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A. FAUNTLEROY

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Invariant theory for linear algebraic groups II (char k arbitrary)

A. FAUNTLEROY
Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205, USA

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Given a linear algebraic group G and an action of it on a quasi-projective variety X, all defined over an algebraically closed field k, there will in general be no quasi-projective orbit space. When the group G is reductive, Mumford in GIT gave a reasonable criteria for the existence of a quasi-projective quotient using his notion of stable points. In order to generalize his concepts to arbitrary linear groups it is necessary to treat the case of unipotent group actions. If H is the unipotent radical of G, then one first must construct $Y = X/H$ and assuming Y is well behaved, apply the technique of Mumford to the action of the reductive group G/H on Y.

There are two technical problems involved in this program. The first is to find reasonable conditions which guarantee that Y exists and is quasi-projective. The second is to insure that the pair $(Y, G/H)$ satisfies the hypothesis required in order to apply the methods of GIT.

The results of [1] give a method for handling these problems when the ground field k has characteristic zero. The purpose of this note is to extend the key results on unipotent actions given in [1; Section 1] to the case of arbitrary characteristics. It is then a straightforward matter to extend to arbitrary algebraic groups G over arbitrary fields k the notions of stability given in [1].

Let X be a quasi-factorial variety over k, i.e., $B = \Gamma(X, O_X)$ is a unique factorization domain and the canonical map: $X \to \operatorname{Spec} B$ is an open immersion. Let H be a connected unipotent algebraic group defined over k. We assume throughout that H acts on X and that the isotropy group in H of each point of X is finite. We recall here some definitions from [1].

(1) A point $x \in X$ is *semi-stable* if $\dim c^{-1}(cx) = \dim H$ where c: $X \to \operatorname{Spec} A$, $A = B^H$, is the natural map. If X^{ss} denotes the set of semi-stable points of X then X^{ss} is open and H-stable (c.f. [1]). Moreover there exists a quasi-factorial variety Q and an H-equivariant map π: $X \to Q$ making Q an s-categorical quotient of X by H. This means that for any morphism f:

$X \to Y$, Y a *separated* algebraic scheme, with f constant on H-orbits, there is a unique map g: $Q \to Y$ with $f = g\pi$. Further $Q = c(X^{ss})$ is open in Spec A.

(2) A point $x \in X$ is *stable* if there is an open neighborhood U of x with $HU = U$ and such that U/H exists and is affine. We denote by X^s the open set of stable points. It is evidently invariant under the action of H and a geometric quotient X^s/H exists as an algebraic scheme. A point x is *properly stable* if it is stable and there exists an open H-invariant neighborhood V of x with $V \subseteq X^s$ such that the action of H on V is proper. The set of properly stable points X^{ps} is evidently open and H-stable.

PROPOSITION 1.1. *Let $U \subseteq X^s$ be an open H stable subset and suppose $Y = U/H$. If the quotient morphism $U \to Y$ is affine then $U \subseteq X^{ss}$. If further, Y is separated then the natural map $Y \to Q$ is an open immersion so Y is quasi-factorial.*

Proof. First assume that Y and U are affine. Let $A = B^H$. Then A is factorial and $k[U] = B[a^{-1}]$, $k[Y] = A[a^{-1}]$ for some $a \in A$. The triangle below clearly commutes

$$\begin{array}{ccc} U & \xrightarrow{c} & \text{Spec } A \\ & {\scriptstyle q}\searrow \quad \nearrow & \\ & Y & \end{array}$$

The non empty fibres of $c|_U$ are of dimension $l = \dim H$. If $x \in U \subset X$ then $c(x) = q(x) = y$ so $a(y) \neq 0$ and hence $a(x) \neq 0$. Thus each point of the fibre $c^{-1}(c(x))$ lies in U and it follows that the dimension of each fiber is l so $U \subset X^{ss}$. Now in the general case U is covered by H-stable open affine subsets with affine quotients so $U \subseteq X^{ss}$. If Y is separated, then $Y \to Q$ is a birational quasi-finite map, hence an open immersion by Zariski's Main Theorem.

In [1] it was shown that when char $k = 0$, X^{ps} is the set of points in X for which the action of H is locally trivial and that $H \times X^{ps} \to X^{ps} \times X^{ps}$, $(h, x) \to (hx, x)$ is proper so the morphism $X^{ps} \to Y = X^{ps}/H$ is affine and Y is separated hence quasi-factorial. The main purpose of this note is to give the appropriate generalization of this result in arbitrary characteristics. Since H contains a normal series with successive quotients isomorphic to the additive group G_a, one would expect the answer to lie in G_a-actions. This is indeed the case. A first guess might be to replace *locally trivial by* locally trivial in the finite radical topology. However, the example 3 of [2] gives a counterexample to this conjecture.

It is important to note here that without the hypothesis that X be quasi-factorial, the action of H on X^{ps} need not be proper! (See Example 2, p. 727 in [2].)

A point $x \in X$ will be called *finitely-stable* or *f-stable* if there exists an open affine neighborhood V of x invariant under the action of H and an H-equivariant finite morphism $H \times S \to V$ for some affine variety S. Let X^{fs} denote the set of finitely stable points of X. In the definition we may assume without loss of generality that S is normal.

LEMMA 1.2. *X^{fs} is contained in X^s.*

Proof. It suffices to show that if $H \times S \to V$ is a finite surjective H-morphism with V normal then V/H exists and is affine. By [3; p. 539] we can find a Seshadri cover $Z \to V$ of V with respect to H such that $k(Z)$ is the normal closure of $k(H \times S)$ in an algebraic closure of $k(V)$. It follows that Z is the normalization of $H \times S$ in $k(Z)$ so in particular is affine. Moreover, $Z \to H \times S$ is a Seshadri cover of $H \times S$. The action of $H \times S$ is easily seen to be proper so the action of H on Z is proper. Then $W = Z/H \to S$ is finite so W is affine. By Theorem 7.1 of [3] V/H exists and is affine.

LEMMA 1.3. *Let $Z \to X$ be a Seshadri cover of X. If the action of H on Z is proper, then the action of H on X is proper.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H \times Z & \longrightarrow & Z \times Z \\ \downarrow & & \downarrow \\ H \times X & \xrightarrow{\Phi} & X \times X \end{array}$$

The vertical and upper horizontal maps are finite hence Φ is finite hence proper.

The following lemma is the key to our description of X^{ps}. It describes the situation locally when $H = G_a$.

LEMMA 1.4. *Let V be a factorial affine variety on which G_a acts. Let R denote the coordinate ring of V. Then the following conditions are equivalent:*

(1) *There exists a variety S and a finite surjective G_a-equivariant morphism $p: G_a \times S \to V$*

(2) *There is an element $g \in R$ such that $\tilde{\sigma}(g)$ is monic in $R(\lambda) = k[G_a \times V]$ where $\tilde{\sigma}$ is the comorphism for the action of G_a on V.*

Proof. Suppose first that (1) holds. Let G_a act diagonally on $G_a \times V$. Then $1 \times p: G_a \times S \to G_a \times V$ is finite and G_a equivariant. Let W be the image of $1 \times p$. Then W is a G_a-stable subvariety of $G_a \times V$ of codimension one. Thus W is defined by a single irreducible invariant polynomial $F(T)$ in $R[T]$. The composition of $1 \times p$ with the second projection $G_a \times V \to V$ is the original morphism p. Hence the restriction of the second projection to W is a finite morphism. It follows that $F(T)$ can be taken monic in T. Write $F(T) = a_0 + a_1 T + \cdots + T^n$ with $a_i \in R$.

Let $\hat{\sigma}: R[T] \to R[T][\lambda] = R[T, \lambda]$ denote the comorphism for the action of G_a on $G_a \times V$. If $\Sigma\, b_i T^i \in R[T]$. Then $\sigma(\Sigma\, b_i T^i) = \Sigma\, \tilde{\sigma}(b_i)(T + \lambda)^i$. Using the fact that $\hat{\sigma}(F(T)) = F(T)$ we find

$$(T + \lambda)^n + \hat{\sigma}(a_{n-1})(T + \lambda)^{n-1} + \cdots + \hat{\sigma}(a_0)$$
$$= T^n + a_{n-1} + \cdots + a_1 T + a_0$$

Taking for T the value $-\lambda$ we see that $\hat{\sigma}(a_0) = (-\lambda)^n + a_{n-1}(-\lambda)^{n-1} + \cdots + a_0$. Thus $g = (-1)^n a_0$ satisfies $\hat{\sigma}(g)$ is monic in λ and (2) holds.

Conversely suppose $g \in R$ with $\hat{\sigma}(g) = g + g_1 \lambda + \cdots + g_{n-1}\lambda^{n-1} + \lambda^n$. Let W be the closed subset of $G_a \times V$ defined by

$$G(T) = g + g_1(-T) + \cdots + g_{n-1}(-T)^{n-1} + (-T)^n = 0.$$

Note that if $(\mu, p) \in W$ then

$$g(-\mu \cdot p) = g(p) + g_1(p)(-\mu) + \cdots + g_{n-1}(p)(-\lambda)^{n-1} + (-\mu)^n$$
$$= G(T)(\mu, p) = 0.$$

Conversely if $g(-\mu \cdot p) = 0$ then $(\mu, p) \in W$. Now let G_a act diagonally on $G_a \times V$ then if $(\mu, p) \in W$ and $\lambda \in G_a$ we have

$$G(T)(\lambda \cdot (u, p)) = G(T)(\lambda + u, \lambda \cdot p)$$
$$= g((-\lambda - u) \cdot (\lambda p))$$
$$= g(-\mu \cdot p) = 0.$$

Thus W is G_a-stable. The mapping $W \to V$ obtained by restricting the second projection $G \times V \to V$ to W is finite since $G(T)$ is monic. Replacing W by a suitable irreducible component if necessary, we obtain a G_a-stable closed subvariety W of $G_a \times V$ such that the mapping $W \to V$ is finite and G_a-stable closed subvariety W of $G_a \times V$ such that the mapping $W \to V$ is finite and G_a-equivariant. Finally, since $G_a \times V$ is trivial as a G_a-space so also is any G_a-stable subvariety so that $W \simeq G_a \times S$ for some variety S. This gives the desired implication (2) implies (1) and completes the proof of the lemma.

THEOREM 1.5. *Let X be a quasi-factorial variety on which the connected unipotent group H acts. Then H acts properly on $X^{fs}(H)$. In particular, $Y = X^{fs}(H)/H$ is quasi-factorial and q: $X^{fs}(H) \to Y$ is an affine morphism.*

Proof. We argue by induction on dim H. Assume the result holds for connected subgroups $N \subseteq H$ with $0 < \dim N < \dim H$ and let N be such a subgroup which is normal in H. Recall, [1; Sec. 3] that $H \simeq N \times (H/N)$ as an N-space. It follows that $X^{fs}(H) \cong X^{fs}(N)$ and by the inductive assumption H/N acts properly on $Y_N^{fs}(H/N)$ where $Y_N = X^{fs}(N)/N$.

Let Z be a Seshadri cover of $X^{fs}(H)$. We have a commutative diagram:

$$\begin{array}{ccc} Z & \longrightarrow & X^{fs}(H) \\ \downarrow & & \downarrow \\ W_1 = Z/N & \longrightarrow & Y_N^{fs}(H/N) \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

where W and Y are quotients under the action of H/N. Since H/N acts properly on $Y_N^{fs}(H/N)$ it also acts properly on W_1. Thus W is quasi-affine. But $W = Z/H$ and Z is locally trivial. By [1, 1.9] H acts properly on Z. By Lemma 1.3, H acts properly on $X^{fs}(H)$.

To complete the proof we need only establish the result in the case $H = G_a$. By Lemma 1.4 we can find an affine open cover $\{X_\alpha\}$ of $X^{fs}(H)$ consisting of H-stable open affines and an element $g_\alpha = R_\alpha = k[X_\alpha]$ with $\tilde{\sigma}(g_\alpha)$ monic in $R_\alpha[\lambda]$. The map Φ will be proper if it's finite. We consider the cover $\{X_\alpha \times X_\beta\}$ of $X^{fs}(H) \times X^{fs}(H)$. Then $\Phi^{-1}(X_\alpha \times X_\beta) = H \times X_\alpha \cap X_\beta$ so Φ is affine. Let $B = \Gamma(X^{fs}(H), O_X)$ so that $R_\alpha = B[f_\alpha^{-1}]$ with $f_\alpha \in A = B^H$. Then $k[X_\alpha \cap X_\beta] = B[f_\alpha^{-1} \cdot f_\beta^{-1}]$. I claim the map

$$B[f_\alpha^{-1}] \otimes B[f_\beta^{-1}] \xrightarrow{1 \otimes \tilde{\sigma}} B[f_\alpha^{-1} \cdot f_\beta^{-1}][\lambda]$$

is finite. If $b \in B[f_\alpha^{-1} \cdot f_\beta^{-1}]$ and $b = s/f_\alpha^n f_\beta^m$ then $b = (1 \otimes \tilde{\sigma})(s/f_\alpha^n \otimes 1/f_\beta^m)$ so $B[f_\alpha^{-1} f_\beta^{-1}]$ is in the image of $1 \otimes \tilde{\sigma}$. Since $(1 \otimes \tilde{\sigma})(1 \otimes g_\beta) = \tilde{\sigma}(g_\beta)$ is monic in λ the ring $B[f_\alpha^{-1} \cdot f_\beta^{-1}][\lambda]$ is integral over the image of $1 \otimes \tilde{\sigma}$. It follows that Φ is finite and the theorem is proved.

COROLLARY 1.6. *$X^{fs}(H)$ contains every H-stable open subset of X on which H acts properly stably. In particular $X^{fs}(H) = X^{ps}(H)$.*

Proof. Let $U \subseteq X$ be H-stable open and assume H acts properly stably on U. It follows that we can replace U by an affine open subset and assume $Y = U/H$ is affine. If Z is a Seshadri cover of U then Z and W are affine and hence $Z \simeq H \times W$. Since $Z \to U$ is finite, $U \subset X^{fs}(H)$. The theorem asserts that the action of H on X^{fs} is properly stable hence $X^{fs}(H) \subset X^{ps}(H)$ and equality follows.

The extension of the results of [1] to arbitrary characteristics depends on the invariance under G of the properly stable points of $R_u G$ for actions of arbitrary connected algebraic groups G on quasi-factorial varieties. The following lemma is a key technical tool for this.

LEMMA 1.7 *Let G be a linear algebraic group, N a closed normal subgroup of G and X a quasi-factorial variety on which G acts. If $U \subseteq X$ is an N-stable open subset on which N acts properly then N acts properly on gU for all g in G.*

Proof. $NgU = gNU = gU$ so gU is N-stable. Now $\Phi: N \times U \to U \times U$ is proper so finite. Let $ad(g)$ denote conjugation by g in G so $ad(g)(n) = gng^{-1}$ and denote by λ_g left multiplication by g. The following diagram is commutative with vertical arrows representing isomorphisms.

$$\begin{array}{ccc} N \times U & \xrightarrow{\ \Phi\ } & U \times U \\ \Big\downarrow{\scriptstyle ad(g) \times \lambda_g} & & \Big\downarrow{\scriptstyle \lambda_g \times \lambda_g} \\ N \times gU & \xrightarrow[\ \Phi_g\ ]{} & gU \times gU \end{array}$$

Thus Φ_g is finite hence proper.

Note that if G is unipotent and $U \subset X^s$ (for the action of N) then $gU \subset X^s$ for all $g \in G$. For a proof see [1; Proposition 2.4].

THEOREM 1.7. *Let N be a closed connected normal subgroup of the unipotent group G and X a quasi-factorial variety on which G acts. Let $X_0 = X^{ps}(N)$, $Y_0 = X_0/N$ and q: $X_0 \to Y_0$ the quotient map. Then X_0 is G stable and $X^{ps}(G) = q^{-1}(Y_0^{ps}(G/N)$.*

Proof. The lemma and the preceding note imply $GX_0 = X_0$. We saw in the proof of Theorem 1.5 that $X^{ps}(G) \subseteq X_0$ and it is evidently N-stable. Its image Y_1 in Y_0 is thus open, G/N stable and easily seen to be contained in $Y_0^{ps}(G/N)$ (cf. [1; 2.4]). But if $X_1 = q^{-1}(Y_0^{ps}(G(N))$, then X_1 is G-stable and clearly $Y_0^{ps}(G/n)/(G/n) \simeq X_1/G$. It remains only to show that G acts properly on X_1. This can be seen as follows:

Let $T = X_1/G = Y^{ps}(G/N)/(G/N)$. Then $X_1 \to Y^{ps}(G/N)$ and $Y^{ps}(G/N) \to T$ are affine maps because $X_1 \subseteq X^{ps}(N)$ and N act properly on X_1 and G/N acts properly on $Y^{ps}(G/N)$. Let $T_2 \subset T$ be affine, $Y_2 = \alpha^{-1}(T_2)$ where α: $Y^{ps}(G/N) \to T$ is the quotient map and finally let X_2 be the inverse image of Y_2 in X_1. If Z is a Seshadri cover of X_2 and $W = Z/G$ is its quotient then we have a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S & \dashrightarrow & Y_2 \\ \downarrow & & \downarrow \\ W & \longrightarrow & T_2 \end{array}$$

Since $Z \to X_2$ is finite and N acts properly on X_2 it acts properly on Z. But Z is locally trivial for the action of G hence also N so $S = Z/N$ exists and is separated. But then the canonical map $S \to Y_2$ is finite so S is actually affine. Since $W = S/(G/N)$ is separated and $W \to T_2$ is finite, W is affine. But $Z \to W$ being locally trivial gives $Z \simeq G \times W$. Thus $X_2 \subset X^{fs}(G) \subset X^{ps}(G)$. Since X_1 can be covered by such open affines it follows that $X_1 \subseteq X^{ps}(G)$ and hence $X^{ps}(G) = q^{-1}(Y_0^{ps}(G/N))$.

Corollary 1.8. *Let X be a quasi-factorial variety on which the connected unipotent group G acts. If all stability groups for the action of G are finite then $X^{ps}(G)$ is non-empty.*

Proof. It clearly suffices to establish the result when $G = G_a$. If $f \in \Gamma(X, O_X)$ is a nonconstant non-invariant function then $\sigma(f) = f + f_1 T + \cdots + f_k T^k$ with f_k invariant. Let $X_0 = X_{f_k}$. Lemma 2.4 implies $X_0 \subseteq X^{fs}(G_a) = X^{ps}(G_a)$.

Remark. The results of [1] contained in Sections 3 and 4 now follow essentially from the arguments given there without any assumption on the characteristic of the ground field.

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