

COMPOSITIO MATHEMATICA

F. J.-B. J. CLAUWENS

***K*-theory, λ -rings, and formal groups**

Compositio Mathematica, tome 65, n° 2 (1988), p. 223-240

<http://www.numdam.org/item?id=CM_1988__65_2_223_0>

© Foundation Compositio Mathematica, 1988, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

***K*-theory, λ -rings, and formal groups**

F.J.B.J. CLAUWENS

*Mathematical Institute, Catholic University, Toernooiveld, 6525 ED Nijmegen,
 The Netherlands*

Received 29 June 1987; accepted 15 September 1987

Introduction

In [C] the author showed how one can compute the algebraic *K*-group $K_2(R, I)$ when R has a structure of λ -ring; this relied on a presentation of that group found by H. Maazen and J. Stienstra [Ms] and F. Keune [K]. The purpose of this paper is to show that the same technique works for a more general group $K_{2,F}(R, I)$ in which also a formal group F is involved. An instance of this more general group occurs in §17 of [S], which inspired the present paper.

DEFINITION 0.1. Let F be a one-dimensional formal group over \mathbf{Z} . Let R be a commutative ring and let I be a nilpotent ideal. Then $K_{2,F}(R, I)$ is the abelian group with as generators the symbols $\langle a, b \rangle_F$ for $(a, b) \in I \times R \cup R \times I$ and relations

$$\begin{aligned} \langle a, 1 \rangle_F & \qquad \qquad \qquad \text{for } a \in I \\ \langle a, b \rangle_F + \langle a, c \rangle_F - \langle a, a^{-1}F(ab, ac) \rangle_F & \text{ for } (a, b, c) \in I \times R \times R \\ & \qquad \qquad \qquad \cup R \times I \times I \\ \langle a, bc \rangle_F + \langle b, ac \rangle_F - \langle ab, c \rangle_F & \text{ for } (a, b, c) \in I \times R \times R \\ & \qquad \qquad \qquad \cup R \times I \times R \cup R \times R \times I \end{aligned}$$

where $a^{-1}F(ab, ac)$ has to be interpreted in the obvious way.

If F is the multiplicative formal group defined by $F(X, Y) = X + Y - XY$ then $K_{2,F}(R, I)$ coincides with the group $K_2(R, I)$ from algebraic *K*-theory. If F is the additive formal group defined by $F(X, Y) = X + Y$ then $K_{2,F}(R, I)$ coincides with the group $K_{2,L}(R, I)$ considered in [C] and [LQ]. As explained there it is a cyclic homology group of (R, I) ; it is isomorphic to $\Omega_{R,I}/\delta I$ if the natural projection $R \rightarrow R/I$ splits.

To formulate the main theorem we introduce the notation $G(R, I)^{\text{top}}$ for the inverse limit $\lim G(R/J^N, (I + J^N)/J^N)$ if G is a functor from pairs (R, I) as above to abelian groups and J is an ideal describing a topology on R .

THEOREM 0.2. *Let F be a one dimensional formal group over \mathbf{Z} which is strongly isomorphic to a special one. Let R be a λ -ring and let I and J be ideals such that (R, J, I) is admissible. Then there is a homomorphism $L_F: K_{2,F}(R, I)^{\text{top}} \rightarrow K_{2,L}(R, I)^{\text{top}}$ such that*

$$L_F \langle a, b \rangle_F = \langle a, b \rangle_L + \text{higher order terms.}$$

The notion of special formal group and of strong isomorphism are defined in the next section. For the definition of admissible and other notions connected with λ -rings we refer to [C].

§1. Generalities about formal groups

Let A be a commutative ring. A one-dimensional commutative formal group F over A is a formal power series $F(X, Y) = \sum f_{ij} X^i Y^j$ with $f_{ij} \in A$ such that

$$F(X, Y) = X + Y + \text{terms of degree } > 1$$

$$F(X, Y) = F(Y, X),$$

$$F(X, F(Y, Z)) = F(F(X, Y), Z).$$

By substitution into F one can define a structure of abelian group on any topologically nilpotent ideal in a complete commutative A -algebra. In particular this applies to the formal power series over A with vanishing constant term. In this paper we use the words “formal group” in the understanding that we always mean a one-dimensional commutative one.

If F and \tilde{F} are both formal groups then an isomorphism G from \tilde{F} to F means a formal power series $G(X) = \sum g_n X^n$ with $g_n \in A$ such that

$$G(X) = X + \text{terms of degree } > 1,$$

$$G\tilde{F}(X, Y) = F(G(X), G(Y)).$$

In the case that $A = \mathbf{Z}$ one can find integers γ_n such that G is an infinite sum of terms X^n , each occurring γ_n times; the sum being taken in the F sense.

DEFINITION 1.1. The isomorphism G is called a strong isomorphism if n divides γ_n for every n .

PROPOSITION 1.2. Let \tilde{F} and F be formal groups over \mathbf{Z} , and let $G: \tilde{F} \rightarrow F$ be a strong isomorphism. Then G induces an equivalence $G_*: K_{2,\tilde{F}} \rightarrow K_{2,F}$ such that

$$G_* \langle a, b \rangle_{\tilde{F}} = \langle a, a^{-1}G(ab) \rangle_F$$

Proof. We show that the three relations are satisfied.

- 1) $G_* \langle a, 1 \rangle_{\tilde{F}} = \langle a, a^{-1}G(a) \rangle_F = \sum \gamma_n \langle a, a^{-1}a^n \rangle_F = \sum n^{-1} \gamma_n \langle a^n, 1 \rangle_F = 0$ since it follows from the third relation that $\langle a^n, 1 \rangle_F = n \langle a, a^{n-1} \rangle_F$.
- 2) $G_* \langle a, b \rangle_{\tilde{F}} + G_* \langle a, c \rangle_{\tilde{F}} = \langle a, a^{-1}G(ab) \rangle_F + \langle a, a^{-1}G(ac) \rangle_F = \langle a, a^{-1}F(G(ab), G(ac)) \rangle_F = G_* \langle a, a^{-1}G^{-1}F(G(ab), G(ac)) \rangle_{\tilde{F}} = G_* \langle a, a^{-1}\tilde{F}(ab, ac) \rangle_{\tilde{F}}$
- 3) $G_* \langle a, bc \rangle_{\tilde{F}} + G_* \langle b, ac \rangle_{\tilde{F}} = \langle a, a^{-1}G(abc) \rangle_F + \langle b, b^{-1}G(abc) \rangle_F = \langle ab, a^{-1}b^{-1}G(abc) \rangle_F = G_* \langle ab, c \rangle_{\tilde{F}} \quad \bullet$

For any formal group over \mathbf{Q} there is an isomorphism f from F to the additive formal group; this is called the logarithm of the formal group. Let p be a prime number and let F be a formal group over \mathbf{Q} with logarithm $F(X) = \sum f_n X^n$; then the formal group with logarithm $\sum f_{p^n} X^{p^n}$ is called the p -typical formal group associated to F . If F is defined over \mathbf{Z} then its p -typification is (for these facts see [H]).

If $f = \sum f_n X^n$ is a logarithm for F and $\tilde{f} = \sum \tilde{f}_n X^n$ is one for \tilde{F} and if G is an isomorphism as above then we have $\tilde{f} = fG = \sum \gamma_n f(X^n)$. Writing this out yields $\tilde{f}_n = \sum_{mk=n} \gamma_m f_k$ hence $(\sum \tilde{f}_n n^s) = (\sum \gamma_m m^s)(\sum f_k k^s)$ in the language of formal Dirichlet series introduced in [C]. So if F and \tilde{F} are strongly isomorphic then their p -typifications are.

DEFINITION 1.3. A formal group F is called special if its logarithm $f(X) = \sum f_n X^n$ satisfies $f_{mk} = f_m f_k$ for every m, k .

So the p -typification of a special formal group is again special. Therefore the p -typification of a formal group satisfying the conditions of Theorem 0.2 again satisfies those conditions.

§2. The λ -operations associated to F

In [C] we introduced certain λ -operations λ^n, θ^n and η^n . In this § we introduce F -twisted versions λ^n_F, θ^n_F and η^n_F of these operations.

Recall that the element a of a λ -ring is called one-dimensional if $\lambda^n(a) = 0$ for $n > 1$.

PROPOSITION 2.1. *Let F be a formal group over \mathbf{Z} . There exists a unique sequence $\{\lambda_F^n\}$ of λ -operations such that $\lambda_i^F = \sum_{n=1}^\infty t^n \lambda_F^n$ satisfies*

- 1) $\lambda_i^F(a + b) = F(\lambda_i^F(a), \lambda_i^F(b))$
- 2) $\lambda_i^F(a) = ta$ if a is one-dimensional.

Proof. Recall that the universal λ -ring U can be embedded in the inverse limit of polynomial rings $\mathbf{Z}[s_1, s_2, \dots, s_n]$ so that the canonical element $u \in U$ corresponds to $\sum_{n=1}^\infty s_n$. Now $F(ts_1, ts_2, \dots)$ is the required element λ_i^F . ●

PROPOSITION 2.2. *Let F be a formal group over \mathbf{Q} with logarithm f . Let R be a λ -ring containing \mathbf{Q} , and let $a \in R$. Then*

$$f(\lambda_i^F(a)) = \sum_{n=1}^\infty f_n t^n \psi^n(a).$$

Proof. From the definition of logarithm and the first property of λ_i^F it follows that $f\lambda_i^F$ is additive. Moreover the statement is true on one-dimensional elements. Therefore it is true on $\mathbf{Q}[s_1, s_2, \dots]$ and thus on $U \otimes \mathbf{Q}$. ●

Now we define the F -twisted version of the operations η^n in §3 of [C].

LEMMA 2.3. *If R is a λ -ring and $a \in R$ is not a zero-divisor then*

$$\lambda_{ta}^F = \sum_{n=1}^\infty t^n a^n \lambda_F^n: R[[t]] \rightarrow taR[[t]] \text{ is a bijection.}$$

Proof. This follows easily from the first property of λ_i^F together with the fact that

$$\lambda_F^n(t^m R[[t]]) \subseteq t^{nm} R[[t]].$$
 ●

DEFINITION 2.4. Let F be a formal group over \mathbf{Z} . The λ -operations η_F^n in two variables are defined by the condition that

$$\lambda_{ta}^F \left(\sum_{n=1}^\infty t^{n-1} \eta_F^n(a, b) \right) = tab.$$

We write $\theta_F^n(b)$ for $\eta_F^n(1, b)$.

By applying f on both sides in this definition and using Proposition 2.3 we get the following generalisation of Proposition 3.3 of [C]:

OBSERVATION 2.5. For elements a, b in a λ -ring R containing \mathcal{Q} one has

$$f_n a^{n-1} b^n = \sum_{m|n} f_m a^{m-1} \psi^m(\eta_F^{n/m}(a, b)).$$

From this it is clear that $\eta_F^n(a, b)$ is of degree $n - 1$ in a and of degree n in b . Therefore we can rewrite the formula 2.4 as

$$\lambda_{ta}^F(\eta_F(ta, b)) = tab \quad \text{where} \quad \eta_F(ta, b) = \sum_{n=1}^{\infty} \eta_F^n(ta, b).$$

PROPOSITION 2.6. *If F is a formal group over Z and a, b, c are elements of a λ -ring then*

$$\eta_F(ta, b) + \eta_F(ta, c) = \eta_F(ta, (ta)^{-1}F(tab, tac)).$$

Proof. This follows from Lemma 2.3 since both sides have the same image under λ_{ta}^F by Definition 2.4 and the first property of λ^F . ●

Using the formalism of formal Dirichlet series introduced in §8 of [C] we can rewrite some of the foregoing identities as relations between such series.

DEFINITION 2.7. Let F be a formal group over Z . Then $Y_F \in DS(U)$ and $H_F \in DS(U_2)$ are defined by the formulas

$$Y_F(a) = \sum n f_n a^{n-1} n^s, \quad H_F(a, b) = \sum \eta_F^n(a, b) n^s$$

Now Observation 2.5 can be reformulated as $b Y_F(ab) = Y_F(a) \cdot TH_F(a, b)$. The formula at the end of §1 can be rewritten as $Y_{\tilde{F}}(1) = T\Gamma \cdot Y_F(1)$. Henceforth we write Y_F for $Y_{\tilde{F}}(1)$. ●

Since we have taken a generalisation of Proposition 3.8 of [C] as definition of the η_F^n , the generalisation of Definition 3.1 of [C] becomes a proposition:

PROPOSITION 2.8. *If F is a formal group over Z and a, b are elements of a λ -ring then*

$$\eta_F^n(a, b) = \sum_m \eta_F^m(a, a^{(n-m)/m}) \psi^m \theta_F^{n/m}(b) \quad \text{for } n > 1.$$

Here the sum extends over all $m \neq n$ dividing n .

Proof. Property 2.5 determines the operations η_F^n uniquely: we may assume that a, b are the canonical elements in U_2 , and the right hand side is of the form $\eta_F^n(a, b) + \text{terms involving } \eta_F^k(a, b) \text{ with } k < n$.

Therefore it is sufficient to show that the right hand side of the above statement satisfies the same identity. This follows easily by rearranging sums and using the induction hypothesis. ●

COROLLARY 2.9. *The operations η_F^n are elements of the subring V_2 of the ring U_2 of λ -operations in two variables as defined in §2 in [C].*

Therefore the theory of §5 of [C] tells us the following. If (R, J, I) is admissible then $\eta_F(a, b) = \sum_{n=1}^\infty \eta_F^n(a, b)$ converges in the J -topology for $a \in I$ or $b \in I$. Moreover Proposition 2.6 implies that in that situation one has

$$\eta_F(a, b) + \eta_F(a, c) = \eta_F(a, a^{-1}F(ab, ac)).$$

§3. The relation between η_F and η

In this § the operations η_F^n are expressed in terms of the η^m .

DEFINITION 3.1. Let F be a formal group over \mathbf{Q} with logarithm f . Then the λ -operations $C_F^{n,m} \in U \otimes \mathbf{Q}$ are defined by

$$\sum_{k|m} f_k a^{k-1} \psi^k(C_F^{n,m/k}(a)) = n f_{nm} a^{m-1}$$

In particular $C_F^{n,1}(a) = n f_n$; and if F is special then $C_F^{n,m} = 0$ for $m > 1$.

PROPOSITION 3.2. *Let F be a formal group over \mathbf{Q} and let a, b be elements of a λ -ring containing \mathbf{Q} . Then*

$$\eta_F^n(a, b) = \sum_{m|n} C_F^{n/m,m}(a) \psi^m(\eta^{n/m}(a, b)).$$

Proof. Again it is sufficient to show that the right hand side satisfies identity 2.5. This follows easily by rearranging the sum and applying Definition 3.1 and Proposition 2.5 for the η^m . ●

We now show that in fact $C_F^{n,m} \in U$ if F is a formal group over \mathbf{Z} .

LEMMA 3.3

- a) Suppose that $\xi^m \in U \otimes \mathbf{Q}$ and that the operation $(a, b) \rightarrow \sum_{m|n} \xi^m(a)\psi^m(\theta^{n/m}(b))$ is element of U_2 ; then in fact $\xi^m \in U$.
- b) Suppose that $\xi \in U \otimes \mathbf{Q}$ and that the operation $a \rightarrow \xi(a)\psi^k(a')$ is in U . Then in fact $\xi \in U$.

Proof

- a) If the operations ξ^m are integral for $m < M$ then we may suppose that they vanish in that range. Now apply the above operation to the ring $U[t_1, t_2]/(t_1^{M+1})$ and take for a the canonical element $u \in U$ and $b = t_1 + t_2$ or t_2 respectively, and take the difference of the results. Then the m th term vanishes also for $m > M$, and the M th term is

$$\xi^M(u)\{\theta^{n/M}(t_1^M + t_2^M) - \theta^{n/M}(t_2^M)\} = \xi^M(u)t_1^M t_2^{n-M}.$$

If this is integral then $\xi^M(u)$ must be.

- b) The universal λ -ring U is the polynomial ring over \mathbf{Z} freely generated by the $\lambda^n(u)$. Therefore a product of two elements can only be a multiple of an integer > 1 if one of the factors is. But the element of U corresponding to the operation $a \rightarrow \psi^k(a')$ is not divisible by any integer > 1 as is easily seen by applying this operation on the ring $\mathbf{Z}[t]$ and the element t . ●

PROPOSITION 3.4. Let F be a formal group over \mathbf{Z} . Then the operations $C_F^{n,m}$ are in U_2 .

Proof. By substituting Definition 3.1 of [C] into Proposition 3.2 and rearranging terms we get

$$\eta_F^n(a, b) = \sum_{m|n} \xi^m(a)\psi^m(\theta^{n/m}(b))$$

where

$$\xi^m(a) = \begin{cases} \sum_{k|m} C_F^{n/k,k}(a)\psi^k(\eta^{m/k}(a, a^{n-m/m})) & \text{for } m < n \\ C_F^{1,m} & \text{for } m = n \end{cases}$$

So by Lemma 3.3a these operations are integral. From this it follows by induction on n and Lemma 3.3b that the operations $C_F^{n,m}$ are integral ●

COROLLARY 3.5. Let F be a formal group over \mathbf{Z} and let a, b be elements of a λ -ring. Then again

$$\eta_F^n(a, b) = \sum_{m|n} C_F^{n/m,m}(a)\psi^m(\eta^{n/m}(a, b)).$$

REMARK 3.6. Proposition 3.4 implies that the numbers $C_F^{n,m}(1) \in \mathcal{Q}$ are integers. We will abbreviate these to $C_F^{n,m}$. They are determined by the formula $\sum_{k|m} f_k C_F^{n,m/k} = n f_{nm}$ which is exactly the formula in [D] expressing that $n f_n$ is a “lexoid function”. The meaning of these numbers for us becomes clear by putting $a = 1$ in Proposition 3.2; that yields the formula

$$\theta_F^n(b) = \sum_{m|n} C_F^{n/m,m} \psi^m(\theta^{n/m}(b)).$$

§4. Some identities for the numbers C_F

In this § the prime p is fixed. First we prove some general relations between the numbers C_F^{p',p^j} ; then we use these to draw some consequences for these numbers from the hypothesis of Theorem 0.2.

LEMMA 4.1 (see [D]). *Let F be a formal group over \mathcal{Q} . Then*

$$C_F^{p',p^j} = p C_F^{p'-1,p^{j+1}} + C_F^{p'-1,1} C_F^{p,p^j}$$

for every (i, j) with $i > 0$.

Proof. Let $f(X) = \sum_n f_n X^n$ be the logarithm of F . If we substitute the identities

$$C_F^{p',p^k} = p C_F^{p'-1,p^{k+1}} + C_F^{p'-1,1} C_F^{p,p^k} \text{ for } 0 \leq k \leq j - 1$$

into the identity

$$\sum_{k=0}^j C_F^{p',p^k} f_{p^j-k} = p^j f_{p^j} = p C_F^{p'-1,1} f_{p^{j+1}} + p \sum_{k=0}^j C_F^{p'-1,p^{k+1}} f_{p^j-k}$$

then we get

$$C_F^{p',p^j} + C_F^{p'-1,1} \sum_{k=0}^{j-1} C_F^{p,p^k} f_{p^j-k} = p C_F^{p'-1,p^{j+1}} + p C_F^{p'-1,1} f_{p^{j+1}}.$$

Now the statement follows by comparing this with the identity $p f_{p^{j+1}} = \sum_{k=0}^j C_F^{p,p^k} f_{p^j-k}$ ●

LEMMA 4.2. *Let F be a formal group over \mathcal{Q} . Then*

$$C_F^{p^i, p^j} = \sum_{k=0}^{i-1} p^k C_F^{p^i, p^{j+k}} C_F^{p^{i-k-1}, 1} \text{ for every } (i, j).$$

Proof. Immediate from Lemma 4.1 by induction on i . ●

LEMMA 4.3. *Let F be a formal group over \mathcal{Q} and let p be prime. Then for every (i, j, k) with $k \leq i$ one has*

$$C_F^{p^i, p^j} - \sum_{m=0}^k p^m C_F^{p^i-k, p^m} C_F^{k-m, p^j} = \begin{cases} p^k C_F^{p^i-k, p^{k+j}} & \text{if } j > 0 \\ 0 & \text{if } j = 0 \end{cases}$$

Proof. We use induction on k ; for $k = 0$ the statement is empty. For $k > 0$ we apply Lemma 4.1 for (i, j) , the induction hypothesis for $(i - 1, 0, k - 1)$, Lemma 4.1 for $(k - m, j)$ and the induction hypothesis for $(i - 1, j + 1, k - 1)$. ●

Now we deduce from the condition on F in theorem 0.2. a relation for the integers $C_F^{n, m}$. First we recall a few notations from §9 of [C].

DEFINITION 4.4. If $\Xi = \sum \zeta_n n^s$ is a Dirichlet series then we write

$$\Xi_p = \sum \zeta_{p^e} (p^e)^s, \quad T_p \Xi = \sum p^{v_p(n)} \zeta_n n^s.$$

PROPOSITION 4.5. *Let F be a p -typical formal group over \mathcal{Z} strongly isomorphic to a special one. Then p^h divides $C_F^{p^i, p^h}$ for every h .*

Proof. By Remark 3.6 one has $p^i f_{p^{e+i}} = \sum_{d=0}^e C_F^{p^i, p^{e-d}} f_{p^d}$. Therefore

$$Y_F - \sum_{j=0}^{i-1} p^j f_{p^j} (p^j)^s = Y_F \sum_h p^h C_F^{p^i, p^h} (p^{h+i})^s,$$

so

$$\sum_h p^h C_F^{p^i, p^h} (p^{h+i})^s = 1 - Y_F^{-1} \sum_{j=0}^{i-1} p^j f_{p^j} (p^j)^s.$$

If F is strongly isomorphic to a special formal group with logarithm \tilde{f} then one can rewrite Y_F^{-1} as

$$T\Gamma_p \cdot Y_{\tilde{F}}^{-1} = T\Gamma_p \cdot (1 - p\tilde{f}_p p^s)$$

and the right hand side is of the form

$$1 - \sum_{k=0}^{\infty} p^k \gamma_{p^k} (p^k)^s \cdot \sum_{m=0}^i \xi_m (p^m)^s$$

where

$$\sum_{m=0}^i \xi_m (p^m)^s = (1 - p \tilde{f}_p p^s) \cdot \sum_{j=0}^{i-1} p^j f_{p^j} (p^j)^s$$

has integral coefficients. Now the statement follows from the fact that $p^k \gamma_{p^k}$ is divisible by p^{2k} and thus by p^{2h} if $k + m = h + i$. ●

LEMMA 4.6. *Let F be any formal group, and a element of a λ -ring. Then*

$$Y_F(a) = \sum a^{p^h q^{-1}} Y_{F,P}(a^{p^h q}) p^h C_F^{q,p^h} (p^h q)^s.$$

In particular $Y_F = Y_{F,P} \cdot Y_{F,C} = Y_{F,C} \cdot Y_{F,P}$ where $Y_{F,C} = \sum p^h C_F^{q,p^h} (p^h q)^s$. Here the sums are over all q indivisible by p and over all h .

Proof. According to Remark 3.6 one has $q f_{p^e q} = \sum_{d=0}^e C_F^{q,p^{e-d}} f_{p^d}$. Substituting this into the formula $Y_F(a) = \sum p^e q f_{p^e q} a^{p^e q^{-1}} (p^e q)^s$ and writing h for $e - d$ gives the result. ●

PROPOSITION 4.7. *Let F be a formal group over \mathbf{Z} strongly isomorphic to a special one. Then p^h divides C_F^{q,p^h} if q is indivisible by p .*

Proof. Suppose that F is strongly isomorphic to \tilde{F} and that \tilde{F} is special. Then according to 2.7 one has $Y_{\tilde{F}} = T\Gamma \cdot Y_F$ where $T\Gamma$ is in the image of T_p^2 . This implies that $Y_{\tilde{F},P} = T\Gamma_p \cdot Y_{F,P}$ where $T\Gamma_p$ is in the image of T_p^2 . By lemma 4.6 one has $Y_{F,C} = Y_F \cdot Y_{F,P}^{-1} = T\Gamma^{-1} \cdot Y_{\tilde{F}} \cdot T\Gamma_p \cdot Y_{\tilde{F},P}^{-1} = T\Gamma^{-1} \cdot \Gamma_p \cdot Y_{\tilde{F},C}$. However $Y_{\tilde{F},C}$ is of the form $\sum_q C_{\tilde{F}}^{q,1} q^s$ and thus in the image of T_p^2 . Therefore $Y_{F,C}$ is in the image of T_p^2 . Writing this out one gets the statement. ●

§5. Some p -primary congruences

Now we use Proposition 4.5 to generalise the contents of §10 of [C] to the F -twisted case.

DEFINITION 5.1. Let F be a formal group over \mathbf{Z} . If $i \leq e$ then we write $\chi_F^{p^e, p^i}$ for the λ -operation defined by

$$\chi_F^{p^e, p^i}(a) = \theta_F^{p^e}(a) - \sum_{m=0}^i p^m C_F^{p^{e-i}, p^m} a^{p^e - p^{i-m}} \theta_F^{p^{i-m}}(a)$$

PROPOSITION 5.2. Let F be a p -typical formal group over \mathbf{Z} which is strongly isomorphic to a special one. Let a be the canonical element of U . Then

$$\chi_F^{p^e, p^i}(a) \equiv \sum_{j=0}^i \sum_{m=0}^{i-j} p^m C_F^{p^{e-i}, p^m} C_F^{p^{i-j-m}, p^j} \psi^{p^j}(\chi_F^{p^{e-i}, p^{i-j-m}}(a)) \text{ modulo } p^i V.$$

Proof. If one expresses the θ_F in terms of the θ as in Remark 3.6 then the right hand side of Definition 5.1 becomes

$$\begin{aligned} & \sum_{j=i+1}^e C_F^{p^{e-i}, p^j} \psi^{p^j}(\theta^{p^{e-i}}(a)) \\ & + \sum_{j=0}^i \left\{ C_F^{p^{e-i}, p^j} - \sum_{m=0}^{i-j} p^m C_F^{p^{e-i}, p^m} C_F^{p^{i-j-m}, p^j} \right\} \psi^{p^j}(\theta^{p^{e-i}}(a)) \\ & - \sum_{j=0}^i \sum_{m=0}^{i-j} p^m C_F^{p^{e-i}, p^m} C_F^{p^{i-j-m}, p^j} \{ a^{p^e - p^{i-m}} - \psi^{p^j}(a^{p^{e-i} - p^{i-j-m}}) \} \psi^{p^j}(\theta^{p^{i-j-m}}(a)) \\ & + \sum_{j=0}^i \sum_{m=0}^{i-j} p^m C_F^{p^{e-i}, p^m} C_F^{p^{i-j-m}, p^j} \psi^{p^j} \{ \theta^{p^{e-i}}(a) - a^{p^{e-i} - p^{i-j-m}} \theta^{p^{i-j-m}}(a) \}. \end{aligned}$$

The last sum is the right hand side of the statement. From Propositions 4.3 and 4.5 and the fact that the λ -operation $x \rightarrow x^{p^i+k} - \psi^{p^i}(x^{p^k})$ is in $p^{k+1}V$ it follows that the first three terms are in $p^i V$. ●

COROLLARY 5.3. In the situation of Proposition 5.2 one has

$$\chi_F^{p^e, p^i}(a) \equiv \begin{cases} 0 & \text{if } p \text{ is odd} \\ C_F^{2^{e-i}, 1} \sum_{j=0}^i C_F^{2^{i-j}, 2^j} \psi^{2^j}(\chi^{2^{e-i}, 2^{i-j}}(a)) & \text{if } p = 2 \end{cases}$$

Proof. It follows from Lemma 10.5 of [C] that χ^{p^e, p^i} is in $p^i V$ if p is odd and in $p^{i-1} V$ if $p = 2$. ●

DEFINITION 5.4. Let F be a formal group over \mathbf{Z} . Then the λ -operations τ_F^n are defined by $\sum \tau_F^n(a, b)n^s = H_F(1, a) \cdot TH_F(a, b)$. In other words

$$\tau_F^n(a, b) = \sum_{m|n} \frac{n}{m} \theta_F^m(a) \psi^m \eta_F^{n/m}(a, b).$$

From this one deduces easily that

$$\tau_F^n(a, b) = \theta_F^n(a) \psi^n(b) + \sum \frac{n}{m} \tau_F^m(a, a^{(n-m)/m}) \psi^m \theta_F^{n/m}(b)$$

where the sum extends over all $m < n$ dividing n .

DEFINITION 5.5. Let p be a prime, and let F be a formal group over \mathbf{Z} . Then the operation $\varepsilon_F^{p^e}$ is defined by

$$\varepsilon_F^{p^e}(a, b) = \sum_{i=0}^e C_F^{p^{e-i} p^i} \psi^{p^i}(\varepsilon_F^{p^{e-i}}(a, b))$$

where ε is defined as in Definition 10.7 of [C] i.e.

$$\varepsilon^{p^e}(a, b) = \begin{cases} 0 & \text{if } p > 2 \text{ or } e \leq 1 \\ 2^{e-1}(ab)^{2^e-4} \theta^2(a)^2 \theta^2(b)^2 & \text{if } p = 2 \text{ and } e \geq 2 \end{cases}$$

PROPOSITION 5.6. Let p be a prime and let F be a p -typical formal group strongly isomorphic to a special one. Let a, b be the canonical elements of U_2 ; then one has

$$\tau_F^{p^e}(a, b) = \theta_F^{p^e}(a) b^{p^e} + \varepsilon_F^{p^e}(a, b) \text{ modulo } p^e W_2.$$

Proof. We use induction in e ; the case $e = 0$ is trivial so assume $e > 0$. We claim that

$$\tau_F^{p^e}(a, b) = \theta_F^{p^e}(a) b^{p^e} + \sum_{k=0}^e [C_F^{p^{e-k}, 1} \varepsilon_F^{p^k}(a, a^{p^{e-k}}) - \chi_F^{p^e, p^k}(a)] p^{e-k} \psi^{p^k} \theta_F^{p^{e-k}}(b).$$

To prove the claim we apply the induction hypothesis and get

$$\begin{aligned} \tau_F^{p^e}(a, b) &= \theta_F^{p^e}(a)\psi^{p^e}(b) + \sum_{i=0}^{e-1} p^{e-i} \tau_F^{p^i}(a, a^{p^{e-i}-1})\psi^{p^i} \theta_F^{p^{e-i}}(b) \\ &= \theta_F^{p^e}(a)\psi^{p^e}(b) + \sum_{i=0}^{e-1} p^{e-i} [\theta_F^{p^i}(a)a^{p^e-p^i} + \varepsilon_F^{p^i}(a, a^{p^{e-i}-1}) \text{ modulo } p^i W] \psi^{p^i} \theta_F^{p^{e-i}}(b) \\ &\equiv \sum_{i=0}^e p^{e-i} \theta_F^{p^i}(a)a^{p^e-p^i} \psi^{p^i} \theta_F^{p^{e-i}}(b) + \sum_{i=0}^{e-1} p^{e-i} \varepsilon_F^{p^i}(a, a^{p^{e-i}-1}) \psi^{p^i} \theta_F^{p^{e-i}}(b). \end{aligned}$$

The first sum can be rewritten as

$$\begin{aligned} &\sum_{i=0}^e \sum_{j=0}^{e-i} p^{e-i} \theta_F^{p^i}(a)a^{p^e-p^i} C_F^{p^{e-i-1}, p^j} \psi^{p^{i+j}} \theta^{p^{e-i-j}}(b) \\ &= \sum_{k=0}^e \left[\sum_{j=0}^k p^j \theta_F^{p^{k-j}}(a)a^{p^e-p^{k-j}} C_F^{p^{e-k}, p^j} \right] p^{e-k} \psi^{p^k} \theta^{p^{e-k}}(b) \\ &= \sum_{k=0}^e [\theta_F^{p^e}(a) - \chi_F^{p^e, p^k}(a)] p^{e-k} \psi^{p^k} \theta^{p^{e-k}}(b) \end{aligned}$$

and by Lemma 2.2 of [C] the terms involving $\theta_F^{p^e}(a)$ add up to $\theta_F^{p^e}(a)b^{p^e}$.

In the second sum the value of $\psi^{p^i} \theta_F^{p^{e-i}}(b)$ only matters modulo pW since its cofactor is in $p^{e-1}W$. So we may replace it by $C_F^{p^{e-i-1}, 1} \psi^{p^i} \theta^{p^{e-i}}(b)$. This establishes the claim.

Now consider the bracket expression in the claim, modulo $p^k W$. For $p > 2$ it vanishes according to Corollary 5.3 and the definition of ε_F , and the proof is finished. So let $p = 2$. For $k > 1$ the χ_F term yields

$$\begin{aligned} &- C_F^{2^e-k, 1} \left[C_F^{2, 2^{k-1}} \psi^{2^{k-1}} (a^{2^{e-k+1}-4} \theta^2(a)^2) \right. \\ &\quad \left. + \sum_{j=0}^{k-2} 2^{k-j-1} C_F^{2^{k-j-1}, 2^j} \psi^{2^j} (a^{2^{e-j}-4} \theta^2(a)^2 + a^{2^{e-j}-8} \theta^2(a)^4) \right] \end{aligned}$$

and the ε_F term yields

$$\begin{aligned} &C_F^{2^e-k, 1} \sum_{j=0}^{k-2} C_F^{2^{k-j-1}, 2^j} \psi^{2^j} \varepsilon_F^{2^{k-j}}(a, a^{2^{e-k}-1}) \\ &= C_F^{2^e-k, 1} \sum_{j=0}^{k-2} C_F^{2^{k-j-1}, 2^j} \psi^{2^j} (2^{k-j-1} (a^{2^{e-k}})^{2^k-4} \theta^2(a)^2 \theta^2(a^{2^{e-k}-1})^2). \end{aligned}$$

Again the value of $\theta^2(a^{2^{e-k}-1})^2$ only matters modulo $2W$ so we may replace it by $a^{2^{e-k+2}-8}\theta^2(a)^2$. The expression thus becomes

$$- C_F^{2^{e-k},1} \sum_{j=0}^{k-1} C_F^{2^{k-j},2^j} 2^{k-j-1} \psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$$

For $k = 1$ one gets the same result; for $k = 0$ one gets zero. Finally we may replace $C_F^{2^{e-k},1} C_F^{2^{k-j},2^j}$ by $C_F^{2^{e-j},2^j}$ since it is the same modulo 2^{j+1} . We thus find the cofactor of $\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$ in $\tau_F^{p^e}(a, b)$ to be

$$- 2^{e-j-1} C_F^{2^{e-j},2^j} \sum_{k=j+1}^{e-1} \psi^{2^k} \theta^{2^{e-k}}(b)$$

and according to Lemma 10.6 of [C] this may be rewritten as

$$- 2^{e-j-1} C_F^{2^{e-j},2^j} (b^{2^{e-j}-4}\theta^2(b)^2)$$

which is exactly the cofactor of $\psi^{2^j}(a^{2^{e-j}-4}\theta^2(a)^2)$ in $\varepsilon_F^{2^e}(a, b)$. ●

§6. Proof of the theorem

LEMMA 6.1. *Let p be a prime. Let F be a p -typical formal group over \mathbf{Z} strongly isomorphic to a special one. If a, b are the canonical elements in U_2 then*

$$\theta_F^{p^e}(ab) \equiv \theta_F^{p^e}(a)b^{p^e} + a^{p^e}\theta_F^{p^e}(b) \text{ modulo } p^e W_2$$

Proof. In view of Proposition 4.5 and the fact that $\theta_F^{p^e} = \sum_{i=0}^e C_F^{p^{e-i},p^i} \psi^{p^i} \theta^{p^{e-i}}$ we only have to show that

$$\psi^{p^j}(\theta^{p^{e-j}}(ab)) \equiv \psi^{p^j}(\theta^{p^{e-j}}(a))b^{p^e} + a^{p^e}\psi^{p^j}(\theta^{p^{e-j}}(b)) \text{ modulo } p^{e-j} W_2$$

But this follows from the fact that

$$\theta^{p^j}(ab) = \theta^{p^j}(a)b^{p^j} + a^{p^j}\theta^{p^j}(b) - p^j \theta^{p^j}(a)\theta^{p^j}(b)$$

for all j . ●

PROPOSITION 6.2. *Let p be a prime. Let F be a p -typical formal group over \mathbf{Z} strongly isomorphic to a special one. Then for each natural number d there exist $B_F \in DS(W_d)$ such that*

$$TB_F(a_1, \dots, a_d) = \sum_{i=1}^d H_F(1, a_i) \cdot TH_F\left(a_i, \prod_{j \neq i} a_j\right) - H_F\left(1, \prod_{i=1}^d a_i\right)$$

if a_1, \dots, a_d are elements of a λ -ring R .

Proof. For $d = 2$ we have to prove that there exists a sequence of operations $\beta_F^{p^e}$ of two variables such that $p^e \beta_F^{p^e}(a, b) = \tau_F^{p^e}(a, b) + \tau_F^{p^e}(b, a) - \theta_F^{p^e}(ab)$. But Proposition 5.6 and 6.1 accomplish just that.

The statement for $d > 2$ follows from the one for $d = 2$ as was proven in Proposition 9.1 of [C]. ●

To prove a similar result in the general case we need a generalisation of Lemma 9.2 of [C].

LEMMA 6.3. *Let F be a formal group over \mathbf{Z} . If R is a λ -ring and $a, b \in R$ then*

$$\begin{aligned}
 TH_F(1, a) &= Y_{F,C}^{-1} \cdot \sum TH_{F,P}(1, a^{p^h q}) p^h C_F^{q,p^h}(p^h q)^s, \\
 TH_F(a, b) &= \left[\sum TH_{F,P}(a, a^{p^h q-1}) p^h C_F^{q,p^h}(p^h q)^s \right]^{-1} \\
 &\quad \times \left[\sum TH_{F,P}(a, a^{p^h q-1} b^{p^h q}) p^h C_F^{q,p^h}(p^h q)^s \right]
 \end{aligned}$$

Here the sum is over all q prime to p and over all h .

Proof. One has $TH_F(1, a) = Y_F^{-1} \cdot a Y_F(a)$ and $TH_{F,P}(1, a^{p^h q}) = Y_{F,P}^{-1} \cdot a^{p^h q} Y_{F,P}(a^{p^h q})$ according to Remark 2.7. The substitution of these identities into Lemma 4.6 yields the first statement. According to 2.7 one has also

$$\begin{aligned}
 TH_F(1, ab) &= Y_F^{-1} \cdot ab Y_F(ab) = [Y_F^{-1} \cdot a Y_F(a)] \cdot [Y_F(a)^{-1} \cdot b Y_F(ab)] \\
 &= TH_F(1, a) \cdot TH_F(a, b).
 \end{aligned}$$

Similarly one has

$$TH_{F,P}(1, a^{p^h q} b^{p^h q}) = TH_{F,P}(1, a) \cdot TH_{F,P}(1, a^{p^h q-1} b^{p^h q}).$$

The combination of these identities with the first statement yields the second one. ●

PROPOSITION 6.4. *Let F be a formal group over \mathbf{Z} strongly isomorphic to a special one. Then there exist $B_F \in DS(W_d)$ such that*

$$TB_F(a_1, \dots, a_d) = \sum_{i=1}^d H_F(1, a_i) \cdot TH_F\left(a_i, \prod_{j \neq i} a_j\right) - H_F\left(1, \prod_{j=1}^d a_j\right)$$

if a_1, \dots, a_d are elements of a λ -ring R .

Proof. (Essentially Proposition 9.3 of [C]). Let a, b be the canonical elements in U_2 . Let p be a prime. By Lemma 6.3 one has

$$\begin{aligned} & TH_F(1, a) \cdot T^2 H_F(a, b) - Y_{F,C}^{-1} \cdot TH_{F,P}(1, a) \\ & \quad \times \sum p^{2h} q C_F^{q,p^h} T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) (p^h q)^s \\ & = Y_{F,C}^{-1} \cdot \left[\sum [TH_{F,P}(1, a^{p^h q}) - p^h q TH_{F,P}(1, a) \right. \\ & \quad \times T^2 H_{F,P}(a, a^{p^h q - 1})] p^h C_F^{q,p^h} (p^h q)^s \left. \right] \\ & \quad \times \left[\sum T^2 H_{F,P}(a, a^{p^h q - 1}) p^{2h} q C_F^{q,p^h} (p^h q)^s \right]^{-1} \\ & \quad \times \left[\sum T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) p^{2h} q C_F^{q,p^h} (p^h q)^s \right]. \end{aligned}$$

Here all four factors are in the image of T_p^2 ; for this one needs Proposition 4.7 for the first two factors and the $d = p^h q$ case of Proposition 6.2 for the second factor. So the whole difference is in the image of T_p^2 . Therefore modulo $T_p^2 DS(W_2)$ one has

$$\begin{aligned} & TH_F(1, a) \cdot T^2 H_F(a, b) + TH_F(1, b) \cdot T^2 H_F(b, a) - TH_F(1, ab) \\ & \equiv Y_{F,C}^{-1} \cdot \sum [p^h q TH_{F,P}(1, a) \cdot T^2 H_{F,P}(a, a^{p^h q - 1} b^{p^h q}) + p^h q TH_{F,P}(1, b) \\ & \quad \times T^2 H_{F,P}(b, a^{p^h q} b^{p^h q - 1}) - TH_{F,P}(1, a^{p^h q} b^{p^h q})] p^h C_F^{q,p^h} (p^h q)^s \end{aligned}$$

and by using Proposition 6.2 for $d = 2p^h q$ one sees that this is in $T_p^2 DS(W_2)$. Thus the coefficient of n^s in this Dirichlet series is in $p^{2\gamma_r(n)} W_2$. Since that is

the case for every prime p the Chinese Remainder Theorem says that the coefficient is in $n^2 W_2$. In other words the Dirichlet series is in $T^2 DS(W_2)$. This finishes the proof for $d = 2$; the cases $d > 2$ follow from the case $d = 2$ as in Proposition 9.1 of [C]. ●

DEFINITION 6.5. Let F be a formal group over \mathbf{Z} . Then the operations v_F^n and the Dirichlet series N are defined by

$$\sum v_F^n(a, b)n^s = N_F(a, b) = H_F(1, a) \cdot \delta H_F(a, b).$$

In other words

$$v^n(a, b) = \sum_{mk=n} \theta_F^m(a) \phi^m(\delta \eta_F^k(a, b)).$$

Here the ϕ^m are the operations on differential forms introduced in §4 of [C].

PROPOSITION 6.6. Let F be a formal group over \mathbf{Z} which is strongly isomorphic to a special one, and let a_1, \dots, a_d be elements of a λ -ring R . Then

$$\sum_{i=1}^d N_F \left(a_i, \prod_{j \neq i} a_j \right) = \delta B_F(a_1, \dots, a_d)$$

Proof. This is an easy consequence of Proposition 6.4. For the details of the proof see Proposition 8.6 of [C]. ●

Now we prove the main theorem. Consider the expression $v_F(a, b) = \sum_{n=1}^\infty v_F^n(a, b)$. Using exactly the same arguments as in §6 of [C] one finds that it converges if $(a, b) \in I \times R \cup R \times I$ and that Corollary 2.9 and Proposition 6.6 imply that it maps the relations in Definition 0.1 to the zero element of $(\Omega_{R,I}/\delta I)^{\text{op}}$. By the same reasoning as in §7 of [C] one can lift the resulting map to $K_{2,L}(R, I)^{\text{op}}$.

References

[C] F. Clauwens, The K -groups of λ -rings, Part I, Construction of the logarithmic invariant, *Comp. Math.* 61 (1987) 295–328.
 [D] B. Ditters, Fonctions lexoides et produits euleriens, *C.R. Acad. Sci. Paris* 276 (1973) 531–534.
 [H] M. Hazewinkel, Formal groups and Applications, Academic Press, New York (1978).
 [K] F. Keune, The relativisation of K_2 , *J. of Alg.* 54 (1978) 159–177.

- [LQ] J.-L. Loday and D. Quillen, Cyclic homology and the Lie algebra homology of matrices, *Comm. Math. Helv.* 59 (1984) 565–591.
- [MS] H. Maazen and J. Stienstra, A presentation for K_2 of split radical pairs, *J. of Pure and Appl. Algebra* 10 (1977) 271–294
- [S] J. Stienstra, Cartier-Dieudonné theory for Chow groups, *J. Reine Angew. Math.* 355 (1985) 1–66.