P}^5$ lies on a unique such secant, and $\mathbf{P}(\chi)$ is birationally equivalent, as an S_5-space, to the variety of these secants. This variety is in turn birationally equivalent, as an S_5-space, to the symmetric square $\Sigma^{(2)}$, the variety of unordered pairs of points on Σ. This completes the proof of Proposition 9.

REMARK: The proof of Proposition 9 shows that a quintic Del Pezzo surface Σ defined over any infinite field k has a bisecant L defined over k. This gives an immediate proof, via projection from L, of the theorem of Enriques-Manin-Swinnerton-Dyer that Σ is rational over k.

THEOREM 10. *$\Sigma^{(2)}/S_5$ is rational.*

Proof: Points of $\Sigma^{(2)}/S_5$ correspond to unordered pairs of points on Σ, modulo Aut Σ. In turn, these correspond to cubic surfaces with a chosen unordered pair of skew lines, modulo automorphism.

Recall that given two skew lines M_1 and M_2 on a smooth cubic surface F, there are exactly five skew lines $L_1, \ldots, L_5$ on F meeting M_1 and M_2. Blowing down $L_1, \ldots, L_5$ maps F to a quadric Q, and M_1, M_2 are mapped to twisted cubics meeting in five points, and so lying in opposite families. I.e., one is of bidegree (1, 2) and the other of bidegree (2, 1). Conversely, given two general twisted cubics C_1 and C_2 on Q in opposite families, we recover five points as $C_1 \cap C_2$; blowing up these five points leads back to the configuration $M_1, M_2, L_1, \ldots, L_5$ on F.

Hence $\Sigma^{(2)}/S_5 \sim (A \times B)/\mathrm{Aut}\, Q$, where A is one family of twisted cubics on Q and B is the other. So we need to prove the following result.

PROPOSITION 11. *$(A \times B)/\mathrm{Aut}\, Q$ is rational.*

Proof: Let p_1, p_2 denote the projections of Q onto $\mathbf{P}^1$. Set $V(i) = H^0(\mathcal{O}_{\mathbf{P}^1}(i))$ and $V(i, j) = H^0(p_1^* \mathcal{O}_{\mathbf{P}^1}(i) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(j))$. Let G denote Aut Q and G^0 its connected component. We have $A = \mathbf{P}(V(1, 2))$ and $B = \mathbf{P}(V(2, 1))$, and $G = G^0 \times \langle \tau \rangle$, where $\tau^2 = 1$. G^0 acts on each of A and B, and so diagonally on $A \times B$, while τ acts on $A \times B$ by interchanging the factors.

To prove that $A \times B/G$ is rational, we shall use the slice method, as follows. Via the symbolical method [3] we shall construct a G-equivariant rational map $\sigma\colon A \times B \to \mathbf{P}(V(1, 1)) = \mathbf{P}^3$, which we shall prove to be dominant. Let $P \in \mathbf{P}^3$ be a generic point whose stabilizer in G is H; then $A \times B/G \sim \sigma^{-1}(P)/H$ (this is the slice method). We shall prove that $\sigma^{-1}(P)/H$ is rational by a further application of the slice method.

LEMMA 12. *There is a dominant G-equivariant rational map $\sigma\colon A \times B \to \mathbf{P}^3$ given by a linear system of bidegree* (1,1) *on $A \times B$.*

Proof: Let $x = (x_1, x_2)$ be homogeneous co-ordinates on one copy of $\mathbf{P}^1$ and $y = (y_1, y_2)$ co-ordinates on the other. Suppose that $f \in V(1, 2)$ and $g \in V(2, 1)$; then symbolically we write

$$f = a_x \otimes A_y^2 \quad \text{and} \quad g = b_x^2 \otimes B_y,$$

where $a_x = a_1 x_1 + a_2 x_2$, etc. We define σ by

$$\sigma(f, g) = (ab)(AB) b_x \otimes A_y,$$

where $(ab) = a_1 b_2 - a_2 b_1$ and $(AB) = A_1 B_2 - A_2 B_1$. Clearly σ is equivariant under τ; it is thus G-equivariant. To check that σ is dominant, we shall compute it explicitly. In non-symbolical terms, we can write

$$f = \sum_{i,j} \binom{2}{j} \alpha_{ij} x_1^{1-i} x_2^i y_1^{2-j} y_2^j$$

and

$$g = \sum_{k,l} \binom{2}{k} \beta_{kl} x_1^{2-k} x_2^k y_1^{1-l} y_2^l,$$

where the coefficients α_{ij}, β_{kl} are given in terms of the symbols a_i etc. by the relations

$$\alpha_{ij} = a_1^{1-i} a_2^i A_1^{2-j} A_2^j$$

and

$$\beta_{kl} = b_1^{2-k} b_2^k B_1^{1-l} B_2^l.$$

Then expansion of the formula for σ followed by these substitutions shows that

$$\begin{aligned}\sigma(f, g) = {} & (\alpha_{00}\beta_{11} - \alpha_{01}\beta_{10} - \alpha_{10}\beta_{01} + \alpha_{11}\beta_{00})x_1 y_1 \\ & + (\alpha_{01}\beta_{11} - \alpha_{02}\beta_{10} - \alpha_{11}\beta_{01} + \alpha_{12}\beta_{00})x_1 y_2 \\ & + (\alpha_{00}\beta_{21} - \alpha_{01}\beta_{20} - \alpha_{10}\beta_{11} + \alpha_{11}\beta_{10})x_2 y_1 \\ & + (\alpha_{01}\beta_{21} - \alpha_{02}\beta_{20} - \alpha_{11}\beta_{11} + \alpha_{12}\beta_{10})x_2 y_2.\end{aligned}$$

Choose f, g given by the conditions $\alpha_{01} = 0$, $\alpha_{11} = 2$, other $\alpha_{ij} = 1$, $\beta_{11} = 2$ and other $\beta_{kl} = 1$. Then a trivial check shows that $\sigma(f, g) = 3x_1 y_1 - 4x_2 y_2 = P$, say. Since P is irreducible, its G-orbit in $\mathbf{P}^3$ is dense, and so σ is dominant. This completes the proof of Lemma 12.

We can replace P by any other point in the same orbit, and so we may assume that $P = x_1 y_2 - x_2 y_1$; then up to isogeny, the connected component H^0 of the stabilizer H of P is the diagonal subgroup of $SL_2 \times SL_2$, and then $H = H^0 \times \langle \tau \rangle$. As H^0-spaces, we have $V(1, 2) \cong V(2, 1) \cong V(1) \oplus V(3)$, and so an H^0-equivariant projection θ: $A \times B \to \mathbf{P}(V(1)) \times \mathbf{P}(V(1))$. If τ acts on the right hand side by permuting the factors, then θ is in fact H-equivariant. Put $Y = \sigma^{-1}(P)$.

LEMMA 13: *The restriction $\theta|_Y$ is dominant.*

Proof: Since the G-orbit of P is dense, it is enough to show that θ is dominant. This is clear from the construction of θ.

Now let $y \in \mathbf{P}(V(1)) \times \mathbf{P}(V(1))$ be a point not on the diagonal, and $K \subset H$ its stabilizer. Then $K \cap H^0 = T$, a maximal torus in SL_2. Let N denote the normalizer of T in SL_2; suppose that $w \in N - T$. Then if $y = (q_1, q_2)$, we have $w(y) = (q_2, q_1)$ and so $\tau w(y) = y$. Hence K is generated by T and τw, and so is isomorphic to N. Put $Z = \theta^{-1}(y) \cap Y$; since the H-orbit of y is dense, we have $Y/H \sim Z/K$, by the slice method, and so $A \times B/G \sim Z/K$.

LEMMA 14. *Z/K is rational.*

Proof: Recall what we have established: we took $P = x_1 y_2 - x_2 y_1 \in \mathbf{P}^3$, whose stabilizer $H = H^0 \times \langle\tau\rangle$, where H^0 is (up to isogeny) the diagonal subgroup of $SL_2 \times SL_2$. In the formula for σ given above, let F_{ij} denote the coefficient of $x_i y_j$; then the equations defining $Y = \sigma^{-1}(P)$ in $A \times B$ are $F_{11} = 0$, $F_{22} = 0$ and $F_{12} = F_{21}$. Next, the rational H-equivariant map θ: $A \times B \to \mathbf{P}(V(1)) \times \mathbf{P}(V(1))$ is given symbolically by

$$\theta(a_x \otimes A_y^2, b_x^2 \otimes B_y) = ((aA)A_y, (bB)b_x)$$

$$= ((\alpha_{01} - \alpha_{10})y_1 + (\alpha_{02} - \alpha_{11})y_2, (\beta_{01} - \beta_{10})x_1 + (\beta_{11} - \beta_{20})x_2),$$

where $\alpha_{ij} = a_1^{1-i}a_2^i A_1^{2-j} A_2^j$ and $\beta_{kl} = b_1^{2-k} b_2^k B_1^{1-l} B_2^l$, as before. Then we can take $y = (y_1, x_2) \in \mathbf{P}(V(1)) \times \mathbf{P}(V(1))$, so that $\mathrm{Stab}(y) = K$ and $Z = Y \cap \theta^{-1}(y)$ is given by the five equations

$$F_{11} = 0, \quad F_{22} = 0, \quad F_{12} = F_{21}, \quad \alpha_{02} - \alpha_{11} = 0$$

and

$$\beta_{01} - \beta_{10} = 0.$$

The action of the torus T on $A \times B$ is given by

$$\alpha_{ij} \mapsto t^{-3+2(i+j)}\alpha_{ij}, \; \beta_{kl} \mapsto t^{-3+2(k+l)}\beta_{kl},$$

and for a suitable choice $\tilde{\tau}$ of a generator of K/T we have

$$\tilde{\tau}(\alpha_{ij}, \beta_{kl}) = -(\beta_{2-j,1-i}, -\alpha_{1-l,2-k}).$$

Using the equations above we can eliminate α_{02} and β_{01}, and then project Z to

$$Z' \subset \mathbf{P}^3 \times \mathbf{P}^3 = \mathrm{Proj}\, \mathbf{C}[\alpha_{00}, \alpha_{01}, \alpha_{11}, \alpha_{12}] \times \mathrm{Proj}\, \mathbf{C}[\beta_{21}, \beta_{11}, \beta_{10}, \beta_{00}];$$

this is K-equivariant and birational. The equation defining Z' is

$$-3\alpha_{11}^2\beta_{10}^2 + \alpha_{11}\alpha_{12}\beta_{00}\beta_{10} - \alpha_{00}\alpha_{11}\beta_{10}\beta_{21} + \alpha_{01}\beta_{10}(\alpha_{01}\beta_{21} + \alpha_{12}\beta_{10})$$

$$+\alpha_{11}\beta_{11}(\alpha_{00}\beta_{11} - \alpha_{01}\beta_{10} + \alpha_{11}\beta_{00}) = 0.$$

Consider the locus $Z^0 \subset \mathbf{A}^6 = \operatorname{Spec} \mathbf{C}[\alpha'_{00}, \beta'_{21}, \alpha'_{01}, \beta'_{11}, \alpha'_{12}, \beta'_{00}]$, the open subvarieties of Z and $\mathbf{P}^3 \times \mathbf{P}^3$ given by the conditions $\alpha_{11} \neq 0$ and $\beta_{10} \neq 0$, where $\alpha'_{ij} = \alpha_{ij}/\alpha_{11}$ and $\beta'_{kl} = \beta_{kl}/\beta_{10}$. The group K acts linearly on $\mathbf{A}^6$, and the action of T is given by $(\alpha'_{ij}, \beta'_{kl}) \to (t^{-4+2(i+j)}\alpha'_{ij}, t^{-2+2(k+l)}\beta'_{kl})$.

Embed $\mathbf{A}^6 \hookrightarrow \mathbf{P}^6$ and $Z^0 \hookrightarrow Z^*$ by adjoining a homogeneous coordinate w, invariant under K. The equation defining Z^* is

$$-3w^3 + w\alpha'_{12}\beta'_{00} - w\alpha'_{00}\beta'_{21} + \alpha'_{01}(\alpha'_{01}\beta'_{21} + \alpha'_{12}w)$$

$$+ \beta'_{11}(\alpha'_{00}\beta'_{11} - w\alpha'_{01} + w\beta'_{00}) = 0;$$

i.e. Z^* is cubic.

Let ζ denote the centre of SL_2, and put $\bar{K} = K/\zeta$, $\bar{T} = T/\zeta$. Then $\bar{K}$ acts on $\mathbf{P}^6$ and the sheaf $\mathcal{O}(1)$ is $\bar{K}$-linearized. Consider the 3-plane Π in $\mathbf{P}^6$ given by the equations $w = \alpha'_{01} = \beta'_{11} = 0$; then inspection of the equation defining Z^* shows that $\Pi \subset Z^*$, so that projection away from Π expresses Z^* birationally as a three-fold quadric bundle $\tilde{Z}$ over $\mathbf{P}^2 = \operatorname{Proj} \mathscr{C}[w, \alpha'_{01}, \beta'_{11}]$. We have

$$\begin{tikzcd} \tilde{Z} \arrow[rr] \arrow[dr, "\varrho"'] & & \mathbf{P}^2 \times \mathbf{P}^4, \arrow[dl] \\ & \mathbf{P}^2 & \end{tikzcd}$$

a commutative diagram on which $\bar{K}$ acts equivariantly. The factor $\mathbf{P}^4$ is Proj $\mathbf{C}[\alpha'_{00}, \alpha'_{12}, \beta'_{21}, \beta'_{00}, v]$, where v is $\bar{K}$-invariant. Since T acts on $\mathbf{P}^2$ via $(w, \alpha'_{01}, \beta'_{11}) \to (w, t^{-2}\alpha'_{01}, t^2\beta'_{11})$, if follows that $\bar{K}$ acts generically freely on $\mathbf{P}^2$. Moreover, the sheaf $\mathcal{O}(1)$ on $\mathbf{P}^4$ is $\bar{K}$-linearized, and so generically $\tilde{Z}/\bar{K}$ is embedded in $(\mathbf{P}^2/\bar{K}) \times \mathbf{P}^4$ as a 3-fold quadric bundle over the rational curve $\mathbf{P}^2/\bar{K}$. Then by Tsen's theorem $\tilde{Z}/\bar{K}$ is rational over $\mathbf{P}^2/\bar{K}$, and so rational.

This completes the proof of Proposition 11, and so of Theorem 10.

THEOREM 15. *$\mathbf{P}(\psi)/S_5$ and $\mathbf{P}(\chi)/S_5$ are rational.*

Proof: This follows from Proposition 9 and Theorem 10.

Arithmetic rationality of M_6

In this section we prove that in fact M_6 is rational over any field. In fact we prove something slightly stronger, to state which we need a definition.

DEFINITION – LEMMA 16. *If X, Y are $\mathbf{Z}$-schemes of finite type that are geometrically reduced and irreducible, then they are arithmetically birational (denoted $X \sim_{\mathbf{Z}} Y$) if there are open subschemes X^0 of X* and Y^0 *of Y that are isomorphic and faithfully flat over $\mathbf{Z}$. The relation of arithmetical birationality is an equivalence relation.*

Clearly, if $X \sim_{\mathbf{Z}} Y$, then for any field k the k-varieties $X \otimes k$ and $Y \otimes k$ are birationally equivalent; it is not clear, however, that the converse is true.

THEOREM 17: *M_6 is arithmetically rational.*

Proof: Let Σ be the scheme obtained from $\mathbf{P}^2_{\mathbf{Z}}$ by blowing up the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$; as before, Σ is embedded anticanonically in $\mathbf{P}^5_{\mathbf{Z}}$ as a quintic Del Pezzo surface. Put $U_2 = H^0(\mathcal{O}_\Sigma(2))$, a $\mathbf{Z}$-lattice, and let $U^0 \subset \mathbf{P}(U_2)$ be the locus whose geometric points correspond to smooth curves. As before, the group S_5 acts on both of those, via automorphisms of Σ. We shall prove the result by showing that $\mathbf{P}(U_2)/S_5 \sim_{\mathbf{Z}} \mathbf{P}^{15}$ and that $U^0/S_5 \sim_{\mathbf{Z}} M_6$.

Define a functor $\mathfrak{F}$: *Schemes* $\to$ *Sets* by $\mathfrak{F}(S) = \{$isomorphism classes of flat projective morphisms $\pi: C \to S |$ all geometric fibres of π are smooth curves of genus six with exactly five g^1_4's$\}$. We shall show that $U^0/S_5 = M$, say, coarsely represents $\mathfrak{F}$.

First, we construct a morphism $\phi: \mathfrak{F} \to h_M = \mathrm{Hom}\,(-, M)$ as follows: for any scheme S, suppose that $[\pi: C \to S] \in \mathfrak{F}\,(S)$. There is an étale Galois cover $\tilde{S} \to S$, with Galois group $H \hookrightarrow S_5$ corresponding to the monodromy on the g^1_4's on the geometric fibres of π. Put $\tilde{\pi}: \tilde{C} = C \times_S \tilde{S} \to \tilde{S}$. There are five distinct line bundles on $\tilde{C}$ inducing a g^1_4 on the fibres; pick one, say $\mathcal{Q}$, and put $\mathcal{L} = \omega_{\tilde{C}/\tilde{S}} \otimes \mathcal{Q}^{-1}$. Then $H^1(C_s, \mathcal{L}_s)$ is two-dimensional for all geometric points s of $\tilde{S}$, and so $\tilde{\pi}_*\mathcal{L}$ is locally free of rank three, by the base-change theorem, and induces a g^2_6 on each fibre of $\tilde{\pi}$. Moreover, $\tilde{\pi}_*\mathcal{L}$ generates $\mathcal{L}$, and so gives a morphism $C \to \mathbf{P}(\tilde{\pi}_*\mathcal{L})$, a $\mathbf{P}^2$-bundle over $\tilde{S}$.

Over each geometric point s of $\tilde{S}$, the g^1_4's on $\tilde{C}_s$ besides $\mathcal{Q}_s$ are cut out by the systems of lines through the nodes of the plane model of $\tilde{C}_s$ given by $\mathcal{L}_s$; since these g^1_4's are defined globally over $\tilde{S}$, it follows that $\mathbf{P}(\tilde{\pi}_*\mathcal{L})$ has four disjoint sections, and is therefore trivial. Blow up along these sections to get $\tilde{C} \hookrightarrow \Sigma \times \tilde{S}$, a relative quadric section. So there is a classifying map $\tilde{S} \to U^0$, and so a morphism $S = \tilde{S}/H \to U^0/H \to U^0/S_5 = M$. This defines an element of $h_M(S)$, and so gives us a map $\mathfrak{F}(S) \to h_M(S)$. It is clear that these maps, as S varies, are given by a morphism ϕ: $\mathfrak{F} \to h_M$ of functors.

According to Mumford's definition [4, Definition 5.6] we must prove two things:

(i) for all algebraically closed fields Ω, the map $\phi(\mathrm{Spec}\ \Omega)$: $\mathfrak{F}\ (\mathrm{Spec}\ \Omega) \to h_M(\mathrm{Spec}\ \Omega)$ is an isomorphism, and

(ii) for all schemes N and for all morphisms ψ: $\mathfrak{F} \to h_N$, there is a unique morphism χ: $h_M \to h_N$ such that $\psi = \chi \circ \phi$.

Proof of (*i*): By a theorem of Seshadri [5, Theorem 4], the natural map $(\mathbf{P}(U_2)/S_5) \otimes \Omega \to (\mathbf{P}(U_2) \otimes \Omega)/S_5$ is an isomorphism; then (i) follows from Proposition 4, whose statement and proof are valid over any Ω.

Proof of (*ii*): Suppose that ψ: $\mathfrak{F} \to h_N$ is given. Suppose that $[\alpha: S \to M] \in h_M(S)$. Put $\tilde{S} = S \times_M U^0$. From the family $C \to U^0$, induced from the universal family of quadric sections of Σ, we get $\tilde{\pi}$: $\tilde{C} = C \times_{U^0} \tilde{S} \to \tilde{S}$; i.e., $[\tilde{\pi}] \in \mathfrak{F}(\tilde{S})$. We define the morphism χ by $\chi(S)(\alpha) = \psi(\tilde{S})(\tilde{\pi})$.

So $M = U^0/S_5$ does indeed coarsely represent $\mathfrak{F}$, which is an open subfunctor of the moduli functor; hence $U^0/S_5 \sim_{\mathbf{Z}} M_6$.

It remains to show that $\mathbf{P}(U_2)/S_5 \sim \mathbf{P}^{15}$.

Recall that all complex representations of S_5 are defined over $\mathbf{Q}$. Let Λ denote the root lattice A_4; by Corollary 3 there is an S_5-equivariant surjection $U_2 \otimes \mathbf{Q} \to \Lambda \otimes \mathbf{Q} = \phi$. It is well known (and easy to see) that the only non-zero $\mathbf{Z}S_5$-sublattices of Λ are isomorphic to either Λ or $\Lambda^\vee$, and so we have an S_5-equivariant surjection $U_2 \to \Lambda_1$, where Λ_1 is one or other of Λ and $\Lambda^\vee$. Let V denote the kernel, and β: $\tilde{\mathbf{P}} = Bl_{\mathbf{P}(V)}\mathbf{P}(U_2) \to \mathbf{P}(\Lambda_1)$ the induced morphism. (For any lattice L, we define $\mathbf{P}(L) = \mathrm{Proj}\ (\mathrm{Symm}^\bullet(L^\vee))$.)

The action of S_5 on both Λ and $\Lambda^\vee$ is generated by reflexions; let $\Delta \subset \mathbf{P}(\Lambda_1)$ denote the discriminant locus, which is the union of the reflexion hyperplanes. Put $\mathbf{P}^0 = \mathbf{P}(\Lambda_1) - \Delta$, then $\mathbf{P}^0$ is faithfully flat over $\mathbf{Z}$, and S_5 acts on $\mathbf{P}^0$ with trivial geometric stabilizers. So the natural map $\mathbf{P}^0 \to \mathbf{P}^0/S_5$ is étale, and if γ: $X = \beta^{-1}(\mathbf{P}^0)/S_5 \to Y = \mathbf{P}^0/S_5$, then all the geometric fibres of γ are isomorphic to $\mathbf{P}^{11}$. We want to show that γ is a trivial $\mathbf{P}^{11}$-bundle.

Since Λ_1 is either Λ or $\Lambda^\vee$, there is a non-zero S_5-invariant pairing $\Lambda_1 \times \Lambda_1 \to \mathbf{Z}$, i.e., an S_5-invariant element of $\Lambda_1^\vee \otimes \Lambda_1^\vee$, and so an invariant element of $\Lambda_1^\vee \otimes U_2^\vee$, which we interpret as an element λ of $\mathrm{Symm}^\bullet\ (\Lambda_1^\vee) \otimes U_2^\vee$. We can assume that the coefficients of λ have no common factor in $\mathbf{Z}$. Then the zero-locus of λ is a divisor, flat over $\mathbf{Z}$, on the regular scheme X. Let L be the corresponding line bundle. L is very ample, with vanishing higher cohomology, on each geometric fibre of γ, and so by the base change theorem, L is very ample relative to γ and gives an isomorphism $X \to \mathbf{P}^{11} \times Y$.

Hence $\mathbf{P}(U_2)S_5 \sim_{\mathbf{Z}} (\mathbf{P}(\Lambda_1)/S_5) \times \mathbf{P}^{11}$. Now similar arguments, involving projection onto each factor, show that

$$(\mathbf{P}(\Lambda) \times \mathbf{P}(\Lambda_1))/S_5 \sim_{\mathbf{Z}} \mathbf{P}(\Lambda) \times (\mathbf{P}(\Lambda_1)/S_5)$$

and

$$\sim_{\mathbf{Z}} (\mathbf{P}(\Lambda)/S_5) \times \mathbf{P}(\Lambda_1).$$

Now the ring of invariants $\mathrm{Symm}^{\bullet}\,(\Lambda^{\vee})^{S_5}$ is a polynomial ring, by Newton's theorem on symmetric functions, and so $\mathbf{P}(\Lambda)/S_5 \sim_{\mathbf{Z}} \mathbf{P}^4$. Hence $\mathbf{P}^4 \times \mathbf{P}(\Lambda_1)/S_5) \sim_{\mathbf{Z}} \mathbf{P}^4 \times \mathbf{P}^4$, and so

$$\mathbf{P}(U_2)/S_5 \sim_{\mathbf{Z}} (\mathbf{P}(\Lambda_1)/S_5) \times \mathbf{P}^{11} \sim_{\mathbf{Z}} (\mathbf{P}(\Lambda_1)/S_5)$$

$$\times\, \mathbf{P}^4 \times \mathbf{P}^7 \sim_{\mathbf{Z}} \mathbf{P}^4 \times \mathbf{P}^4 \times \mathbf{P}^7 \sim_{\mathbf{Z}} \mathbf{P}^{15} \qquad \text{Q.E.D.}$$

The character table of S_5

	1	10	20	30	15	20	24
	1^5	$1^3 2$	$1^2 3$	14	12^2	23	5
1	1	1	1	1	1	1	1
ϕ	4	2	1	0	0	-1	-1
χ	5	1	-1	-1	1	1	0
ψ	6	0	0	0	-2	0	1
χ'	5	-1	-1	1	1	-1	0
ϕ'	4	-2	1	0	0	1	-1
σ	1	-1	1	-1	1	-1	1

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