COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 68, nº 2 (1988), p. 203-220
<http://www.numdam.org/item?id=CM_1988__68_2_203_0>

Compositio Mathematica 68: 203–220 (1988)

Abundance conjecture for 3-folds: case $\nu = 1$

Dedicated to Professor F. Hirzebruch on his 60th birthday

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Received 11 January, 1988

Introduction

A normal projective variety is said to be *minimal* if it has only terminal singularities and its canonical divisor $K_X \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ is nef. A recent result of S. Mori [Mr] asserts the existence of a minimal model for a given complex algebraic 3-fold except for uniruled ones.

In [My] the author proved a minimal 3-fold has non-negative Kodaira dimension; when combined with Mori's theorem mentioned above, this amounts to the following characterization of 3-folds with $\kappa = -\infty$:

THEOREM. *A complex algebraic 3-fold has Kodaira dimension $-\infty$ if and only if it is uniruled.*

A natural question now arises: *What is the characterization of 3-folds with* $\kappa = 0$? More specifically:

(*) *Does a 3-fold with $\kappa = 0$ have a minimal model with numerically trivial canonical divisor?*

To make things more explicit, let us introduce an invariant $\nu(X)$, the *numerical Kodaira dimension*, of a minimal variety X. By definition,

$$\nu(X) = \min \{d \in \mathbb{Z};\ c_1(K_X)^{d+1} = 0 \in \mathrm{H}^{2d+2}(X, \mathbb{Q})\}.$$

Clearly ν takes values in $\{0, 1, \ldots, \dim X\}$. For example, $\nu(X) = 0$ is equivalent to the numerical triviality of K_X; $\nu(X) = \dim X$ if and only if K_X is *big*, i.e. $K_X^{\dim X} > 0$.

As is easily seen, the question $(*)$ would be affirmatively answered if we could verify

$$(**) \text{ (Abundance conjecture) } \quad \kappa(X) = \nu(X).$$

The inequality $\kappa(X) \leqq \nu(X)$ follows from a formal argument, yet the inequality of the converse direction is not so trivial. Furthermore $(**)$ involves an important implication; via his powerful "base point freeness theorem", Y. Kawamata [Kw] pointed out that the linear system $|mK_X|$ is free from base points for sufficiently divisible m, provided the abundance conjecture $(**)$ is true.

In an extremal case $\nu = 0$ or 3, the equality $\kappa = \nu$ for a minimal 3-fold can be checked rather easily. The objective of the present paper is to show the equality in one of the intermediate cases: $\nu = 1$.

Main Theorem. *Let X be a minimal* 3-*fold with $\nu(X) = 1$. Then $\kappa(X) = 1$ and there is a positive integer m such that $\mathcal{O}_X(mK_X)$ is generated by global sections.*

Our proof is based on the analysis of an effective Cartier divisor $D \in |mK_X|$ $(m > 0)$, the existence of which is guaranteed by $\kappa(X) \geqq 0$ [My]. We are interested in the analytic and infinitesimal neighbourhoods of D as well as D itself. A direct analysis of them seems a little bit too tough; to simplify the situation, we need three reduction steps described below.

Let $U \subset X$ be a sufficiently small analytic neighbourhood of D. Then we have:

(0.1) (Gorenstein reduction, see §1) *There is a finite covering $\gamma\colon V \to U$, étale off* Sing (U), *such that $K_V = \gamma^* K_U$ is Cartier.*

(0.2) (Semi-stable reduction, see §2) *There is a proper generically finite covering $\sigma\colon W \to V$, étale off* supp $(\gamma^* D)$, *such that W is smooth and that $\sigma^*\gamma^* D$ is a multiple of a reduced divisor $\tilde{D}$ with only simple normal crossings.*

(0.3) (Minimal model à la Kulikov–Persson–Pinkham, §3) *After finitely many contractions of components of $\tilde{D}$ and elementary transformations, a smooth "minimal model" $(W_0, \tilde{D}_0)$ of $(W, \tilde{D})$ is reached. The natural image $\tilde{D}_0$ of $\tilde{D}$ in W_0 is still a divisor with only simple normal crossings and $\tilde{D}_0 | \tilde{D}_0 \approx K_{W_0} | \tilde{D}_0 \approx 0$.*

Once we come across this situation, it is combinatorics to determine the structure of $\tilde{D}_0$ as an analytic space. A theorem of R. Friedman shows that

$\tilde{D}_0$ is actually a degeneration of smooth surfaces with $\kappa = 0$. This implies that $K_{W_0}|\tilde{D}_0$ and $\tilde{D}_0|\tilde{D}_0$ are both torsion in $\mathrm{Pic}(\tilde{D}_0)$ so that there exists an étale covering $\tau: M \to W_0$ such that $K_M|S \sim S|S \sim 0$, where $S = \tau^*\tilde{D}_0$. Finally, we study the infinitesimal neighbourhoods of S in M:

(0.4) *The infinitesimal displacements of S in M are not obstructed. In particular,*

$$\dim \mathrm{H}^0(nS, \mathcal{O}_{nS}(kS)) = n \quad \text{for } n \in \mathbb{N}, k \in \mathbb{Z},$$

whence it follows that

$$\dim \mathrm{H}^0(nD, \mathcal{O}_{nD}(nD)) \sim \mathrm{O}(n).$$

The Main Theorem is a direct consequence of (0.4), see §4.

In this paper, we work in the category of analytic spaces.

1. Gorenstein reduction

In order to show the Gorenstein reduction (0.1), let us start with some elementary observations.

(1.1) LEMMA. *Let $(Z, 0)$ be a germ of a terminal 3-fold singularity of index r. Then*

$$\mathrm{H}_1(Z, \mathbb{Z}) = 0,$$

$$\mathrm{H}_1(Z - 0, \mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z},$$

$$\mathrm{Pic}(Z) = (1),$$

$$\mathrm{Pic}(Z - 0)_{\mathrm{tor}} \cong \mathrm{Hom}(\mathrm{H}_1(Z - 0, \mathbb{Z}), \mathbb{C}^*)_{\mathrm{tor}} \cong \boldsymbol{\mu}_r.$$

Proof. $(Z, 0)$ is a $\boldsymbol{\mu}_r$-quotient of a compound Du Val singularity $(\tilde{Z}, \tilde{0})$ and $\pi_1(\tilde{Z} - \tilde{0}) = (1)$ by Milnor's theorem [Ml, Theorem 6.6]. □

(1.2) LEMMA. *Let $(Z, 0)$ be as above and S an effective Cartier divisor passing through 0. Then the restriction mapping*

$$\mathrm{Pic}(Z - 0)_{\mathrm{tor}} \to \mathrm{Pic}(S - 0)_{\mathrm{tor}}$$

is injective.

Proof. Let $f\colon \tilde{Z} \to Z$ be the "canonical" $\boldsymbol{\mu}_r$-covering as in the proof of (1.1). $\tilde{S} = f^*S$ is a *connected* Cartier divisor on $\tilde{Z}$, while $\tilde{0} = f^{-1}(0)$ is a single point and hence of codimension 2 in $\tilde{S}$. Therefore $\tilde{S} - \tilde{0}$ is connected, which implies the surjectivity of $\pi_1(S - 0) \to \pi_1(Z - 0)$ and of $H_1(S - 0, \mathbb{Z}) \to H_1(Z - 0, \mathbb{Z})$. Thus we infer the injectivity of

$$\begin{aligned} \operatorname{Pic}(Z - 0)_{\text{tor}} &\cong \operatorname{Hom}(H_1(Z - 0, \mathbb{Z}), \mathbb{C}^*) \\ \longrightarrow F &= \operatorname{Hom}(H_1(S - 0, \mathbb{Z}), \mathbb{C}^*)). \end{aligned}$$

The group F is naturally identified with that of flat line bundles $\subset \operatorname{Pic}(S - 0)$.

(1.3) Corollary. *In the same notation as in* (1.2), $\alpha K_Z|_S$ *is Cartier on S if and only if* $r|\alpha$ ($\alpha \in \mathbb{Z}$, r = *index of* $(Z, 0)$).

Proof. $\alpha K_Z|_S$ is Cartier if and only if $\mathcal{O}_{S-0}(\alpha K_Z) \cong \mathcal{O}_{S-0}$, which means that αK_Z is trivial on $Z - 0$ by (1.2), i.e. αK_Z is Cartier on Z. □

Let $\bar{U}$ be an analytic 3-fold with only finitely many terminal singularities and $D \subset \bar{U}$ an effective Cartier divisor which contains the singular locus $\operatorname{Sing}(U)$.

(1.4) Lemma. *Let r denote the index of* $\bar{U}$, *viz. the L.C.M. of the indices at the singular points. Assume that* $c_1(rK_{\bar{U}})|D \in \mathrm{H}^2(D, \mathbb{Z})$ *is torsion. Then there are a small neighbourhood* $\bar{U}' \subset \bar{U}$ *of D and a finite étale covering* $g\colon \bar{U}'' \to \bar{U}'$ *such that* $c_1(rK_{\bar{U}''})|g^*D = 0 \in H^2(g^*D, \mathbb{Z})$.

Proof. Immediate consequence of the natural isomorphism

$$\mathrm{H}^2(D, \mathbb{Z})_{\text{tor}} \cong \mathrm{H}_1(D, \mathbb{Z})_{\text{tor}} \cong \mathrm{H}_1(\bar{U}, \mathbb{Z})_{\text{tor}}$$

for a tubular neighbourhood $\bar{U}'$ of D. □

(1.5) Lemma. *Let the notation and the assumption be as in* (1.4). *Then there exists a finite cyclic* $\boldsymbol{\mu}_r$*-covering* $h\colon D^* \to g^*D$ *which has the following two properties:*

(1.5.1) *h is étale off* $\operatorname{Sing}(\bar{U}'') \subset g^*D$;

(1.5.2) *The branch index of h at* $P \in g^*D$ *is exactly the local index of* $\bar{U}''$ *at P; in other words,* D^* *is locally a disjoint union of canonical covers over P.*

Proof. Since $\operatorname{Pic}^0(g^*D) \cong \mathrm{H}^1(g^*D, \mathcal{O})/\mathrm{H}^1(g^*D, \mathbb{Z})$ is a divisible group, we can find $\tau \in \operatorname{Pic}^0(g^*D)$ such that $rK_{\bar{U}''} - r\tau = 0 \in \operatorname{Pic}^0(g^*D)$. Fix a non-zero section $s \in \mathrm{H}^0(g^*D, \mathcal{O}_{g^*D}(rK_{\bar{U}''} - r\tau))$ and construct a $\boldsymbol{\mu}_r$-cover

$$D^* = \operatorname{Specan} \{\mathcal{O}_{g^*D} \oplus \mathcal{O}_{g^*D}(\tau - K_{\bar{U}''}) \oplus \cdots \oplus \mathcal{O}_{g^*D}((r-1)(\tau - K_{\bar{U}''}))\}$$

in a standard manner. Then D^* satisfies our requirements by (1.4) since $\mathcal{O}(\tau)$ is locally isomorphic to $\mathcal{O}$. □

Now we have the following theorem which is slightly more general than (0.1):

(1.6) THEOREM. *Let $\bar{U}$ be an analytic 3-fold with only finitely many terminal singularities and D an effective Cartier divisor. Let r be the index of $\bar{U}$ and assume that $c_1(rK_{\bar{U}})|_D \in \mathrm{H}^2(D, \mathbb{Z})$ is torsion. Then, for a sufficiently small neighbourhood $\bar{U}' \subset \bar{U}$ of D, there is a finite covering $\gamma\colon V \to \bar{U}'$ which satisfies the following conditions:*

(1.6.1) *γ is étale off* $\operatorname{Sing}(\bar{U}')$;
(1.6.2) *The branch index of γ at $P \in D$ is exactly the local index of $\bar{U}$ at P;*
(1.6.3) *V is a normal Gorenstein analytic space with only terminal singularities.*

Proof. Fix a small neighbourhood $\Delta \subset \bar{U}$ of $\operatorname{Sing}(\bar{U})$. Then choose a sufficiently small neighbourhood $\bar{U}' \subset \bar{U}$ of D in such a way that $D_0 = D - (D \cap \Delta)$ is a deformation retract of $\bar{U}_0' = \bar{U}' - (\bar{U}' \cap \Delta)$. By (1.5), we have a finite étale covering

$$\bar{\gamma}\colon D_0^* = D^* - h^{-1}(g^{-1}(D \cap \Delta)) \to D_0.$$

Since $\pi_1(D_0) \cong \pi_1(\bar{U}_0')$, there is an étale covering

$$\gamma_0\colon V_0 \to \bar{U}_0'$$

which induces $\bar{\gamma}$. On the other hand, we have the canonical covering $\tilde{\Delta} \to \Delta$. Recalling that $g \circ h\colon D^* \to D$ is locally the canonical covering, we can patch up V_0 with finitely many copies of components of $\tilde{\Delta}$ to get a finite covering

$$\gamma\colon V \to \bar{U}' \cup \Delta.$$

This construction implies (1.6.1–3). □

2. Semi-stable reduction

Let Y be a complex 3-manifold, $E \neq 0$ an effective, projective Cartier divisor on Y and $V \subset Y$ a small open neighbourhood of E. Throughout this section, we fix this notation and assume the following extra conditions:

(a) The reduced part E_{red} of E is a divisor with only simple normal crossings;
(b) $E|E$ is numerically trivial on E;
(c) There exists a divisor H on Y such that $H|E$ is ample.

Let E $= \Sigma_{i=1}^{s} a_i S_i$ be the decomposition into distinct irreducible components.

(2.1) LEMMA.·*The restriction maps and the degree maps give natural isomorphisms*

$$\mathrm{H}^4(E, \mathbb{Z}) \xrightarrow[\sim]{\text{rest.}} \bigoplus_{i=1}^{s} \mathrm{H}^4(S_i, \mathbb{Z}) \xrightarrow[\sim]{\deg} \mathbb{Z}^s.$$

Proof. Consider the exact sequence

$$0 \to \mathbb{Z}_E \to \bigoplus_{i=1}^{s} \mathbb{Z}_{S_i} \to \bigoplus_{i<j} \mathbb{Z}_{S_i \cap S_j} \to \bigoplus_{i<j<k} \mathbb{Z}_{S_i \cap S_j \cap S_k} \to 0.$$

From the fact that the real dimension of $S_i \cap S_j = 2$, the assertion easily follows. □

We denote by δ the natural isomorphism $\mathrm{H}^4(E, \mathbb{Z}) \xrightarrow{\approx} \mathbb{Z}^s$. Let ϱ: $\mathrm{H}_c^4(V, \mathbb{Z}) \to \mathrm{H}_c^4(E, \mathbb{Z}) = \mathrm{H}^4(E, \mathbb{Z})$ be the restriction map, where the subscript c stands for the cohomology with compact support.

(2.2) LEMMA. $\mathrm{Im}\,(\delta \circ \varrho) \subset \{(x_1, \ldots, x_s) \in \mathbb{Z}^s; \Sigma\, a_i x_i = 0\}$.

Proof. Let $\eta \in \mathrm{H}_c^4(V, \mathbb{Z})$. Then $\deg(\eta | S_i) = \deg(\eta \cup S_i)$, so that

$$\Sigma\, a_i \deg(\eta | S_i) = \Sigma\, a_i \deg(\eta \cup S_i) = \deg(\eta \cup \Sigma\, a_i S_i)$$
$$= \deg \eta \cup E.$$

By the Lefschetz duality $\mathrm{H}_c^4(V, \mathbb{Z}) \cong \mathrm{H}_2(V, \mathbb{Z}) \cong \mathrm{H}_2(E, \mathbb{Z})$, η can be regarded as a 2-cycle η' on E and we have

$$\deg \eta \cup E = \deg E|\eta'.$$

Since E is numerically trivial on E, $\deg E|\eta' = 0$ which proves the lemma. □

(2.3) COROLLARY. $\ker\{\mathrm{H}_1(V - E, \mathbb{Z}) \to \mathrm{H}_1(V, \mathbb{Z})\}$ *has positive rank.*

Proof. By the Lefschetz duality we have

$$\ker\{H_1(V - E, \mathbb{Z}) \to H_1(V, \mathbb{Z})\} \cong \ker\{H_c^5(V, E; \mathbb{Z}) \to H_c^5(V, \mathbb{Z})\}$$
$$\cong \operatorname{Coker}\{H_c^4(V, \mathbb{Z}) \to H^4(E, \mathbb{Z})\},$$

and the third term has positive rank by (2.2). □

(2.4) Definition. Let $L \subset Y$ be a compact effective divisor such that

(2.4.a) L is projective with an ample divisor H and that
(2.4.b) $L|L$ is numerically trivial.

Let $L = \Sigma e_i L_i$ be the decomposition into irreducible components. L is said to be *primitive* if L is connected and G.C.D. $\{e_i\} = 1$.

(2.5) Lemma. *Suppose that an effective divisor $L = \Sigma e_i L_i$ satisfies* (2.4.a) *and* (2.4.b). *Assume that L is connected. If $(\Sigma e_i' L_i) \cdot H|L$ is numerically trivial, then $e_i' = ce_i$ for some constant $c \in \mathbb{Q}$ independent of i. In particular, L can be uniquely decomposed into $\Sigma l_i L_i$, where L_i's are primitive and disjoint with each other.*

The proof is easy and left to the reader. Applying this to our original situation, we have

(2.6) Corollary. *E can be uniquely decomposed into $\Sigma b_i E_i$, where E_i's are primitive divisors which are mutually disjoint and b_i's are positive integers.*

Thus the small neighbourhood $V \subset Y$ is a disjoint union of neighbourhoods V_i of E_i. Therefore, without loss of generality, we may assume that E is connected in the argument below. Let $E = e\Sigma_{i=1}^s a_i' S_i$ be the decomposition into irreducible components, where $e \in \mathbb{N}$, G.C.D. $\{a_i'\} = 1$.

(2.7) Lemma. *Assume that E is connected. Then*

$$\operatorname{Im} \delta \circ \varrho \subset \{(x_1, \ldots, x_s) \in \mathbb{Z}^s;\quad \Sigma a_i' x_i = 0\}$$

is a sublattice of finite index.

Proof. It suffices to show $\operatorname{Im}(\delta \circ \varrho \otimes \mathbb{Q}) = \{(x_1, \ldots, x_s) \in \mathbb{Q}^s; \Sigma a_i' x_i = 0\}$. Consider the $\mathbb{Q}$-vector subspace $\Pi \subset \operatorname{Im}(\delta \circ \varrho \otimes \mathbb{Q})$ generated by $S_1 H|E, \ldots, S_s H|E$. (Note that $S_i \in H_c^2(V, \mathbb{Z})$, $H \in H^2(V, \mathbb{Z})$ so that $S_i \cdot H \in H_c^4(V, \mathbb{Z})$.) Then, by (2.5), the unique relation between the $S_i \cdot H|E \in H^4(E, \mathbb{Q})$ is

$$\Sigma (a_i' S_i \cdot H)|E = 0.$$

Hence

$$\dim_{\mathbb{Q}} \operatorname{Im}(\delta \circ \varrho \otimes \mathbb{Q}) = \dim_{\mathbb{Q}} \operatorname{Im}(\varrho \otimes \mathbb{Q})$$

$$\geqq \dim_{\mathbb{Q}} \Pi = s - 1 = \dim_{\mathbb{Q}} \{(x_1, \ldots, x_s) \in \mathbb{Q}^s;\ \Sigma a_i' x_i = 0\}.$$

This shows the assertion. □

(2.8) COROLLARY. *If E is connected, then*

$$\ker\{H_1(V - E, \mathbb{Z}) \to H_1(V, \mathbb{Z})\}/\text{tor} \cong \operatorname{Coker}\{H_c^4(V, \mathbb{Z}) \to H^4(E, \mathbb{Z})\}/\text{tor}$$

$$\cong \delta^{-1}(\mathbb{Z}(a_1', \ldots, a_s')) \subset H^4(E, \mathbb{Z}).$$

(2.9) COROLLARY. *For each positive integer l, there exists a canonical $\boldsymbol{\mu}_l$-covering σ_l: $V_l \to V$ branching along E whose branch index along S_i is exactly $l/(l, a_i')$. If l is divisible by $a_1', \ldots, a_s'$, then $(\sigma_l^* E)/l$ is a reduced Cartier divisor.*

The normal analytic space V_l has toric singularities over the double curves of E_{red}. However, it is known that V_l has a nice resolution:

(2.10) THEOREM (G. Kempf et al. [KKMS]). *If l is sufficiently divisible, then V_l has a resolution $\pi = \pi_l$: $W = W_l \to V_l$ such that $\pi^* \sigma_l^* E/l$ is a reduced divisor with only simple normal crossings.*

(2.11) REMARK. The integer l above is not L.C.M. $\{a_i'\}$ in general.

Putting things together, we obtain

(2.12) THEOREM. *There exists a proper, generically finite covering σ: $W \to V$ such that*

(2.12.a) *W is non-singular and that*
(2.12.b) *$\sigma^* E$ is a multiple of a reduced divisor with only simple normal crossings.*

To show (0.2), we apply (2.12) to a suitable resolution (Y, E) of the Gorenstein reduction of $(\bar{U}, D)$. Since D comes from X, its total transformation E is projective; H is easily constructed from the pull-back of an ample divisor on X and the exceptional divisors with respect to the resolution.

3. Minimal model

Let N be an analytic 3-manifold with an effective, projective, reduced divisor T on it. Assume the following two conditions:

(a) *$T|T$ is numerically trivial (on T);*
(b) *There are positive integers m_i such that $K_N|T \approx (\Sigma m_i T_i)|T$, where T_i's stand for the irreducible components of T.*

(3.1) REMARK. In this situation, $K_N|T$ is nef $\Leftrightarrow K_N|T \approx 0 \Leftrightarrow m_i = m_j$ for every i, j. If we start with $D \in |mK_X|$ for a minimal 3-fold X and take a Gorenstein reduction $\gamma\colon V \to U$ of a small neighbourhood U of D and then a semi-simple reduction $\sigma\colon W \to V$, then the pair $(W, \sigma^*\gamma^*D/\deg \sigma)$ satisfies the conditions (a) and (b) above. (Without Gorenstein reduction, the coefficient m_i might be a rational number.) Furthermore, we have in this case

$$K_W \sim \Sigma m_i \tilde{D}_i, \qquad m_i \in \mathbb{N}$$

where $\tilde{D}_i$ is an irreducible component of $\tilde{D} = \sigma^*\gamma^*D/\deg \sigma$.

(3.2) THEOREM (Kulikov [Kl], Persson–Pinkham [PP]). *Let N and T be as above. Then, after finitely many smooth contractions of components of T and/or Kulikov's elementary transformations (or "symmetric flops") we come across a minimal model (M, S); the pair (M, S) has the following properties:*

(3.2.A) *M is non-singular and $K_M|S \approx 0$;*
(3.2.B) *The proper transformation S of T is a reduced divisor with only simple normal crossings and $S|S \approx 0$;*
(3.2.C) *If $K_N \sim \Sigma m_i' T_i$, then $K_M \sim (\min\{m_i'\}) \cdot S$.*

The original papers deal with a degeneration of smooth surfaces, but their numerical proof works in our setting.

(3.3) REMARK. The assumption that m_i is integral is essential. If we allow rational numbers as coefficients, certain quotient singularities appear on a minimal model. S is not necessarily projective; however, contractions of finitely many curves on S gives a normal 3-fold $\hat{M}$ in which the image $\hat{S}$ of S is projective.

It is not too difficult to classify S as an analytic space; the result is essentially given in Friedman–Morrison [FM, p. 15 ff.].

(3.4) THEOREM. *S is isomorphic to one of the following surfaces:*

(0) *A smooth surface (S is either a $K3$, Enriques, abelian of hyperelliptic surface);*

$(1)_s$ *A cycle of (relatively) minimal elliptic ruled surfaces* S_i ($i \in \mathbb{Z}/s\mathbb{Z}$, $s \geqq 2$) *and* S_i *meets only* $S_{i\pm 1}$ *along two disjoint sections;*

$(1')_s$ *A chain of minimal elliptic ruled surfaces* $S_1, \ldots, S_s$ ($s \geqq 2$) *such that*

(α) S_i *meets only* $S_{i\pm 1}$ *along two disjoint sections for* $1 < i < s$,

(β) S_1 [*resp.* S_r] *meets only* S_2 [*resp.* S_{r-1}] *along an étale double section;*

$(2)_s$ *A chain of surfaces* $S_1, \ldots, S_s$ ($s \geqq 2$) *such that*

(α) S_i *is a minimal elliptic ruled surface and meets only* $S_{i\pm 1}$ *along two disjoint sections for* $1 < i < s$,

(β) S_1 [*resp.* S_s] *is a rational surface and* $S_2|S_1$ [*resp.* $S_{s-1}|S_s$] *is a smooth elliptic curve* $\sim -K_{S_1}$ [*resp.* $-K_{S_s}$];

$(2')_s$ *A chain of surfaces* $S_1, \ldots, S_s$ ($s \geqq 2$) *such that*

(α) S_i *is a minimal elliptic ruled surface and meets only* $S_{i\pm 1}$ *along two disjoint sections for* $1 < i < s$,

(β) S_1 *is a minimal elliptic ruled surface with* $S_2|S_1$ *being an étale double section,*

(γ) S_s *is a rational surface with* $S_{s-1}|S_s$ *being a smooth elliptic curve* $\sim -K_{S_s}$;

(3) *Configuration of rational surfaces whose dual graph is a triangulation of either a* 2*-sphere* S^2, *a real projective plane* $\mathbb{P}^2(\mathbb{R})$, *a torus* $S^1 \times S^1$ *or a Klein bottle.*

(3.5) REMARK. A surface of type $(1')_s$ [resp. $(2')_s$] is an étale $\boldsymbol{\mu}_2$-quotient of that of type $(1)_{2s-2}$ [resp. $(2)_{2s-1}$].

(3.6) PROPOSITION. *If S is of type* (0) *or* $(1)_s$ *or* $(1')_s$ [*resp.* $(2)_s$ *or* $(2')_s$ *or* (3)], *then* $4K_S$ *or* $6K_S \sim 0$ [*resp.* $2K_S \sim 0$]. *Hence, by adjunction,*

$$12(K_M + S)|_S \sim 0.$$

(3.7) COROLLARY. *If* $K_M \sim nS$, $n \in \mathbb{Z}\backslash\{-1\}$, *then* $S|S$ *is torsion. For a tubular neighbourhood* $M' \subset M$ *of* S, *there is an étale covering* $\varepsilon\colon \tilde{M}' \to M'$ *such that*

$$\varepsilon^*S|\varepsilon^*S \sim K_{\tilde{M}'}|\varepsilon^*S \sim 0.$$

(3.8) THEOREM (Friedman [F]). *Under the notation and assumption as in* (3.7), $\tilde{S} = \varepsilon^*S$ *has a versal deformation*

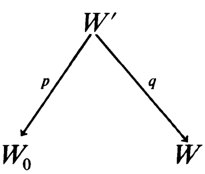
$$\phi\colon (\mathscr{X}, \tilde{S}) \to (\mathscr{Y}, 0).$$

Here $\mathscr{X}$ and $\mathscr{Y}$ are complex manifolds, $0 \in \mathscr{Y}$ is a reference point, and ϕ is a proper flat morphism with central fibre $\tilde{S} = \phi^{-1}(0)$. The relative canonical sheaf $\omega_{\mathscr{X}/\mathscr{Y}} = \omega_{\mathscr{X}} \otimes \phi^*\omega_{\mathscr{Y}}^{-1}$ is trivial around $\tilde{S}$.

(3.9) Remarks. Since contractions and elementary transformations commute with étale covering, we can replace the semistable-Gorenstein reduction $\gamma \circ \sigma: W \to U$ by a suitable étale covering of W so that the image $\tilde{D}_0$ of $\tilde{D} = (\gamma \circ \sigma)^* D/\deg \sigma$ on a minimal model W_0 satisfies

$$\tilde{D}_0 | \tilde{D}_0 \sim K_{W_0} | \tilde{D}_0 \sim K_{\tilde{D}_0} \sim 0.$$

It goes without saying that $\tilde{D}_0$ is a degeneration of $K3$ or abelian surfaces. As an immediate consequence of the construction of the minimal model W_0, there exists a diagram of proper bimeromorphic morphisms



such that $p^*\tilde{D}_0 = q^*\tilde{D}$.

4. Formal neighbourhoods

In this section, we give the proof of Main Theorem. Let us start with an elementary observation.

(4.1) Lemma. *Let S be a compact analytic space with the underlying reduced structure $T = S_{\mathrm{red}}$. Let $\mathscr{L}$ be an invertible sheaf on S. If $\mathscr{L} \otimes \mathcal{O}_T \cong \mathcal{O}_T$ and $\mathscr{L}^{\otimes n} \cong \mathcal{O}_S$ for some positive integer n, then $\mathscr{L} \cong \mathcal{O}_S$. In other words,* $\ker\{\mathrm{Pic}(S) \to \mathrm{Pic}(T)\}$ *has no torsion.*

Proof. Without loss of generality, we may assume that S is connected. Since T is compact and reduced,

$$\mathrm{H}^0(T, \mathcal{O}_T) = \mathbb{C}, \qquad \mathrm{H}^0(T, \mathcal{O}_T^*) = \mathbb{C}^*.$$

Hence the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ gives rise to a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & H^1(S, \mathbb{Z}) & \xrightarrow{i} & H^1(S, \mathcal{O}) & \longrightarrow & \mathrm{Pic}(S) & \longrightarrow & H^2(S, \mathbb{Z}) \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \downarrow \wr \\ 0 & \longrightarrow & H^1(T, \mathbb{Z}) & \xrightarrow{j} & H^1(T, \mathcal{O}) & \longrightarrow & \mathrm{Pic}(T) & \longrightarrow & H^2(T, \mathbb{Z}). \end{array}$$

Since j is injective, so is i and we see that

$$\ker\{\mathrm{Pic}(S) \to \mathrm{Pic}(T)\} \cong \ker\{H^1(S, \mathcal{O}) \to H^1(T, \mathcal{O})\}$$

is a $\mathbb{C}$-vector space. □.

The main ingredient of this section is the following:

(4.2) THEOREM. *Let S be a connected, compact, reduced analytic subspace of pure codimension* 1 (*hence an effective Cartier divisor*) *on an analytic manifold M. Assume the following three conditions:*

(4.2.a) $\mathcal{O}_S(S) \cong \mathcal{O}_S$;
(4.2.b) $\mathcal{O}_M(aK_M) \cong \mathcal{O}_M(bS)$ *for some* $a, b \in \mathbb{Z}$, $a > 0$, $b \neq -2a, -3a, -4a, \ldots$
(4.2.c) *There exists a versal deformation*

$$\phi: (\mathcal{X}, S) \to (\mathcal{Y}, 0)$$

of S such that $\mathcal{X}$ is smooth and $\omega_{\mathcal{X}/\mathcal{Y}} \cong \mathcal{O}_{\mathcal{X}}$ around S. Then, for every positive integer n, we have

$$(4.2.1)_n \quad \mathcal{O}_{nS}(S) \cong \mathcal{O}_{nS}$$

and there exists a natural morphism

$$\phi_n: \mathrm{Spec}\ \mathbb{C}[\varepsilon]/(\varepsilon^n) \to (\mathcal{Y}, 0)$$

which induces an isomorphism

$$(4.2.2)_n \quad nS \cong \mathrm{Spec}\ (\mathbb{C}[\varepsilon]/(\varepsilon^n)) \times_{\mathcal{Y}} \mathcal{X}.$$

Moreover,

$(4.2.3)_n$ $H^0(nD, \mathcal{O}(mD)) \to H^0(n'D, \mathcal{O}(mD))$ *is surjective for every $n' < n$ and $m \in \mathbb{Z}$.*

The proof of (4.2) is by induction on n. $(4.2.1)_1$ is nothing but (4.2.a), while $(4.2.3)_1$ is vacuous. The morphism ϕ_1: Spec $\mathbb{C} \to (\mathcal{Y}, 0)$ is trivially defined as the constant map to 0, which establishes $(4.2.2)_1$.

Let us fix the notation. Let $\{U_i\}$ be an open Stein covering of M and $f_i \in \Gamma(U_i, \mathcal{O}_M)$ a local defining equation of S. On $U_i \cap U_j$, there is a non-vanishing function $\varphi_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_M^*)$ such that

$$f_i = \varphi_{ij} f_j.$$

Thus $\{f_i\}$ defines a global section of the invertible sheaf $\mathcal{O}_M(S)$ associated with the transition functions $\{\varphi_{ij}\}$.

(4.3) Proof of (4.2) for $n = 2$. Take an everywhere non-vanishing section $s = \{s_i\} \in H^0(S, \mathcal{O}_S(S))$, where

$$s_i \in \Gamma(U_i \cap S, \mathcal{O}_S^*), \quad s_i = \varphi_{ij} s_j.$$

Let $\tilde{s}_i \in \Gamma(U_i, \mathcal{O}_M)$ be a local lifting of s_i and $\tilde{S}_i$ the divisor on Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2) \times U_i$ defined by

$$f_i - \varepsilon \tilde{s}_i = 0.$$

Then we have $\tilde{S}_i = \tilde{S}_j$ on Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2) \times (U_i \cap U_j)$. Indeed,

$$\begin{aligned} \mathcal{I}_{\tilde{S}_j} &= (f_j - \varepsilon \tilde{s}_j)\mathcal{O}_M[\varepsilon] = \varphi_{ij}(f_j - \varepsilon \tilde{s}_j)\mathcal{O}_M[\varepsilon] \\ &= (f_i - \varepsilon \varphi_{ij} \tilde{s}_j)\mathcal{O}_M[\varepsilon] = \{(f_i - \varepsilon \tilde{s}_i) + \varepsilon(\tilde{s}_i - \varphi_{ij}\tilde{s}_j)\}\mathcal{O}_M[\varepsilon] \\ &\subset \mathcal{I}_{\tilde{S}_i} + \varepsilon(\tilde{s}_i - \varphi_{ij}\tilde{s}_j)\mathcal{O}_M. \end{aligned}$$

On the other hand, since $\{\tilde{s}_i\}$ is a lift of $\{s_i\}$,

$$\tilde{s}_i - \varphi_{ij}\tilde{s}_j \in \mathcal{I}_S = f_i \mathcal{O}_M,$$

so that

$$\begin{aligned}\mathscr{I}_{\tilde{S}_j} &\subset \mathscr{I}_{\tilde{S}_i} + \varepsilon f_i \mathcal{O}_M \\ &= \mathscr{I}_{\tilde{S}_i} + \varepsilon(f_i + \varepsilon \tilde{s}_i)\mathcal{O}_M \\ &= \mathscr{I}_{\tilde{S}_i}\end{aligned}$$

thanks to $\varepsilon^2 = 0$. By the symmetry between i and j, we have $\mathscr{I}_{\tilde{S}_i} = \mathscr{I}_{\tilde{S}_j}$ on Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2) \times (U_i \cap U_j)$. Thus $\{\tilde{S}_i\}$ defines an effective divisor $\tilde{S}$ on Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2) \times M$. There are natural projections p: $\tilde{S} \to$ Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ and q: $\tilde{S} \to M$. The ring homomorphism

$$q^{-1}: \mathcal{O}_M \to \mathcal{O}_{\tilde{S}}$$

is surjective. In fact, noting $\tilde{s}_i \in \mathcal{O}_M^*$, we have $\varepsilon = f_i \tilde{s}_i^{-1}$. Thus q is a closed immersion. In the mean time

$$\begin{aligned}\ker q^{-1} &= \mathcal{O}_M \cap \{(f_i - \varepsilon \tilde{s}_i)(\mathcal{O}_M \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2))\} \\ &= f_i^2 \mathcal{O}_M = \mathscr{I}_{2S},\end{aligned}$$

so that q gives an isomorphism $\tilde{S} \cong 2S$. On the other hand, since $\varepsilon \mathcal{O}_{\tilde{S}} = f_i \tilde{s}_i^{-1} \mathcal{O}_{\tilde{S}} = f_i \mathcal{O}_{\tilde{S}} \neq 0$, $\tilde{S}$ is flat over Spec $\mathbb{C}[\varepsilon]/(\varepsilon^2)$, with central fibre S. Hence there exists a natural morphism

$$\phi_2: \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2) \to (\mathscr{Y}, 0)$$

such that

$$(4.2.2)_2 \quad 2S \cong \tilde{S} \cong \text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2) \times_{\mathscr{Y}} \mathscr{X}.$$

In particular, it gives isomorphisms of dualizing sheaves:

$$\begin{aligned}\omega_{2S} \cong \omega_{\tilde{S}} &\cong p^* \omega_{\text{Spec } \mathbb{C}[\varepsilon]/(\varepsilon^2)} \otimes \phi_2^* \omega_{\mathscr{X}/\mathscr{Y}} \\ &\cong \mathcal{O}_{\tilde{S}} \underset{\mathcal{O}_{\tilde{S}}}{\otimes} \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{\tilde{S}} \cong \mathcal{O}_{2S};\end{aligned}$$

while the adjunction formula shows

$$\omega_{2S} \cong \mathcal{O}_{2S}(K_M + 2S)$$

whence follows

$$\mathcal{O}_{2S} \cong \omega_{2S}^{\otimes a} \cong \mathcal{O}_{2S}(aK_M + 2aS) \cong \mathcal{O}_{2S}((2a + b)S).$$

Since $b \neq -2a$, this implies that $\mathcal{O}_{2S}(S)$ is torsion in Pic$(2S)$. Now, by (4.1) and (4.2.a) we conclude:

$$(4.2.1)_2 \quad \mathcal{O}_{2S}(S) \cong \mathcal{O}_{2S}.$$

$(4.2.3)_2$ is easy. In fact, a non-vanishing section of $\mathcal{O}_{2S}(mS) \cong \mathcal{O}_{2S}$ gives a $\mathbb{C}$-basis of $H^0(S, \mathcal{O}_S(mS)) \cong \mathbb{C}$.

(4.4) Proof of (4.2) for $n \geqq 3$. Suppose that $(4.2.2)_{n-1}$, $(4.2.2)_{n-1}$ and $(4.2.3)_{n-1}$ hold ($n \geqq 3$). By $(4.2.2)_{n-1}$, we can identify $\mathcal{O}_{(n-1)S}$ with the flat $\mathbb{C}[\varepsilon]/(\varepsilon^{n-1})$-algebra

$$\mathbb{C}[\varepsilon]/(\varepsilon^{n-1}) \underset{\mathcal{O}_{\mathscr{Y}}}{\otimes} \mathcal{O}_{\mathscr{X}}$$

via ϕ_{n-1}. Note that $\varepsilon\mathcal{O}_{(n-1)S} = f_i\mathcal{O}_{(n-1)S} \subset \mathcal{O}_{(n-1)S}$ on $U_i \cap (n-1)S$:

$$\varepsilon \equiv f_i\alpha_i \mod f_i^{n-1}\mathcal{O}_M,$$

where $\alpha_i \in \Gamma(U_i, \mathcal{O}_M^*)$. Then

$$f_i(\alpha_i - \varphi_{ij}^{-1}\alpha_j) = f_i\alpha_i - f_j\alpha_j \equiv \varepsilon - \varepsilon = 0 \mod f_i^{n-1}\mathcal{O}_M;$$

or, equivalently

$$\alpha_i \equiv \varphi_{ij}^{-1}\alpha_j \mod f_i^{n-2}\mathcal{O}_M$$

so that $\{\alpha_i\}$ gives rise to a global section $\alpha \in H^0((n-2)S, \mathcal{O}(-S))$. (We need here the hypothesis $n \geqq 3$). By $(4.2.3)_{n-1}$, α can be lifted to $\tilde{\alpha} \in H^0((n-1)S, \mathcal{O}(-S))$. $\tilde{\alpha}$ is represented by $\tilde{\alpha}_i \in \Gamma(U_i, \mathcal{O}_M)$ such that

$$\tilde{\alpha}_i \equiv \varphi_{ij}^{-1}\tilde{\alpha}_j \mod f_i^{n-1}\mathcal{O}_M.$$

We define a $\mathbb{C}[\varepsilon]/(\varepsilon^n)$-algebra structure on $\mathcal{O}_{ns}$ by the formula

$$\varepsilon g = (f_i\tilde{\alpha}_i)g \quad \text{for} \quad g \in \mathcal{O}_{nS}.$$

This is well-defined because

$$\begin{aligned} f_i\tilde{\alpha}_i - f_j\tilde{\alpha}_j &= (\varphi_{ij}f_j)(\varphi_{ij}^{-1}\tilde{\alpha}_j + \delta_{ij}) - f_j\tilde{\alpha}_j \\ &= \varphi_{ij}f_j\delta_{ij} \in f_j^n\mathcal{O}_M, \end{aligned}$$

where $\delta_{ij} = \tilde{\alpha}_i - \varphi_{ij}^{-1}\tilde{\alpha}_j \in f_j^{n-1}\mathcal{O}_M$. This extends the $\mathbb{C}[\varepsilon]/(\varepsilon^n)$-algebra structure on $\mathcal{O}_{(n-1)S}$ to $\mathcal{O}_{nS}$. Moreover $\mathcal{O}_{nS}$ is flat over $\mathbb{C}[\varepsilon]/(\varepsilon^n)$ by

$$\varepsilon^{n-1}\mathcal{O}_{nS} = (\tilde{\alpha}_i f_i)^{n-1}\mathcal{O}_{nS} = f_i^{n-1}\mathcal{O}_{nS} \neq 0;$$

in other words, we have a proper flat morphism

$$nS \to \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^n)$$

whence derives a morphism

$$\phi_n\colon \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^n) \to (\mathcal{Y}, 0),$$

which extends ϕ_{n-1} and induces an isomorphism

$$(4.2.2)_n \quad nS \cong \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^n) \times_{\mathcal{Y}} \mathcal{X}.$$

Therefore, similarly as in (4.3),

$$\begin{aligned} \omega_{nS} &\cong \mathcal{O}_{nS} && \text{by (4.2.c)},\\ \omega_{nS}^{\otimes a} &\cong \mathcal{O}_{nS}(aK_M + anS) && \text{by adjunction}\\ &\cong \mathcal{O}_{nS}(bS + anS) && \text{by (4.2.b)}. \end{aligned}$$

Since $b \neq -an$, $\mathcal{O}_{nS}(S)$ is a torsion so that

$$(4.2.1)_n \quad \mathcal{O}_{nS}(S) \cong \mathcal{O}_{nS} \quad \text{by (4.1)}.$$

Finally $(4.2.3)_n$ is immediate from $(4.2.1)_n$ and $(4.2.2)_n$. □

(4.5) Corollary. *Under the same assumption as in* (4.2), *we have*

$$\dim H^0(nS, \mathcal{O}_{nS}(kS)) = n$$

for $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

(4.6) Corollary. *Let M, N and U be three analytic spaces and $f\colon N \to M$, $g\colon N \to U$ proper, surjective, generically finite morphisms. Assume that there are compact, effective Cartier divisors $S \subset M$, $T \subset N$ and $D \subset U$ such that $f^*S = T$, $g^*D = kT$ ($k \in \mathbb{N}$). If (M, S) satisfies the hypotheses in* (4.2), *then*

$\dim H^0(nD, \mathcal{O}_{nD}(nD))$ *grows like* n.

Applying this corollary to the original situation, we get

(4.7) COROLLARY. *Let X be a minimal 3-fold with $\nu = 1$. Let D_i be a connected component of $D \in |mK_X|$, $m > 0$, $\mathrm{ind}(X)|m$. Then*

$$\dim \mathrm{H}^0(nD_i, \mathcal{O}_{nD_i}(nD_i)) = O(n).$$

(4.8) *Proof of Main Theorem.* Consider the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(nD) \to \mathcal{O}_{nD}(nD) \to 0$$

and the associated cohomology exact sequence

$$0 \to \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(X, \mathcal{O}_X(nD)) \to \mathrm{H}^0(nD, \mathcal{O}_{nD}(nD)) \to \mathrm{H}^1(X, \mathcal{O}_X).$$

The first and the last terms are independent of n and their dimensions are bounded, so $h^0(nD, \mathcal{O}(nD)) = \Sigma_i h^0(nD_i, \mathcal{O}(nD_i)) \sim O(n)$ implies $h^0(X, \mathcal{O}_X(nD)) \sim O(n)$, i.e. $\kappa(X) = 1$. Similarly, $h^0(X, \mathcal{O}_X(nD_i)) \sim O(n)$. D_i is a multiple of a primitive divisor E_i: $D_i = e_i E_i$. Noting that $D_i|E_i \approxeq 0$, we see that the moving part $|L_i^{(n)}|$ of $|nD_i|$ has no base points and of the form $|n_i' E_i|$, $n_i' > 0$. Hence $|n_i' D_i| = |e_i L_i^{(n)}|$ is base point free; therefore, for $n_0 = \mathrm{L.C.M.}\ \{n_i'\}$, $|n_0 D| = |n_0 m K_X|$ is also base point free. □

(4.9) REMARK. In the assumption in (4.2), the strange condition $b \neq -2a, -3a, \ldots$ is actually necessary. For instance, let A be an abelian variety and consider an non-trivial extension

$$0 \to \mathcal{O}_A \to \mathcal{E} \to \mathcal{O}_A \to 0.$$

Let $M = \mathbb{P}(\mathcal{E})$. $\mathbb{P}(\mathcal{E})$ contains a unique section $S \cong A$. (M, S) satisfies all the hypotheses in (4.2) except that $K_M \sim -2S$. Moreover, $(4.2.2)_2$ holds, too. However, $\mathcal{O}_{2S}(S)$ is not isomorphic to $\mathcal{O}_{2S}$. In fact, since $S \sim \mathbb{1}_{\mathcal{E}}$, the tautological line bundle, we have an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-\mathbb{1}_{\mathcal{E}}) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mathbb{1}_{\mathcal{E}}) \to \mathcal{O}_{2S}(S) \to 0$$

so that $\mathrm{H}^0(2S, \mathcal{O}_{2S}(S)) \cong \mathrm{H}^0(\mathbb{P}(\mathcal{E}), \mathcal{O}(\mathbb{1}_{\mathcal{E}})) \cong \mathrm{H}^0(A, \mathcal{E}) \cong \mathbb{C}$, while $\mathrm{H}^0(2S, \mathcal{O}_{2S}) \cong \mathbb{C}^2$. It is therefore impossible to extend the $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-algebra structure on $\mathcal{O}_{2S}$ to a $\mathbb{C}[\varepsilon]/(\varepsilon^3)$-algebra structure on $\mathcal{O}_{3S}$, i.e. the connected component of $\mathrm{Chow}(M)$ that contains $\{S\}$ is a non-reduced point $\cong \mathrm{Spec}\ \mathbb{C}[\varepsilon]/(\varepsilon^2)$.

(4.10) REMARK. Applying our argument to the minimal surface case, we can prove without complicated dichotomy that $\nu(X) = 1$ implies the existence of an elliptic fibration.

Acknowledgements

This paper was motivated by M. Reid's suggestion that the analysis of $D \in |mK_X|$ eventually leads to the proof of (∗∗); in this sense the present approach owes much to him. The idea was conceived at University of Pisa and worked out at Columbia University and Max-Planck-Institut at Bonn. I am grateful to the three institutions for their hospitality and finantial support. Finally I appreciate the helpful and encouraging conversions with S. Mori and D. Morrison during the preparation of the paper.

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