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A Torelli theorem for osculating cones to the theta divisor

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Let C be a smooth complete curve of genus $g \geqslant 5$ over an algebraically closed field k of characteristic $\neq 2$. Let Θ be the theta divisor on the Jacobian J. Let x be a double point of Θ. Then we may expand a local equation $\theta = 0$ of Θ near x as

$$\theta = \theta_2 + \theta_3 + \text{higher order terms}$$

where θ_i is homogenous of degree i in the canonical flat structure [K3] on J. It is well-known that the tangent cone $R = \{\theta_2 = 0\}$ is a quadric of rank $\leqslant 4$ in the canonical space $\mathbb{P}^{g-1}$ which contains the canonical curve.

If C is not hyperelliptic, trigonal or a plane quintic then for a general double point x the quadric θ_2 has rank 4 and the two rulings of R cut out on C a pair of residual base-point-free distinct g^1_{g-1}'s, $|D|$ and $|K - D|$, (apply Mumford's refinement of Martens' theorem [M]. In the bi-elliptic case x should be a general point of the component of the singular locus of Θ which does not arise from g^1_4's by adding base points or its residual component). Moreover the morphism $\varphi_{|D|} \times \varphi_{|K-D|}: C \to \mathbb{P}^1 \times \mathbb{P}^1$ maps C birationally onto its image C'. Taking the composition with the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ we may regard $\varphi_{|D|} \times \varphi_{|K-D|}$ as the projection of C from the $(g - 5)$-dimensional vertex $V \subseteq \mathbb{P}^{g-1}$ of R.

In this paper we study the geometry of the osculating cone $S = \{\theta_2 = \theta_3 = 0\}$. It is well-known and easy to prove that $C \subseteq S$. Thus if $\sim$ denote the strict transform after blowing up V we have a diagram

$$\begin{array}{ccccccc}
C & \hookrightarrow & \tilde{S} & \hookrightarrow & \tilde{R} & \hookrightarrow & \tilde{\mathbb{P}}^{g-1} \to \mathbb{P}^{g-1} \\
\downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\pi} & & \downarrow \\
C' & \subseteq & \mathbb{P}^1 \times \mathbb{P}^1 & = & \mathbb{P}^1 \times \mathbb{P}^1 & \subseteq & \mathbb{P}^3
\end{array}$$

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where the vertical maps are induced by projecting from V. An important but simple observation is that S contains V and hence that α is a quadric bundle contained in the $\mathbb{P}^{g-4}$ bundle π. We will prove

THEOREM 1. *Suppose that $x \in \Theta \subseteq J$ corresponds to a pair of base-point free residual g^1_{g-1}'s, $|D|$ and $|K - D|$, such that $\varphi_D \times \varphi_{K-D}: C \to C' \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is birational onto its image. Then the fibers of α over $\mathbb{P}^1 \times \mathbb{P}^1 - C'$ are smooth and for a smooth point c' of C' the corresponding point c of C is the only singular point of the fiber of α over c.*

Thus we have a straightforward way to recover the canonical curve from $S \subseteq R \subseteq \mathbb{P}^{g-1}$. Explicitely C is the component of the singular locus of the fibers which projects non-trivially into $\mathbb{P}^1 \times \mathbb{P}^1$.

If char $(k) \neq 2, 3$ then $\{\theta_2 = \theta_3 = \theta_4 = 0\}$ is defined. We note

PROPOSITION 2. *C is not contained in $\{\theta_2 = \theta_3 = \theta_4 = 0\}$.*

This work was done during a visit of UNAM in Mexico. We thank Sevin Recillas for his hospitality and for bringing this question to our attention.

§1. Infinitesimal calculations on the Jacobian via cohomological obstructions

Intrinsically x corresponds to a point of the $g - 1^{\text{st}}$ Picard variety $\mathrm{Pic}_{g-1}(C) \cong J$. The point is the isomorphism class of the invertible sheaf $\mathscr{L} = \mathscr{O}_C(D)$. In our case $\Gamma(C, \mathscr{L})$ is two-dimensional. By the general procedure of [K2] we may locally around x find a 2×2 matrix (f_{ij}) of regular functions vanishing at x such that

$$\theta = \det (f_{ij}).$$

The equation of the tangent cone R is

$$\theta_2 = \det (df_{ij}|_x) = 0.$$

The equations of the vertex V are $df_{ij}|x = 0$ for $1 \leqslant i, j \leqslant 2$.

Using the flat structure on J we may expand

$$f_{ij} = x_{ij} + q_{ij} + \text{higher order terms}$$

where $x_{ij} = df_{ij}|_x$ and q_{ij} are thought of as linear and quadratic functions on the tangent space J at x. Expanding θ as a determinant we have

$$\theta_2 = x_{11}x_{22} - x_{12}x_{21} \text{ and}$$

$$\theta_3 = x_{11}q_{22} + q_{11}x_{22} - x_{12}q_{21} - q_{12}x_{21}.$$

Thus we get:

PROPOSITION 3. *The vertex V is contained in $S = \{\theta_2 = \theta_3 = 0\}$.*

Proof. $V = \{x_{11} = x_{22} = x_{12} = x_{21} = 0\}$. So θ_3 vanishes on V. □

To get deeper results we will have to use the obstruction theory from [K3]. First we will give a cohomological interpretation of the previous material.

The tangent space to J is canonically isomorphic to $H^1(C, \mathcal{O}_C)$. The matrix (x_{ij}) describes the cup product action

$$\cup: H^1(C, \mathcal{O}_C) \to \text{Hom}\,(\Gamma(C, \mathscr{L}), H^1(C, \mathscr{L})),\ \alpha \mapsto \cup\alpha$$

where $H^1(C, \mathscr{L}) \cong \Gamma(C, \Omega_C \otimes \mathscr{L}^{-1})^*$ is also two dimensional. The cone over V is the kernel of $\cup$ and the cone over R corresponds to cohomology classes $\alpha \in H^1(C, \mathcal{O}_C)$ such that $\cup\alpha: \Gamma(C, \mathscr{L}) \to H^1(C, \mathscr{L})$ has rank $\leqslant 1$.

The punctured line over a point c in the canonical curve C consists of the cohomology classes $[\pi_c]$ of the principal part π_c of a rational function with a simple pole at c and otherwise zero. For practice (because these ideas are needed later in a more complicated situation) let us see cohomologically why C is contained in R and let us compute $C \cap V$. Let $\eta_c \in \Gamma(C, \mathscr{L})$ be a section which vanishes at c. Then the principal part $\eta_c\pi_c$ is zero, so $\eta_c \cup [\pi_c] = 0$ and $\cup[\pi_c]: \Gamma(C, \mathscr{L}) \to H^1(C, \mathscr{L})$ has rank $\leqslant 1$. Thus $C \subseteq R$. Also if c is a base-point of $\Gamma(C, \mathscr{L})$ then $[\pi_c] \in \ker(\cup)$ and so $\{\text{base-points of } \mathscr{L}\} \subseteq C \cap V$. If c is not a base-point then there is a section $\gamma_c \in \Gamma(C, \mathscr{L})$ which does not vanish at c. The principal part $\gamma_c\pi_c$ of $\mathscr{L}$ has a pole of order 1 at c and is otherwise zero. Its cohomology class is zero if and only if $\mathscr{L}$ has a rational section σ with a single simple pole at c. By duality σ exists iff c is a base-point of $\Gamma(C, \Omega_C \otimes \mathscr{L}^{-1})$. Thus

$$C \cap V = \{\text{base-points of } \mathscr{L}\} \cup \{\text{base-points of } \Omega_C \otimes \mathscr{L}^{-1}\}.$$

The idea behind the above calculations is the relationship between the matrix (f_{ij}) and the vanishing of cup product. The matrix (f_{ij}) controls the cohomology of all local deformations of $\mathscr{L}$. The above calculation involves a deformation over the infinitesimal scheme $D_1 = \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$. Such a deformation is given by

$$0 \to \varepsilon\mathscr{L} \to \mathscr{L}_2 \to \mathscr{L} \to 0$$

where $\mathscr{L}_2$ is an invertible sheaf on $C \times D_1$. As it is well-known the isomorphism classes of such extensions correspond to cohomology classes $\alpha \in H^1(C, \mathcal{O}_C)$ and a section η of $\mathscr{L}$ lifts to a section of $\mathscr{L}_2$ if and only if $\eta \cup \alpha$ is zero. We will consider similar lifting problems to higher order deformations of $\mathscr{L}$.

Let $D_i = \operatorname{Spec}(A_i)$ with $A_i = k[\varepsilon]/(\varepsilon^{i+1})$. We want to describe the deformation of $\mathscr{L}$ corresponding to a flat curve $D_i \hookrightarrow J$ with support $(D) = x$ [K3]. Such a deformation of $\mathscr{L}$ is determined by its velocity which is a tangent vector of J; i.e., an element in $H^1(C, \mathcal{O}_C)$. Let $\beta = (\beta_c)$ be a $k(C)$-valued function on C such that β_c is regular at c except for finitely many c's. Then β determines a cohomology class $[\beta]$ in $H^1(C, \mathcal{O}_C)$. We want to write the deformation of $\mathscr{L}$ in terms of β. Let $\mathscr{L}_{i+1}(\beta)$ be the sheaf whose stalk at $c \in C \times D_i$ $(= C$ as sets) is given by rational sections $f = f_0 + f_1\varepsilon + \cdots + f_i\varepsilon^i$ of $\mathscr{L} \otimes A_i$ such that $f \exp(\varepsilon\beta_c)$ is regular at c. If $\operatorname{char}(k) \nmid i!$ this expression makes sense as

$$\exp(\varepsilon\beta_c) = \sum_{0 \leqslant j \leqslant i} \beta_c^j \varepsilon^j / j!.$$

Thus if $i = 2$ for $f = f_0 + f_1\varepsilon + f_2\varepsilon^2$ to be a global section of $\mathscr{L}_3(\beta)$ we need (0) f_0 is a regular section of $\mathscr{L}$, (1) $f_1 + f_0\beta_c$ is regular at c for every $c \in C$ and (2) $f_2 + f_1\beta_c + f_0\beta_c^2/2$ is regular at c for every $c \in C$.

Our main tool to study the osculating cone $S = \{\theta_2 = \theta_3 = 0\}$ is the following:

LEMMA 4. *A cohomology class $[\beta] \in H^1(C, \mathcal{O}_C)$ is contained in the cone cover $\{\theta_2 = \theta_3 = \cdots = \theta_{i+1} = 0\} \subseteq \mathbb{P}^{g-1}$ if and only if*

$$\operatorname{length}_{A_i}(\Gamma(C \times D_i, \mathscr{L}_{i+1}(\beta)) \geqslant i + 2.$$

In particular if $k[\beta] \in \mathbb{P}^{g-1}$ is a point which does not lie in V then $k[\beta] \in \{\theta_2 = \theta_3 = \cdots = \theta_{i+1} = 0\}$ if and only if there exists a section f of $\mathscr{L}_{i+1}(\beta)$ such that $f_0 \neq 0$.

Proof. The cohomology of $\mathscr{L}_{i+1}(\beta)$ is controlled by the pullback φ of the matrix (f_{ij}) via $D_i \to J \cong \operatorname{Pic}_{g-1}(C)$, $\varepsilon \mapsto \exp([\beta]\varepsilon)$. Since D_i is just a fat point we have an exact sequence

$$0 \longrightarrow \Gamma(C \times D_i, \mathscr{L}_{i+1}(\beta)) \longrightarrow A_i^{\oplus 2} \xrightarrow{\varphi} A_i^{\oplus 2}$$
$$\longrightarrow H^1(C \times D_i, \mathscr{L}_{i+1}(\beta)) \longrightarrow 0$$

of A_i-modules. The matrix φ is equivalent to a matrix

$$\begin{pmatrix} \varepsilon^a & 0 \\ 0 & \varepsilon^b \end{pmatrix}$$

with $1 \leqslant a \leqslant b \leqslant i + 1$. So length $\Gamma(C \times D_i, \mathscr{L}_{i+1}(\beta)) = a + b$. On the other hand $D_i \hookrightarrow J$ is a flat curve for a non-trivial class $[\beta]$. Hence $k[\beta]$ is contained in $\{\theta_2 = \theta_3 = \cdots = \theta_{i+1} = 0\}$ if and only if $a + b \geqslant i + 2$. If $k[\beta] \notin V$ then $a = 1$ and hence $k[\beta] \in \{\theta_2 = \theta_3 = \cdots = \theta_{i+1} = 0\}$ if and only if $b = i + 1$. But $\varepsilon^{i+1} = 0$ and so there exists a section f of $\mathscr{L}_{i+1}(\beta)$ with $f_0 \neq 0$. □

Thus $k[\beta]$ is contained in R if and only if there is a section $f_0 + f_1\varepsilon$ of $\mathscr{L}_2(\beta)$ with $f_0 \neq 0$. Moreover if $k[\beta] \in R - V$ then f_0 is uniquely determined up to a scalar factor. Also $k[\beta] \in S - V$ if and only if $f_0 + (f_1 + \eta)\varepsilon$ lifts to $\mathscr{L}_3(\beta)$ for a suitable choice of $\eta \in \Gamma(C, \mathscr{L})$. One works out that $[f_0\beta]$ is zero in $H^1(C, \mathscr{L})$ if and only if f_0 lifts to the first order. A second order lifting is possible if and only if $[f_1\beta + f_0\beta^2/2]$ is zero in $H^1(C, \mathscr{L})/\Gamma(C, \mathscr{L}) \cup [\beta]$. (The last division is required because we have to consider η).

Similarly one can compute tangent vectors to a point $k[\beta] \in S$ by computing the sections of a deformation obtained via exp $(([\beta] + t[\gamma])\varepsilon)$ over $k[\varepsilon, t]/(\varepsilon^3, t^2)$.

Using this machinery we will prove:

PROPOSITION 5. (a) *The canonical curve C is contained in S.* (b) *If the rational maps φ_L and $\varphi_{\Omega_C \otimes \mathscr{L}^{-1}} : C \to \mathbb{P}^1$ are distinct, then S is smooth of dimension $g - 3$ at a general point of C.*

Proof. For (a) if c is not a base-point of $\mathscr{L}$ then with the previous notation, π_c the principal part of a rational function with a simple pole at c and η_c a section of $\mathscr{L}$ which vanishes at c, we have that $\eta_c + 0\varepsilon$ is a section of $\mathscr{L}_2(\pi_c)$ and $[\eta_c\pi_c^2/2] \in \Gamma(C, \mathscr{L}) \cup [\pi_c]$. Thus the second obstruction vanishes. So $C - C \cap V \subseteq S$. Hence $C \subseteq S$.

For (b) we will find a tangent vector to R at a general point c which is not tangent to S. Let $T = k[t]/(t^2)$ and d another point of $C - C \cap V$, so

$$c \text{ and } d \text{ are not base-points of } \mathscr{L} \text{ or } \Omega_C \otimes \mathscr{L}^{-1}. \tag{1}$$

$[\pi_c + \pi_d t]$ represents a tangent vector to $k[\pi_c] = c \in \mathbb{P}^{g-1}$. For it to be tangent to R we need $[(\eta_c + rt)(\pi_c + \pi_d t)] = 0$ in $H^1(C, \mathscr{L}) \otimes_k T$ for some regular section r of $\mathscr{L}$; i.e., $[r\pi_c + \eta_c\pi_d] = 0$ in $H^1(C, \mathscr{L})$. If we assume that

$$\varphi_{\mathscr{L}}(c) = \varphi_{\mathscr{L}}(d) \tag{2}$$

then $\eta_c(d) = 0$ and we can take r to be an arbitrary multiple of η_c. Thus we have found a tangent vector to R.

To see that this vector is not tangent to S we consider the obstruction to the second order lifting $[(\eta_c + rt)(\pi_c + \pi_d t)^2/2]$ in $H^1(C, \mathscr{L}) \otimes T/(\Gamma(C, \mathscr{L}) \otimes T) \cup [\pi_c + \pi_d t])$. The question is whether there are sections $f_2 + g_2 t \in \Gamma(C, \mathscr{L}(c + d)) \otimes T$ and $s + s't \in \Gamma(C, \mathscr{L}) \otimes T$ such that $(\eta_c + rt)\pi_c^2/2 - (s + s't)(\pi_c + \pi_d t) - (f_2 + g_2 t)$ is regular at c and d. If we assume that

$$\varphi_{\Omega_C \otimes \mathscr{L}^{-1}}(c) \neq \varphi_{\Omega_C \otimes \mathscr{L}^{-1}}(d) \tag{3}$$

then $\Gamma(C, \mathscr{L}) = \Gamma(C, \mathscr{L}(c + d))$ by duality and the last term is regular in any case. If we assume further that

$$c \text{ is not a ramification point of } \varphi_{\mathscr{L}} \tag{4}$$

then $\eta_c\pi_c^2/2$ has a simple pole at c and one needs $s(c) \neq 0$ to make the expression regular at c. But then $s(d) \neq 0$ by (2) and consequently the term $(r\pi_c^2/2 - s\pi_d - s'\pi_c)t$ has a simple pole at d. So a second lifting is impossible. For a general point $c \in C$ the conditions (1), . . . , (4) are satisfied for every point $d \in \varphi_{\mathscr{L}}^{-1}(\varphi_{\mathscr{L}}(c)) - \{c\}$. □

Next we need to check a simpler fact. Let $c + V$ denotes the linear span of c and V in $\mathbb{P}^{g-1}$.

PROPOSITION 6. *If $c \in C - C \cap V$ then c is a singular point of $S \cap (c + V)$.*

Proof. As $c + V \subseteq R$ we need to compute the derivative of θ_3 along $c + V$ at c. We want to show that any tangent vector in $c + V$ at c is contained in S. Let $[\beta_1], \ldots, [\beta_{g-4}]$ be a basis of $\ker(\cup)$ where the $\beta_{i,c}$ are regular at c. This is possible because each cohomology class is equivalent to one supported off any given point as C-{point} is affine. Let $\beta = \pi_c + \Sigma\, t_i\beta_i$ where the t_i are indeterminates, $T = k[t_1, \ldots, t_{g-4}]/(t_1, \ldots, t_{g-4})^2$. Clearly η_c lifts to the first order as $\eta_c \cup [\beta] = 0$ in $H^1(C, \mathscr{L}) \otimes T$. Let $\eta_c + \gamma\varepsilon$ with $\gamma = 0 + \Sigma\, \gamma_i t_i$ be a lifting to a section of $\mathscr{L}_2(\beta)$. So the γ_i are regular at c. The obstruction to the second order lifting is $[\eta_c\beta^2/2 + \gamma\beta]$ in $H^1(C, \mathscr{L}) \otimes T/(\Gamma(C, \mathscr{L}) \otimes T) \cup [\beta]$. This has the form $[\eta_c\pi_c^2/2 + \Sigma\, t_\iota(\eta_x\pi_c\beta_\iota + \gamma_\iota\pi_c)]$. We already saw that $\eta_c\pi_c^2/2 = \varrho\pi_c$ for some $\varrho \in \Gamma(C, \mathscr{L})$. $\pi_c\beta_i = 0$ as they have different support. $\gamma_i\pi_c$ has at most a simple pole at c, so it does not give an obstruction since there is a section which does not vanish at c. Consequently

$$[\eta_c\pi_c^2/2 + \Sigma t_\iota(\eta_c\pi_c\beta_\iota + \gamma_\iota\pi_c)] \quad = \quad [-\Sigma\, t_i\varrho\beta_i]$$

in $H^1(C, \mathscr{L}) \otimes T/(\Gamma(C, \mathscr{L}) \otimes T) \cup [\beta]$. Since $\beta_i \in \ker(\cup)$ we have $\varrho \cup [\beta_i] = 0$ in $H^1(C, \mathscr{L})$ and the obstruction vanishes for all tangent directions in $c + V$. $\square$

We will now prove Proposition 2. We have to consider a third order lifting problem in $\mathscr{L}_4(\pi_c)$ for a point c of C. Assume that c is not a base-point of $\mathscr{L}$ and $\Omega_C \otimes \mathscr{L}^{-1}$ and that c is not a ramification point of $\varphi_{\mathscr{L}}$ and $\varphi_{\Omega_C\otimes L^{-1}}$. A first order lifting has the form $\eta_c + \varrho\varepsilon$ where ϱ is a regular section of $\mathscr{L}$. A second order lifting has the form $\eta_c + \varrho\varepsilon + \sigma\varepsilon^2$ where σ is regular and $\eta_c\pi_c^2/2 + \varrho\pi_c$ is regular at c. The obstruction to lift to the third order is $[\eta_c\pi_c^3/6 + \varrho\pi_c^2/2 + \sigma\pi_c] \in H^1(C, \mathscr{L})/\Gamma(C, \mathscr{L}) \cup [\pi_c]$. But this is cohomologous to $[-\eta_c\pi_c^3/12]$ which has a double pole at c. Thus the obstruction does not vanish. This proves Proposition 2 and shows that

$$C \cap \{\theta_2 = \theta_3 = \theta_4 = 0\} \subseteq \text{base-points}\,(\mathscr{L}) \cup \text{base-points}\,(\Omega_C \otimes \mathscr{L}^{-1})$$

$$\cup \text{ ramification points}\,(\mathscr{L}) \cup \text{ramification points}\,(\Omega_C \otimes \mathscr{L}^{-1})$$

The reversed inclusion is easily seen as the obstruction clearly vanishes. This means that for a $(C, \mathscr{L})$ general the divisor $C \cap \{\theta_4 = 0\}$ is the sum of the ramification divisors of $\mathscr{L}$ and $\Omega_C \otimes \mathscr{L}^{-1}$ which is in $|4K|$ as it should be. We leave the determination of the multiplicities in special cases open.

§2. Global description of the quadric bundle defined by a cubic hypersurface containing a linear subspace

Let $\mathbb{P}^{h-d} \subseteq \mathbb{P}^h$ be a linear subspace of codimension d. Let $\tilde{\mathbb{P}}$ be $\mathbb{P}^h$ blown up along $\mathbb{P}^{h-d}$. Then we have the projection $\pi\colon \tilde{\mathbb{P}} \to \mathbb{P}^{d-1}$ and an exceptional divisor E in $\tilde{\mathbb{P}}$. Let H be the inverse image of the hyperplane class in $\mathbb{P}^h$ to $\tilde{\mathbb{P}}$ and let L be the inverse image of a hyperplane in $\mathbb{P}^{d-1}$ under π. Then by examining a hyperplane in $\mathbb{P}^h$ which contains the center $\mathbb{P}^{h-d}$ we deduce that

$$H \sim E + L. \tag{A}$$

The next well-known fact describes the $\mathbb{P}^{h-d+1}$-bundle π.

$$\tilde{\mathbb{P}} \cong \mathbb{P}_{\mathbb{P}^{d-1}}(\mathscr{E}) \tag{B}$$

where

$$\mathscr{E} = \pi_*(\mathcal{O}_{\tilde{\mathbb{P}}}(E)) \cong \mathcal{O}_{\mathbb{P}^{d-1}} \oplus \mathcal{O}_{\mathbb{P}^{d-1}}(-1)^{\oplus h-d+1}.$$

Proof. Clearly E gives a hyperplane in each fiber of π. It remains to compute a basis for $\mathscr{E}$. To get a generator e_0 of the first summand we may take π_* of the section 1 of $\mathcal{O}_{\tilde{\mathbb{P}}}(E)$. For the other direct summands we may take $e_i = \pi_*(x_i) \in \Gamma(\mathbb{P}^{d-1}, \mathscr{E}(1))$ where $x_1, \ldots, x_{h-d+1} \in \Gamma(\tilde{\mathbb{P}}, \mathcal{O}(E + L)) \cong \Gamma(\mathbb{P}^h, \mathcal{O}(1))$ are linear forms which induce homogeneous coordinates on $\mathbb{P}^{d-h}$. Looking at the fibers of π we see that these sections generate $\mathscr{E}$ everywhere. $\square$

Let A be a cubic hypersurface in $\mathbb{P}^h$ which contains $\mathbb{P}^{h-d}$. We can write inverse image in $\tilde{\mathbb{P}}$ as $E + \tilde{A}$ where $\tilde{A}$ is an effective divisors. Then

$$\tilde{A} \subseteq \tilde{\mathbb{P}} \xrightarrow{\pi} \mathbb{P}^{d-1} \text{ is a quadric bundle} \tag{C}$$

since $\tilde{A} \sim 3H - E = 2E + 3L$. The equation of $\tilde{A}$ is very simple. Under the identification $\Gamma(\mathbb{P}^h, \mathcal{O}(1)) \cong \Gamma(\tilde{\mathbb{P}}, \mathcal{O}(H)) \cong \Gamma(\mathbb{P}^{d-1}, \mathscr{E}(1))$

$$x_1 = e_1, \ldots, x_{h-d+1} = e_{h-d+1} \quad \text{and}$$

$$x_{h-d+2} = y_0 e_0, \ldots, x_{h+1} = y_{d-1} e_0$$

are homogeneous coordinates on $\mathbb{P}^h$, if $y_0, \ldots, y_{d-1} \in \Gamma(\mathbb{P}^{d-1}, \mathcal{O}(1))$ are homogeneous coordinates in $\mathbb{P}^{d-1}$. Substituting these expressions into the cubic equation $f = f(x_1, \ldots, x_h)$ of A we find

$$f = e_0 \tilde{f} \quad \text{with} \quad \tilde{f} = \sum_{i \leqslant j} a_{ij}\, e_i e_j.$$

so $\tilde{f}$ is a quadratic form in $e_0, \ldots, e_{h-d+1}$ with coefficients

$$a_{00} \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(3)),$$

$$a_{0j} = a_{j0} \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(2)) \quad \text{if } j \geqslant 1 \quad \text{and}$$

$$a_{ij} = a_{ji} \in H^0(\mathbb{P}^{d-1}, \mathcal{O}(1)) \quad \text{of } i, j \geqslant 1.$$

Of course the $(h - d + 2) \times (h - d + 2)$ matrix $a = (a_{ij})$ is just what we obtain from the equation of $\tilde{A}$ under the isomorphism $\Gamma(\tilde{\mathbb{P}}, \mathcal{O}(2E + 3L)) \cong \Gamma(\mathbb{P}^{d-1}, \mathrm{Sym}^2(\mathscr{E})(3)) \subseteq \Gamma(\mathbb{P}^{d-1}, \mathscr{H}om(\mathscr{E}^*(-3), \mathscr{E}))$ using the splitting of $\mathscr{E}$. The fact we will use is

For any point $p \in \mathbb{P}^{d-1}$ the codimension of the singular locus of $\tilde{A} \cap \pi^{-1}(p)$ in $\pi^{-1}(p)$ is the rank of the matrix (a_{ij}) at p. (D)

Thus for a general cubic A containing $\mathbb{P}^{h-d}$ we expect the image of all singular fibers is a divisor of degree $h - d + 4 = \deg \det(a)$.

Let B be a smooth projective variety together with a morphism τ: $B \to \mathbb{P}^{d-1}$. By base change we obtain a quadric bundle

$$\tilde{A}_\tau \hookrightarrow \tilde{\mathbb{P}}_\tau \xrightarrow{\pi'} B.$$

The next result gives what we need abstractly about this quadric bundle.

PROPOSITION 7. *Let X be a closed subvariety of $\tilde{A}_\tau$ such that*
- (a) *$\tilde{A}_\tau \cap \pi'^{-1}(\pi'(p))$ is singular at p for all points $p \in X$,*
- (b) *$\tilde{A}_\tau$ is smooth at a general point of X and*
- (c) *$\pi'(X)$ is a divisor on B with* $\deg(\pi'(X)) \geqslant (h - d - 4) \deg(\tau^{-1}(\text{hyperplane}))$

Then (d) *$\tilde{A}_\tau \cap \pi'^{-1}(q)$ is smooth for all $q \in B - \pi'(X)$ and*
- (e) *p is the only singular point of $\tilde{A}_\tau \cap \pi'^{-1}(\pi'(p))$ for a point p of X such that $\pi'(p)$ is smooth on $\pi'(X)$.*

Proof. Let $q \in B$ be a point. The codimension of the singular locus of $\tilde{A}_\tau \cap \pi'^{-1}(q)$ is equal to the rank of $\tau^*(a_{ij}) \otimes k(q)$. The singular locus simply is

$$\mathbb{P}(\mathrm{coker}(\tau^*(a_{ij}) \otimes k(q)) \subseteq \mathbb{P}(\tau^*(\mathscr{E}) \otimes k(q))$$

if we regard $a = (a_{ij})$ as an homomorphism $\mathscr{E}^*(-3) \to \mathscr{E}$. For the generic point of B, this singular locus has a dense subset of $k(B)$-rational points. In each of them $\tilde{A}_\tau$ is singular. Hence the closure of the singular locus of the generic fiber is contained in the singular locus of $\tilde{A}_\tau$. Therefore by (b) the closure does not contain X.

Next we prove that the singular locus of the generic fiber is empty. Otherwise $\tau^*(a_{ij})$ would have some rank $r \leqslant h - d + 1$. Let x be the generic point of X. Since x is not contained in the closure of the singular locus of the generic fiber and on the other hand the singular locus of $\tilde{A}_\tau \cap \pi'^{-1}(\pi'(x))$ is a linear space which contains x by (a) we have rank $(\tau^*(a_{ij}) \otimes k(\pi'(x))) < r$. Hence all $r \times r$ minors of $\tau^* a$ vanish at $\pi'(x)$. Since not all of them vanish identically on B and all of them are pullbacks of polynomials of degree $< h - d + 4$ on $\mathbb{P}^{d-1}$ this is impossible by (c).

Thus the singular locus of a general fiber is empty and moreover we get that $\pi'(X)$ is scheme-theoretically the zeros of $\det(\tau^*(a_{ij}))$. This proves (d). For (e) just note that

$$\operatorname{rank}(\tau^*(a_{ij}) \otimes k(p)) = h - d + 1$$

if $\pi'(p)$ is smooth in $\pi'(X)$ since otherwise $\pi'(p)$ would be multiple point of $\det(\tau^*(a_{ij})) = 0$. This proves (e). □

Proof *of Theorem 1.*
Let $\mathbb{P}^h$ be the canonical space $\mathbb{P}^{g-1}$ and $\mathbb{P}^{h-d} = V$ the vertex of $R = \{\theta_2 = 0\}$. Let $A = \{\theta_3 = 0\}$. Thus A contains $\mathbb{P}^{h-d}$ by Lemma 3. Next let $B = \mathbb{P}^1 \times \mathbb{P}^1$ and τ the embedding into $\mathbb{P}^3$. Then we have that $\tilde{\mathbb{P}}_\tau$ is R blown up along V and $\tilde{A}_\tau$ is the strict transform $\tilde{S}$ of $S = \{\theta_2 = \theta_3 = 0\}$. Then take $X = C$ the canonical curve. Proposition 6 gives (a) and Proposition 5 gives (b). For (c) we note that $\pi'(C) = C'$ has class $(g - 1, g - 1)$. Hence Theorem 1 follows now from Proposition 7. □

Remarks. (1) For singular points $c' \in C'$ the singular locus of the fiber is higher dimensional. As C' is the zero divisor of the determinant

$$\dim \operatorname{Sing} \alpha^{-1}(c') \leqslant \operatorname{mult}(C', c') - 1.$$

So if $c' \in C'$ is an ordinary double point then $\operatorname{Sing} \alpha'(c')$ is the line spanned by the two preimage points in C and equality holds in the formula above. One might guess that equality always holds. We leave this to the reader.

(2) The canonical curve $C \subseteq \mathbb{P}^{g-1}$ lies in the birational model Y of $\mathbb{P}^1 \times \mathbb{P}^1$ obtained by the rational map defined by the linear series of adjoint curves to C'. Without proving it we mention that Y is a component of the variety defined by the partial derivatives

$$\frac{\partial \tilde{f}}{\partial e_i} = 0 \quad \text{for} \quad i = 1, \ldots, g - 4$$

in R. It is the component which dominates $\mathbb{P}^1 \times \mathbb{P}^1$ and the birational map is just the projection.

(3) In case $g = 5$ Theorem 1 might be slightly misleading. S is a $K3$ surface birational to the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along C'. C' is a divisor of type (4, 4) with 4 (possibly infinitesimally near) double points which lie on a divisor Δ of class (1,1). The last property holds because the two g_4^1's are residual. The double cover has rational double point singularities over the double points of C'. Resolve those. The preimage of Δ has two disjoint components. Both are (-2) curves. In case there are 4 distinct double points S is obtained by contracting one of them, the singular point will give the vertex V. If two or more of the double points of C' are infinitesimally near then S has a more complicated singularity.

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