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Generalized Dirichlet series and *b*-functions

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Introduction

The point of this paper is to indicate how recent results of Cassou-Nogues [3, 4] and Sargos [8], concerning poles of meromorphic continuations of "generalized" Dirichlet series of the form

$$\mathscr{D}_P(s, \varphi) = \sum_m \varphi(m)/P(m)^s \tag{0.1}$$

can be extended by means of a b-function at infinity which can be associated to the data of the polynomial $P(z_1, \ldots, z_n)$ and "test" function φ. (Implicitly, the summation in (0.1) is taken only over the lattice points m in $\mathbf{N}^n$ or $\mathbf{Z}^n$ at which $P(m) \neq 0$.)

In their work, the polynomial P must satisfy the positivity condition that the real part of each coefficient of P is positive. This is for technical reasons, as discussed in Section 1. Moreover, it also allows use of the principal branch of the logarithm to define the quantity $P(m)^s$. On the other hand, it suffices for our purposes to impose a simple growth condition on Re (P) at infinity (cf. (2.1), (2.6)).

The essential idea here is to exploit the Cauchy residue theory, as done by Sargos, but to do so in a convenient conical neighborhood Γ of the divisors at infinity in the compactification $\mathbf{C}^n \hookrightarrow (\mathbf{P}^1\mathbf{C})^n$. Then (0.1) is written as a finite sum of integrals $I_\sigma(s, \varphi)$. The integrand is given by an expression $R^s \psi_\varphi E/(x_1 \ldots x_n)^{N+2}\, \mathrm{d}x_1 \ldots \mathrm{d}x_n$ where:

i) $R(x_1, \ldots, x_n) = 1/P(1/x_1, \ldots, 1/x_n)$ and the principal branch of $\log P$ also determines R^s.
ii) N characterizes the order of growth of φ at infinity inside Γ.
iii) E is the summatory kernel converting lattice points to simple poles.
iv) ψ_φ is a bounded holomorphic function in Γ.

The b-functions are minimal monic non-zero polynomials $b_N(s)$ for which a functional equation is shown to hold for the expression $R^s(1/x_1 \ldots x_n)^{N+2}$, considered as a generator for a suitable module over the Weyl algebra, corresponding to the chart at infinity (cf. Proposition 3).

Thus, if $\mathscr{B}_N = \{\lambda - n: b_N(\lambda) = 0, n = 0, 1, 2, \ldots\}$, the main result of this paper is the following.

THEOREM 1: *If P satisfies the growth condition (2.1), then the poles of (0.1) are contained in $\mathscr{B}_N$.*

The precise connection between poles of the Dirichlet series and roots of the b-function to be introduced here is similar to that encountered in studying the poles of the generalized function $|P|^{2s}$ and their relation to the roots of the standard b-function for P [1]. Note however that much subsequent work extending [1] (works of Barlet, Loeser, Malgrange) has exploited a cohomology theory that is not yet available in this subject.

Section 1 briefly describes the work in [3, 4, 8]. Sections 2–4 describe the meromorphic continuation of (0.1). Section 5 considers a class of real polynomials which satisfy a positivity condition on their coefficients. Theorem 2 characterizes the largest pole of $\mathscr{D}_P(s, \varphi)$ in terms of the polyhedron and is reminiscent of a theorem of Varcenko [11].

Indeed it states, for φ a non-zero constant (a generalization to φ a monomial is also proved but the statement is slightly more technical to state here)

THEOREM 2: *Let $\Gamma_\infty(P)$ be the polyhedron of P at infinity (cf. (5.1)). Assume that P is a real polynomial satisfying the positivity condition and that condition (5.9) holds. Let t_0 be the value of t at which the diagonal $y_1 = \ldots = y_n = t$ intersects $\Gamma_\infty(P)$. Then $1/t_0$ is the largest pole of (0.1).*

REMARK: Sargos has independently proved a more general version of this theorem [10]. On the other hand, we obtain (5.14) a more precise expression for the polar part of $\mathscr{D}_P(s, \varphi)$ at any possible pole, not just the largest. This might be of subsequent interest.

In a second paper on this subject, $\mathscr{D}$ module techniques are used to obtain further information about the poles and their relation to monodromy at infinity.

Section 1

This section briefly summarizes the works of Cassou-Nogues and Sargos.

The techniques used in [3, 4] are based on a theorem of Mellin. As such there is an assumption required on P called the

(+) **condition**: the real part of all non-zero coefficients of the complex polynomial $P(z_1, \ldots, z_n)$ are positive.

The statement of Mellin's theorem can be conveniently split into two parts.

THEOREM A. *Let* $P(z_1, \ldots, z_n) \in C[z_1, \ldots, z_n]$ *satisfy the* (+) *condition. Let* $P(z_1, \ldots, z_n) = \Sigma_{j=1}^{R} a_{I_j} z^{I_j}$.

Let $\theta_1, \ldots, \theta_R$ *be positive numbers. Then, for* $\operatorname{Re}(s) > \theta_1 + \cdots + \theta_R$, *and* $z_i \in (1, \infty)$ *for* $i = 1, \ldots, n$, *one has*

$$\frac{\Gamma(s)}{P(x)^s} = \frac{1}{(2\pi i)^R} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} \cdots \int_{\theta_R - i\infty}^{\theta_R + i\infty} \frac{\Gamma(s - (y_1 + \cdots + y_R))}{a_{I_0}^{s-(y_1+\cdots+y_R)}} \times \frac{\Gamma(y_1)}{a_{I_1}^{y_1}} \cdots \cdot \frac{\Gamma(y_R)}{a_{I_R}^{y_R}} \frac{\mathrm{d}y_1 \ldots \mathrm{d}y_R}{z_1^{u_1} \ldots z_n^{u_n}} \tag{1.1}$$

where, if $I_j = (i_{1,j}, \ldots, i_{n,j})$, $j = 1, \ldots, R$, *one defines* $u_j = i_{j1}(s - y_1 - \cdots - y_R) + i_{j,2}y_2 + \cdots + i_{j,R}y_R$ *for* $j = 1, \ldots, n$. ■

Now, set $S_j(s) = \int_1^\infty z_j^{-s} \, \mathrm{d}z_j$ and $\Sigma_{m_j=1}^{\infty} m_j^{-s} = \zeta(s)$.

Integrating (1.1) with respect to $\mathrm{d}z_1 \ldots \mathrm{d}z_n$ and changing the order of integration (always with $\operatorname{Re}(s) > \theta_1 + \cdots + \theta_R$) gives the following two identities.

$$\Gamma(s) \int_1^\infty \cdots \int_1^\infty P(z_1, \ldots, z_n)^{-s} \, \mathrm{d}z_1 \ldots \mathrm{d}z_n = \frac{1}{(2\pi i)^R} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} \cdots \int_{\theta_R - i\infty}^{\theta_R + i\infty} \frac{\Gamma(s - y_1 - \cdots - y_R)}{a_{I_0}^{s - y_1 - \cdots - y_R}} \cdots \cdot \frac{\Gamma(y_R)}{a_{I_R}^{y_R}} S_1(u_1) \ldots S_n(u_n) \, \mathrm{d}y_1 \ldots \mathrm{d}y_R \tag{1.2}$$

and

$$\Gamma(s) \sum_{m_1, \ldots, m_n \geqslant 1}^{\infty} \frac{1}{P(m_1, \ldots, m_n)^s} = \frac{1}{(2\pi i)^R} \int_{\theta_1 - i\infty}^{\theta_1 + i\infty} \cdots \int_{\theta_R - i\infty}^{\theta_R + i\infty} \frac{\Gamma(s - (y_1 + \cdots + y_R))}{a_{I_0}^{s-(y_1+\cdots+y_R)}} \cdots \frac{\Gamma(y_R)}{a_{I_R}^{y_R}} \zeta(u_1) \ldots \zeta(u_n) \, \mathrm{d}y_1 \ldots \mathrm{d}y_R. \tag{1.3}$$

Similar expressions are obtained if one includes a function, say a polynomial, φ, multiplying the P^{-s} term.

One knows the analytic continuation of $S_j(s)$ and $\zeta(s)$ to contain a single pole at $s = 1$. Thus the possible poles of $S_j(u_j)$ resp. $\zeta(u_j)$ are given by the union of the loci $\{u_j(s, z_1, \ldots, z_R) = 1\}$.

For fixed $z_1, \ldots, z_R$, the locus consists of only finitely many points in any vertical strip of finite width contained in the s-plane. By using this observation and repeating it R times, Mellin was able to show

THEOREM B. *The left hand sides (1.2) and (1.3) are meromorphic functions of s with at most finitely many poles in any vertical strip of finite width. In addition, since for any such vertical strip it is the case that for any $\varepsilon > 0$ $e^{-\varepsilon|s|}S_j(s) \to 0$ (resp. $e^{-\varepsilon|s|}\zeta(s) \to 0$) as $|s| \to \infty$, s inside the strip, it follows that this same "sub-exponential decay" property holds for the left hand sides of (1.2), (1.3).*

What Mellin did not do was determine and thus be able to compare the actual set of poles for the left hand sides of (1.2), (1.3).

This was done 82 years later in the case of $n = 2$ by Cassou-Nogues. Her main result [4] can be summarized as

THEOREM C. *When $n = 2$, the set of possible poles of the left hand side of (1.3) is contained in the set of possible poles of the left hand side of (1.2).*

In fact, the relation between the two sets of poles can be made much more precise.

Sargos [10] has extended this to the case $n > 2$ via a certain Newton polyhedron at infinity (cf. Section 5 for the definition).

Instead of using Mellin's theorem and (1.1), Sargos uses a summatory formula based on Cauchy residue theory.

In one variable this is expressed as follows. Define the region $\Gamma_\theta = a + \{z \in \mathbb{C}: |\mathrm{Arg}(z)| \leqslant \theta\}$ where $a \in (0, 1)$ and $0 \leqslant \theta \leqslant \pi/2$. Set $e(z) = e^{2\pi i z}$. Then, if $f(z)$ is holomorphic in an open set containing the region Γ_θ and f satisfies a bound of the form

$$|f(z)| \leqslant C|z|^{-1-\varepsilon} \quad \text{for all} \quad z \in \Gamma_\theta, \tag{1.4}$$

where $\varepsilon > 0$ and C are independent of z, one has

$$\sum_{m=1}^{\infty} f(m) = \int_{\partial\Gamma_\theta} \frac{f(z)}{e(z) - 1}\, dz. \tag{1.5}$$

In n-variables, (1.4) is extendable if $f(z_1, \ldots, z_n)$ is holomorphic in Γ_θ^n, for some θ, and satisfies a growth property (C, ε independent of z)

$$|f(z_1, \ldots, z_n)| < C|z_1 \ldots z_n|^{-1-\varepsilon} \quad \text{for all} \quad z \in \Gamma_\theta^n. \tag{1.6}$$

If (1.6) is satisfied, then

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} f(m_1, \ldots, m_n) = \int_{\partial\Gamma_\theta} \cdots \int_{\partial\Gamma_\theta} \frac{f(z)}{\prod_{j=1}^{n} (\mathrm{e}(z_j) - 1)} \mathrm{d}z_1 \ldots \mathrm{d}z_n.$$

This is useful by the

LEMMA (1). *When P satisfies the $(+)$ condition there is a $\theta \in (0, \pi/2)$ so that $f(z) = (1/P(z))^s$ is single valued and $1/P(z)$ satisfies (1.6) for some $\varepsilon > 0$ and all $z \in \Gamma_\theta^n$.*

Proof: cf [8, Lemma 5.1]

REMARK: In fact more is proved in [8]. Given any rational function $R(z) = Q(z)/P(z)$, where Q and P satisfy the $(+)$-condition, and given any monomial $z_1^{u_1} \ldots z_n^{u_n}$, $u_j \geqslant 1$ for each j, there exist positive numbers α, θ and ϱ, $\varrho < \pi/2$, so that

$$\max\{\mathrm{Arg}(Q(z)), \quad \mathrm{Arg}(P(z))\} \leqslant \varrho, \tag{1.7)(i}$$

and

$$|R(z)| > C|z_1^{u_1} \ldots z_n^{u_n}|^{\alpha} \tag{ii}$$

for all $z = (z_1, \ldots, z_n) \in \Gamma_\theta^n$. This is generalized in Section 2.

To perform the analytic continuation, one first decomposes the oriented n-chain $\partial\Gamma_\theta \times \cdots \times \partial\Gamma_\theta = \Sigma_\sigma(-\sigma(1))\gamma_{\sigma(1)} \times \cdots \times (-\sigma(n)) \cdot \gamma_{\sigma(n)}$ where

i) σ: $\{1, \ldots, n\} \to \{+, -\}$ is a choice function
ii) $\gamma_+ = \partial(a + \{z \in \Gamma_\theta\colon \mathrm{Arg}\,(z) = \theta\})$
$\gamma_- = \partial(a + \{z \in \Gamma_\theta\colon \mathrm{Arg}\,(z) = -\theta\})$.

Set $\Gamma_\sigma = \gamma_{\sigma(1)} \times \cdots \times \gamma_{\sigma(n)}$.

Define the functions, on $|\gamma_+|$ resp. $|\gamma_-|$ ($|*|$ denotes the support of the

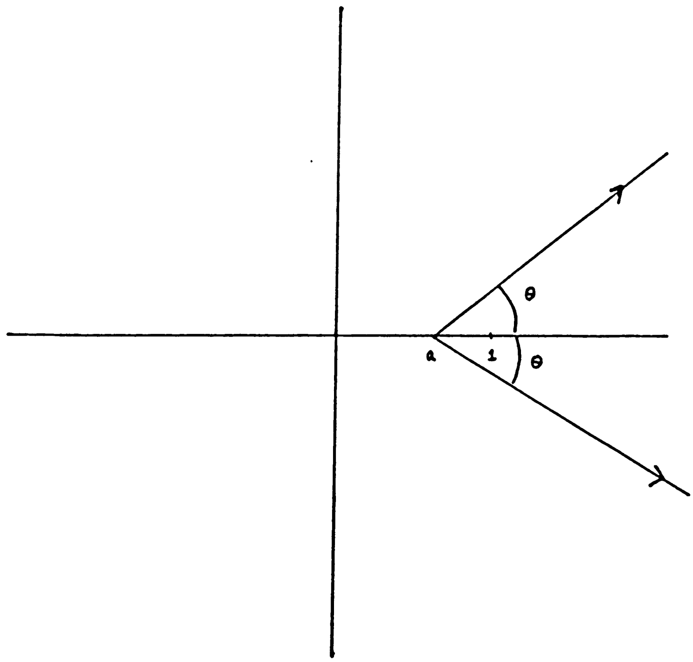

Fig. 1.

chain $*$),

$$E_+(z_j) \;=\; \left.\frac{1}{e(z_j) - 1}\right|_{|\gamma_+|} \;=\; \frac{e(z_j)}{e(z_j) - 1} - 1$$

$$E_-(z_j) \;=\; \left.\frac{1}{e(z_j) - 1}\right|_{|\gamma_-|} \;=\; \frac{e(-z_j)}{1 - e(-z_j)}.$$

These satisfy the crucial convergence properties:
There exists $\delta > 0$ so that

$$E_+(z_j) + 1 \;=\; O(e^{-|z_j|^\delta}) \quad \text{as} \quad |z_j| \to \infty \tag{1.8}$$

$$E_-(z_j) \;=\; O(e^{-|z_j|^\delta}) \quad \text{as} \quad |z_j| \to \infty.$$

For given choice function σ set

$$N(\sigma) \;=\; \#\{j\colon \sigma(j) = +\}$$

$$\mathrm{sgn}(\sigma(j)) \;=\; \begin{Bmatrix} +1 & \text{if} & \sigma(j) & = & + \\ -1 & \text{if} & \sigma(j) & = & - \end{Bmatrix}$$

and define

$$\mathscr{E}_\sigma(z_1, \ldots, z_n) = (-1)^{N(\sigma)} \prod_{j=1}^{n} E_{\sigma(j)}(z_j) \tag{1.9}$$

for $z \in \Gamma_\sigma$.

Then, one has for $f(z)$ satisfying (1.6),

$$\int_{\partial\Gamma_\theta} \cdots \int_{\partial\Gamma_\theta} \frac{f(z_1, \ldots, z_n)}{\prod_1^n (\mathrm{e}(z_j) - 1)} \mathrm{d}z_1 \ldots \mathrm{d}z_n = \sum_\sigma \int_{\Gamma_\sigma} f(z)\mathscr{E}_\sigma(z)\, \mathrm{d}z_1 \ldots \mathrm{d}z_n. \tag{1.10}$$

Using a detailed combinatorial argument (similar to that used in section (5)) to find a "largest monomial" in a Newton polyhedron at infinity for P) Sargos was able to determine the analytic continuation of each integrand in the right side of (1.10) when $f(z) = P(z)^{-s} \cdot z_1^{a_1} \ldots z_n^{a_n}$, $a_n \in N \cup \{0\}$. His result was

THEOREM D. *For P satisfying the $(+)$ condition there is a rational number σ and integer N so that the poles of the analytic continuation of (0.1) (with $\varphi(z) = z_1^{a_1} \ldots z_n^{a_n}$) are contained in the set $\{\sigma - u/N\colon u = 0, 1, 2, \ldots\}$.*

Section 2

There are three parts to the proof of Theorem 1, distributed across the next three sections. Section 2 contains the analytic but preliminary part. Here the growth condition on P is defined and used to establish a summatory formula valid in a halfplane of analyticity between a-tail-of (0.1) and a finite sum of values of "generalized currents" (cf. (2.10)). This establishes the analog of (1.10). Section 3 gives an algebraic derivation of the functional equation that will lead to the proof of the theorem in section 4. Note that in the following we will treat the series (0.1) where the summation is over the tuples m in $\mathbf{N}^n$ at which $P(m) \neq 0$. An obvious extension to the case of $\mathbf{Z}^n$ is left to the reader.

The basic idea in this paper is to use the coordinate inversion $z_i = 1/x_i$, $i = 1, \ldots, n$, to analyze each of the integrals of the summatory formula (2.10). In this way, one thinks of $\mathbf{C}^n$ with coordinates $(z_1, \ldots, z_n)$, denoted $\mathbf{C}^n(z)$, as being compactified via the inclusion $\mathbf{C}^n \hookrightarrow (\mathbf{P}^1\mathbf{C})^n$, and one works in the coordinate chart $\mathbf{C}^n(x)$ with overlap equations as above. The divisors D_i which one adds to $\mathbf{C}^n(z)$ to compactify it are evidently defined in $\mathbf{C}^n(x)$ by $x_i = 0$, $i = 1, \ldots n$. On the other hand for certain polynomials, toroidal compactifications may be more useful in that they allow one to describe the first pole of (0.1) in terms of a combinatorial object, the Newton polyhedron of P at infinity. This is the idea behind Section 5 and [10].

In the following the norm on any copy of $\mathbf{C}^n$ is defined in this way. If, for example, $z = (z_1, \ldots, z_n)$ then $\|z\| = \max\{|z_i|\}$. The growth condition on Re (P) is the following.

There exists a $B \in (0, 1)$ such that (with $z_j = x_j + iy_j$ for each j)

For each differential monomial $D^A = D_{x_1}^{A_1} D_{y_1}^{A_2} \ldots D_{x_n}^{A_{2n-1}} D_{y_n}^{A_{2n}}$,

$$\lim_{\substack{\|x\| \to \infty \\ x \in [B, \infty)^n}} |D^A \operatorname{Re}(P)(x) / \operatorname{Re}(P)(x)| = 0 \tag{2.1}$$

This is the hypoellipticity condition on $[B, \infty)^n$ of Hörmander [pg. 99,6]. Thus, one concludes from the proof of lemma 4.1.1 [6] that there exists a positive rational number ε so that the function

$$\theta: [B, \theta) \to \mathbf{R} \quad \theta(x) = x^\varepsilon$$

defines a region $\Gamma(\theta)$

$$\Gamma(\theta) = \{z = x + iy: |y| \leqslant \theta(x - B), x \geqslant B\}, \tag{2.2}$$

such that Re(P) satisfies the growth condition

$$\lim_{\substack{\|z\| \to \infty \\ z \in \Gamma(\theta)^n}} |\operatorname{Re}(P)(z)| = +\infty. \tag{2.3}$$

In the following, we assume $\varepsilon \leqslant 1$. From (2.3) one concludes

PROPOSITION 1: *If (2.1) is true, then there exist D, c, c', $\alpha > 0$ such that*

$$|\operatorname{Re}(P)(z)| \geqslant c\|z\|^\alpha \geqslant c'|z_1 \ldots z_n|^{\alpha/n} \tag{2.4}$$

for all $z \in \Gamma(\theta)^n$ with $\|z\| \geqslant D$.

Note: We subsequently assume that D is not integral and $D > B$. This is without loss of generality.

Proof: This follows from the argument presented in [6, appendix]. Since $\Gamma(\theta)^n$ is a semialgebraic set the function

$$\mu(r) = \inf \{|\mathrm{Re}\,(P)(z)|^2 : \|z\| = r \quad \text{and} \quad z \in \Gamma(\theta)^n\}$$

has the form

$$\mu(r) = cr^{\alpha}(1 + O(1)) \quad \text{as} \quad r \to \infty.$$

Evidently, (2.3) implies that α must be positive. This shows (2.4). ∎

EXAMPLE: $P(z_1, z_2) = (z_1 - z_2)^2 + z_1$ is a real hypoelliptic polynomial that does not satisfy the non-degeneracy conditions in Section 5.

Setting $(x_1, \ldots, x_n) = (1/z_1, \ldots, 1/z_n)$ (abbreviated as $x = 1/z$ below), (2.4) implies

$$|P(1/x)| \geqslant c\|1/x\|^{\alpha} \geqslant c'|1/x_1 \ldots x_n|^{\alpha/n}.$$

Thus, if $R(x) = 1/P(1/x)$ and $x = 1/z$ with $z \in \Gamma(\theta)^n \cap \{\|z\| \geqslant D\}$, one concludes that constants C, C' exist so that

$$|R(x)| \leqslant C(1/\|1/x\|)^{\alpha} = C(\inf |x_i|)^{\alpha} \leqslant C'|x_1 \ldots x_n|^{\alpha/n}. \tag{2.5}$$

NOTATION: Because θ is fixed in the discussion we subsequently denote $\Gamma(\theta)^n$ by Γ^n. Set $\Gamma^n_D = \{\|z\| \geqslant D\} \cap \Gamma(\theta)^n$.

(2.6) It follows that in the region Γ^n_D a single valued branch for log P exists and is used to define P^s and $(1/P)^s$ in Γ^n_D as single valued holomorphic functions. Inded, because Re (P) satisfies the condition (a) Re $(P)(z) > 0$ for each $z \in \Gamma^n_D$ or (b) Re $(P)(z) < 0$ for each $z \in \Gamma^n_D$, it follows that if (a) holds then the principal value of the logarithm is used to define not only each term $P(m)^s$ in (0.1) but also the functions P^s, $(1/P)^s$ in Γ^n_D. If (b) holds, then one understands $P(m)^s$ to mean the quantity $(-1)^s(-P(m))^s$ (where Arg $(-1) = \pi$) and uses the principal branch of log to define $(-P(m))^s$. Similarly, one defines the functions P^s, $(1/P)^s$ in Γ^n_D by the identities $P(z)^s = (-1)^s(-P(z))^s$, $(1/P(z))^s = (-1)^s(-1/P(z))^s$. As such, all the analysis will be done in Γ^n_D. Thus, with these conventions, our considerations will apply to the "tail" of the series, defined as

$$\mathscr{D}'_P(s, \varphi) = \sum_{\|m\| \geqslant D} \varphi(m)/P(m)^s. \tag{2.7}$$

Because only finitely many terms from (0.1) are lost in (2.7), all the information about the poles of the meromorphic continuation of (0.1) is preserved in (2.7). To proceed, the summatory formula (1.10) needs to be adapted to the situation here.

Let $A = \mathbf{Z} \cap [B, D] = \{1, 2, \ldots, a\}$. For each $i \in A$ set $\gamma(i)$ to be a small circle oriented counterclockwise and centered at i in the plane. The radius of each circle is the same and is taken to be less than 1/2.

Now set $C = (D - B, \ldots, D - B) \in \mathbf{C}^n$ and consider the translation $C + \Gamma^n$. Evidently, $C + \Gamma^n \subset \Gamma_D^n$. One can express $\partial(C + \Gamma^n)$ as follows. In one coordinate plane, one has

$$\partial(C + \Gamma) = \gamma_-(D) - \gamma_+(D)$$

where

$$\gamma_-(D) = \{z = x + iy\colon y = -\theta(x - D), x \geqslant D\}$$

$$\gamma_+(D) = \{z = x + iy\colon y = \theta(x - D), x \geqslant D\}.$$

One orients these chains by increasing x.

Then let $2^n = \{\sigma\colon\{1, \ldots, n\} \to \{+,-\}\}$. One has

$$\partial(C + \Gamma^n) = \sum_{\sigma \in 2^n} (-\sigma(1))\gamma_{\sigma(1)}(D) \times \cdots \times (-\sigma(n))\gamma_{\sigma(n)}(D).$$

Let $\mathscr{C} = \{\gamma(1), \ldots, \gamma(a), -\gamma_+(D), \gamma_-(D)\}$ be a finite collection of oriented arcs in the complex plane. Define

$$\mathscr{T} = \{\sigma\colon\{1, \ldots, n\} \to \mathscr{C}\colon \text{at least one } \sigma(u) \text{ is an unbounded arc}\}.$$

For $\sigma \in \mathscr{T}$ set

$$\Delta_\sigma = \sigma(1) \times \cdots \times \sigma(n).$$

Remark: We do not distinguish notationally between the unbounded chain Δ_σ in the chart $\mathbf{C}^n(z)$ and the compact chain $\bar{\Delta}_\sigma$ in $\mathbf{C}^n(x)$ unless the context does not make clear which is being considered.

The collection $\{\Delta_\sigma\}_{\sigma \in \mathscr{T}}$ is the finite set of oriented n-chains replacing the set $\{\Gamma_\sigma\}$ in (1.10). Observe that the supports of the Δ_σ are mutually pairwise disjoint. Set

$$\Delta = \sum_{\sigma \in \mathscr{T}} \Delta_\sigma .$$

Next, define for each $\sigma \in \mathscr{T}$

$$E_{\sigma(j)}(z_j) = \begin{cases} 1/(\mathrm{e}(z_j) - 1) & \text{if } \sigma(j) \text{ is compact} \\ \dfrac{\mathrm{e}(z_j)}{\mathrm{e}(z_j) - 1} - 1 & \text{if } \sigma(j) = -\gamma_+(D) \\ \dfrac{\mathrm{e}(-z_j)}{1 - \mathrm{e}(-z_j)} & \text{if } \sigma(j) = \gamma_-(D) \end{cases}$$

and

$$\mathscr{E}_\sigma(z) = \prod_{j=1}^{n} E_{\sigma(j)}(z_j).$$

Remark: The class of polynomials P satisfying (2.1) is exactly the class for which the summatory formula (cf. 2.10)) can be obtained via Cauchy residue theory using the product $\Pi(1/(\mathrm{e}(z_j) - 1)$. This is because the function $1/(\mathrm{e}(z) - 1)$ resp. $[1/(\mathrm{e}(z) - 1)] + 1$ is integrable over the arc $\gamma_-(D)$ resp. $\gamma_+(D)$, formed from the function θ, iff the exponent ε in the definition of θ is positive. Indeed, one has that

$$1/(\mathrm{e}(z) - 1)|_{\gamma_-(D)} = o(\mathrm{e}^{-2\pi x^\varepsilon}) \quad \text{as} \quad x \to \infty$$

and

$$1/(\mathrm{e}(z) - 1)|_{\gamma_+(D)} = -1 + o(\mathrm{e}^{-2\pi x^\varepsilon}) \quad \text{as} \quad x \to \infty.$$

In order to define generalized currents associated to each chain Δ_σ, it is first necessary to define the space of test functions $\mathscr{F}$. This is done for each $N \geqslant 0$ as follows.

Definition: Given $N \in \mathbf{N}$, let

$$\mathscr{F}_N = \{\varphi \in C^\infty(\mathbf{C}^n, \mathbf{C}):$$

i) Supp (φ) contains an open neighborhood of $|\Delta|$ in $\mathbf{C}^n(z)$, in which φ is holomorphic

ii) $|\varphi(z)| = O(|z_1 \ldots z_n|^N)$ as $|z| \to \infty$, $z \in |\Delta|$

iii) In an open neighborhood $\mathscr{V}$ of $|\Delta|$ in $\mathbf{C}^n(x)$, there exists a bounded holomorphic function $\psi(x_1, \ldots, x_n)$ such that

$$\varphi(1/x_1, \ldots, 1/x_n) = \psi(x_1, \ldots, x_n)/(x_1 \ldots x_n)^N$$

for all points $(x_1, \ldots, x_n) \in \mathscr{V} - \cup D_i\}$.

Set $\mathscr{F} = \cup \mathscr{F}_N$. One notes that $\mathscr{F}$ contains the ring of rational functions in $(z_1, \ldots, z_n)$. For most purposes, it probably suffices to work with this ring.

For $\sigma \in \mathscr{T}$, set

$$I_\sigma(s, \varphi) = \int_{\Delta_\sigma} (1/P)^s \varphi \mathscr{E}_\sigma \, \mathrm{d}z \quad (\mathrm{d}z = \mathrm{d}z_1 \ldots \mathrm{d}z_n). \tag{2.8}$$

One now shows that there exists a halfplane of analyticity.

PROPOSITION 2: *If P satisfies (2.1), then for each N there exists $B(N)$ such that if* $\mathrm{Re}(s) > B(N)$ *then $I_\sigma(s, \varphi)$ is analytic in s for all $\varphi \in \mathscr{F}_N$ and each $\sigma \in \mathscr{T}$.*

Proof: Let $s = \omega + i\tau$.

a) If $\mathrm{Re}\,(P) > 0$ on Γ_D^n then $|P(z)^{-s}| = |P(z)|^{-\omega} e^{\tau \arg(P(z))}$

b) If $\mathrm{Re}\,(P) < 0$ on Γ_D^n then $|P(z)^{-s}| = |P(z)|^{-\omega} e^{\tau(\arg(P(z)-\pi)}$,

where $\arg\,(P)$ is determined according to (2.6) and is therefore uniformly bounded over Γ_D^n. Now if K is a compact set in $\{\mathrm{Re}\,(s) > B(N)\}$ there exist $C_K > 0$ such that for all $z \in \Gamma_D^n$

$$e^{\tau \arg(P(z))} < C_K.$$

Thus, for $s \in K$ there exists $C_K' > 0$ such that for all $z \in |\Delta_\sigma|$ and each σ

$$|P(z)^{-s}||\varphi \mathscr{E}_\sigma(z)| < C_K'|z_1 \ldots z_n|^{N-\alpha\omega/n}. \tag{2.9}$$

Clearly, $\int_\Delta |z_1 \ldots z_n|^{N-\alpha\omega/n} \mathrm{d}z$ converges and is analytic in s if $N - \alpha\omega/n < -1 - \varepsilon$ for some $\varepsilon > 0$. Choosing $\varepsilon = 1$ and setting $B(N) = [\![n(N + 2)/\alpha]\!]$ suffices to prove the proposition. ■

Thus, if $\varphi \in \mathscr{F}_N$, we obtain an equality between two analytic functions in

Re $(s) > B(N)$

$$\mathscr{D}'_P(s, \varphi) = \sum_{\sigma \in \mathscr{T}} I_\sigma(s, \varphi). \tag{2.10}$$

(2.11) REMARK: Note that the branch for log P used to define the integrand in each $I_\sigma(s, \varphi)$ is the same (cf. (2.6)) as that used to define each term $P(m)^s$. However, if one only imposed the growth condition (2.1) on P, not Re (P), then one could not insure that. Indeed, it would not be so easy to specify a piori one branch of log with respect to which each $P(m)^s$ is defined ans so that (2.10) is an identity between functions of s. This unpleasant prospect forces (2.1) to hold for Re (P).

Section 3

For a polynomial P the existence of a b-function $b(s) \neq 0$ for which there is a functional relation

$$\mathscr{P}(s, x, D_x)P^{s+1} = b(s)P^s \tag{3.1}$$

was proved by Bernstein using purely algebraic techniques [1]. For the reader's convenience this is now briefly summarized.

Let K be a field of characteristic zero. It is not necessary to force K to be algebraically closed but is useful to assume that K is uncountable in cardinality.

Let $\mathscr{D}_n(K) = K[x_1, \ldots, x_n, D_{x_1}, \ldots, D_{x_n}]$ be the Weyl algebra over K [2]. The D_{x_i}, x_j satisfy the relations

$$[x_j, D_{x_i}] = -\delta_{ij}$$

$$[x_i, x_j] = [D_{x_i}, D_{x_j}] = 0.$$

Let

$$\mathscr{F}_j = \{P \in \mathscr{D}_n(K): P = \sum_{|I|+|J| \leqslant j} \alpha_{I,J} x^I D_x^J, \alpha_{I,J} \in K\}$$

be the filtration of $\mathscr{D}_n(K)$ by total order of the operators.

A filtration on a left $\mathscr{D}_n(K)$ module $\mathscr{N}$ is a sequence of finite dimensional

K vector spaces

$$\mathcal{N}_0 \subset \mathcal{N}_1 \subset \ldots$$

so that

i) $\bigcup_{j=0}^{\infty} \mathcal{N}_j = \mathcal{N}$

ii) $\mathcal{F}_j \mathcal{N}_k \subset \mathcal{N}_{k+j}$ for all j, k. (3.2)

A left $\mathcal{D}_n(K)$ module $\mathcal{N}$ is holonomic [2] if $\mathcal{N}$ admits a filtration $\{\mathcal{N}_j\}_0^{\infty}$ such that there are positive integers c, c' so that

$$\dim_K \mathcal{N}_j \leqslant cj^n + c'(j+1)^{n-1} \quad (3.3)$$

holds for all j.

The most basic examples of such modules are the following two. Here, P will be a polynomial in $K[x_1, \ldots, x_n]$ with $\deg P = d$.

A) Set $\mathcal{M} = K[x, 1/P]$ to be the ring of fractions. Define $D_{x_i}(q/P^j)$ via the standard quotient rule formula.

Set $\mathcal{M}_j$ to be the K vector space generated by the elements

$$\frac{q}{P^j} \quad \text{with} \quad \deg(q) \leqslant j \cdot (d+1).$$

Then $\{\mathcal{M}_j\}$ is a filtration of $\mathcal{M}$ satisfying (3.3). So, $\mathcal{M}$ is a holonomic left $\mathcal{D}_n(K)$ module.

B) Let s be a transcendental over K. Let P^s be a generator of a module $\mathcal{N}$ over $K(s)[x_1, \ldots, x_n, 1/P]$. Define a left $\mathcal{D}_n(K(s))$ action on $\mathcal{N}$ where the main ring action is given by the rule

$$D_{x_j}(gP^s) = \left(D_{x_j}(g) + sg \frac{D_{x_j}(P)}{P} \right) P^s.$$

Set $\mathcal{N}_j$ to be the $K(s)$ vector space generated by the elements

$$\frac{g}{P^j} \cdot P^s \quad \text{with} \quad \deg(g) \leqslant j \cdot (d+1).$$

Then $\{\mathcal{N}_j\}$ is a filtration of $\mathcal{N}$ satisfying (3.3).

From the finiteness of the length of $\mathcal{N}$ follows the existence of a polynomial $b(s) \neq 0$ so that (3.1) is satisfied. One need only consider the necessarily stabilizing sequence of submodules $\mathcal{T}_i = \mathcal{D}_n(K(s)) \cdot (P^i \cdot P^s)$.

One can extend B) to any rational function $R = Q/P$ in an evident way. Of interest here however, is a modification of B) which concerns both R^s and another polynomial T. Thus, we show

PROPOSITION 3: Let $R = Q/P$ resp. T *be a rational function resp. polynomial in* $x_1, \ldots, x_n$. *Then for each integer* N *there exist a non-zero polynomial* $b_N(s)$ *and elements* $\mathcal{P}_i(s, x, D_x)$, $i = 0, \ldots m$, *of* $\mathcal{D}_n(K[s])$ *so that*

$$\Sigma_0^m \mathcal{P}_i(T^{-N}R^{s+i+1}) = b_N(s)(T^{-N}R^s). \tag{3.4}$$

Proof: Fix $\mathcal{R}_0 = K(s)[x_1, \ldots, x_n, 1/PQT]$. Set $\mathcal{N}_0$ to be the free $\mathcal{R}_0$ module of rank one generated by R^s. Define a left $\mathcal{D}_n(K(s))$ action on $\mathcal{N}_0$ as follows.

$$D_{x_l}[(g/(PQT)^j)R^s] = \{D_{x_l}(g/(PQT)^j)$$

$$+ s(gT/(PQT)^{j+1})[PD_{x_l}(Q) - QD_{x_l}(P)]\}R^s$$

One checks that it is holonomic by using the filtration of $K(s)$ vector spaces $\mathcal{N}_0(j)$ generated by

$$\{(g/(PQT)^j)R^s: \deg g \leqslant j[\deg P + \deg Q + \deg T + 1]\}.$$

Let $\mathcal{M}_N$ be the $\mathcal{D}_n(K(s))$ submodule generated by $K(s)[x_1, \ldots, x_n, R](T^{-N}R^s)$. It is therefore holonomic. Let $\mathcal{M}_N(j)$ be the decreasing filtration of $\mathcal{M}_N$ defined by

$$\mathcal{M}_N(j) = \mathcal{D}_n(K(s))(K(s)[x_1, \ldots, x_n, R](T^{-N}R^{s+j}).$$

Because it stabilizes, there exists a j such that $\mathcal{M}_N(j) = \mathcal{M}_N(j+1)$. This implies there exist $\mathcal{P}_0 \in \mathcal{D}_n(K(s))$ and elements $a_0(s, x), \ldots, a_m(s, x) \in K(s)[x_1, \ldots, x_n]$ such that

$$\mathcal{P}_0(\Sigma_0^m a_i(s, x)(T^{-N}R^{s+j+i+1})) = T^{-N}R^{s+j}.$$

Clearing denominators of polynomials in s and replacing $s + j$ by s yields a functional equation of the type (3.4). The monic generator $b_N(s)$ of the ideal of polynomials in s for which an identity (3.4) holds is called the b-function for $T^{-N}R^s$. One observes that (3.4) is purely algebraic. There is no implication of any analytic significance to (3.4) when $K = \mathbf{R}$ or $\mathbf{C}$. ■

(3.5) REMARK: It is easy to generalize (3.4) as follows.
There exist a non-zero polynomial $b(s_1, s_2)$ and operators $\mathscr{P}_0, \ldots, \mathscr{P}_m$ in $\mathscr{D}_n(K[s_1, s_2])$ such that

$$\Sigma_0^m \mathscr{P}_i(T^{s_1}R^{s_2+i+1}) = b(s_1, s_2)(T^{s_1}R^{s_2}).$$

Evidently, $b_N(s)$ divides $b(-N, s)$ for each integer N.

(3.6) REMARK: A different functional equation involving only polynimials is this.

If $P_1, \ldots, P_k$ are polynomials in $x = (x_1, \ldots, x_n)$, then there exist a non-zero polynomial $b(s_1, \ldots, s_k)$ and operators $\mathscr{P}_1, \ldots, \mathscr{P}_k \in \mathscr{D}_n(K[s_1, \ldots, s_k])$ such that simultaneously, one has

$$\mathscr{P}_i(P_1^{s_1} \ldots P_i^{s_i} \ldots P_k^{s_k}) = b(s_1, \ldots, s_k)P_1^{s_1} \ldots P_i^{s_i} \ldots P_k^{s_k}. \tag{3.7}$$

Proof: Let $s = (s_1, \ldots, s_k)$. One first shows that

$$\mathscr{N}_0 = K(s)[x, 1/P_1 \ldots P_k]P_1^{s_1} \ldots P_i^{s_i} \ldots P_k^{s_k}$$

determines a holonomic $\mathscr{D}_n(K(s))$ module. This is done as in example (A) above. Thus, the k-fold direct sum

$$\mathscr{M}_k = \mathscr{N} \oplus \mathscr{N} \oplus \cdots \oplus \mathscr{N}$$

is also holonomic. Set

$$\mathbf{e_j} = (P_1^{s_1+j}P_2^{s_2+j-1} \ldots P_k^{s_k+j-1},$$
$$P_1^{s_1+j-1}P_2^{s_2+j} \ldots P_k^{s_k+j-1}, \ldots, P_1^{s_1+j-1}P_2^{s_2+j-1} \ldots P_k^{s_k+j}),$$

an element of $\mathscr{M}_k$. Let $\mathscr{M}_k(j)$ denote the $\mathscr{D}_n(K(s))$ submodule generated by $\mathbf{e_j}$. Multiplication by $P_1 \ldots P_k$ shows that $\mathscr{M}_k(j)$ contains $\mathscr{M}_k(j + 1)$ for each j. Thus, there exist j such that $\mathscr{M}_k(j) = \mathscr{M}_k(j + 1) \ldots$. This implies the existence of an element $\mathscr{S} \in \mathscr{D}_n(K(s))$ such that for each i

$$\mathscr{S}(P_1^{s_1+j} \ldots P_i^{s_i+j+1} \ldots P_k^{s_k+j}) = P_1^{s_1+j-1} \ldots P_i^{s_i+j} \ldots P_k^{s_k+j-1}.$$

For each i, set

$$\mathscr{P}_i = \left(\prod_{j \neq i} P_j\right) \mathscr{S}.$$

Evidently, the denominator of each $\mathscr{P}_i$ is the same. Clearing this one denominator from the k identities

$$\mathscr{P}_i(P_1^{s_1+j} \ldots P_i^{s_i+j+1} \ldots P_k^{s_k+j}) = P_1^{s_1+j} \ldots P_i^{s_i+j} \ldots P_k^{s_k+j}$$

and replacing each $s_i + j$ by s_i yields a set of functional equations of the form (3.7).

Section 4

In the chart $\mathbf{C}^n(x)$ for $(\mathbf{P}^1\mathbf{C})^n$, write

$$R(x_1, \ldots, x_n) = 1/P(1/x_1, \ldots, 1/x_n) = T(x_1, \ldots, x_n)/Q(x_1, \ldots, x_n).$$

Let $U_0 = 1/(x_1 \ldots x_n)^2$ be the Jacobean (up to sign) of the overlap equations for $\mathbf{C}^n(x) \cap \mathbf{C}^n(z)$. For each $N \geqslant 1$ set

$$U(N) = 1/(x_1 \ldots x_n)^{N+2}.$$

For $\varphi \in \mathscr{F}_N$ and $\sigma \in \mathscr{T}$, set

$$\Phi(x_1, \ldots, x_n) = \varphi(1/x_1, \ldots, 1/x_n) = \psi(x_1, \ldots, x_n)/(x_1 \ldots x_n)^N$$

$$E_\sigma(x_1, \ldots, x_n) = \mathscr{E}_\sigma(1/x_1, \ldots, 1/x_n). \tag{4.1}$$

From Section 3, there exists a non-zero minimal polynomial $b_N(s)$ and operators $\mathscr{P}_0, \ldots, \mathscr{P}_m$ in $\mathscr{D}_n(\mathbf{C}[s])$ so that

$$\Sigma_{i=0}^m \mathscr{P}_i(R^{s+i+1}U(N)) = b_N(s)R^s U(N). \tag{4.2}$$

The coordinate inversion $x = 1/z$ maps the chains in $\mathscr{C}$ to compact (taking closures of the infinite arcs in $\mathscr{C}$) chains in the x plane. Denote these chains by $\gamma'(1), \ldots, \gamma'(a), \gamma'_+(D), \gamma'_-(D)$. They are characterized by the following orientations and descriptions.

$\gamma'_+(D)$ is an arc in the halfplane Im $(x) \leqslant 0$ from the (initial) point $x = 1/D$ to the (terminal) point $x = 0$.
$\gamma'_-(D)$ is an arc in the halfplane Im $(x) \geqslant 0$ from the point $x = 0$ to the point $x = 1/D$.
Each $\gamma'(j)$ is a loop traversed counterclockwise about $x = 1/j$.

For $\varepsilon > 0$, set S_ε to be the circle of radius ε and center $x = 0$. Define

$$\gamma'_+(\varepsilon) = \text{subarc of } \gamma'_+(D) \text{ from } x = 1/D \text{ to the point on } S_\varepsilon.$$

$$\gamma'_-(\varepsilon) = \text{subarc of } \gamma'_-(D) \text{ from the point on } S_\varepsilon \text{ to } x = 1/D.$$

(4.3) Let $b_{1,j}$ resp. $b_{2,j}$ denote the initial resp. terminal point for $\gamma'_\pm(\varepsilon)$ when considered as lying in the x_j coordinate plane. Set

$$\mathscr{C}_\varepsilon = \{\gamma'(1), \ldots, \gamma'(a), \gamma'_+(\varepsilon), \gamma'_-(\varepsilon)\}$$

where the orientations are as above. Each choice function $\sigma \in \mathscr{T}$ determines a unique choice function σ_ε on $\mathscr{C}_\varepsilon$ by the rule

$$\sigma(i) = \gamma(k) \text{ implies } \sigma_\varepsilon(i) = \gamma'(k),$$

$$\sigma(i) = -\gamma_+ \text{ implies } \sigma_\varepsilon(i) = \gamma'_+(\varepsilon),$$

$$\sigma(i) = \gamma_- \text{ implies } \sigma_\varepsilon(i) = \gamma'_-(\varepsilon).$$

Let $\Delta_\sigma(\varepsilon) = \sigma_\varepsilon(1) \times \cdots \times \sigma_\varepsilon(n)$. It is an oriented n-chain with boundary. For each $\sigma \in \mathscr{T}$ and j for which $\sigma_\varepsilon(j) = \pm\gamma_\pm(\varepsilon)$ write

$$B_{1,j}(\varepsilon, \sigma) = \sigma_\varepsilon(1) \times \cdots \times \sigma_\varepsilon(j-1) \times \{b_{1,j}\} \times \sigma_\varepsilon(j+1) \times \cdots \times \sigma_\varepsilon(n)$$

$$B_{2,j}(\varepsilon, \sigma) = \sigma_\varepsilon(1) \times \cdots \times \sigma_\varepsilon(j-1) \times \{b_{2,j}\} \times \sigma_\varepsilon(j+1) \times \cdots \times \sigma_\varepsilon(n)$$

One then denotes the component in the x_j plane as $b_{1,j}(\sigma)$ resp. $b_{2,j}(\sigma)$. Set

$$\mathscr{J}_\sigma = \{j: \sigma_\varepsilon(j) = \gamma'_\pm(\varepsilon)\}.$$

One has

$$\partial\Delta_\sigma(\varepsilon) = \sum_{j \in \mathscr{J}_\sigma} (-1)^j (B_{1,j}(\varepsilon, \sigma) - B_{2,j}(\varepsilon, \sigma)).$$

To consider the analytic continuation of each $I_\sigma(s, \varphi)$, $\varphi \in \mathscr{F}_N$, one first defines the integrals over the $\Delta_\sigma(\varepsilon)$. Thus, define

$$I_\sigma(\varepsilon, s, \varphi) \;=\; \int_{\Delta_\sigma(\varepsilon)} R^s U(N)(\psi E_\sigma)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n. \tag{4.4}$$

It follows from the definition that for Re $(s) > B(N)$

$$\lim_{\varepsilon \to 0} I_\sigma(\varepsilon, s, \varphi) \;=\; I_\sigma(s, \varphi) \tag{4.5}$$

(The orientation factor $(-1)^n$ has been absorbed into the definition of the chain $\Delta_\sigma(\varepsilon)$).

(4.2) is now applied as follows. For Re $(s) > B(N)$, (4.2) gives an identity with s a complex variable in this halfplane and $x = (x_1, \ldots, x_n)$ a point outside the locus $\{TQ(x_1 \ldots x_n) = 0\}$. Thus, it gives an identity over each $|\Delta_\sigma(\varepsilon)|$ when Re $(s) > B(N)$. One can now use integration by parts. Let $\mathscr{P}_i^*$ be the adjoint of $\mathscr{P}_i$. Then, for each $\sigma \in \mathscr{T}$ there exists an $n-1$ differential $\Omega_\sigma(s, x)$ so that (with $\mathrm{d}x = \mathrm{d}x_1 \ldots \mathrm{d}x_n$)

$$\Sigma_i \mathscr{P}_i(R^{s+i+1} U(N))(\psi E_\sigma)\, \mathrm{d}x \;=\; \Sigma_i (R^{s+i+1} U(N)) \mathscr{P}_i^*(\psi E_\sigma)\, \mathrm{d}x + \mathrm{d}\Omega_\sigma(s, x).$$

Applied to (4.4), this gives for each sufficiently small positive ε

$$I_\sigma(\varepsilon, s, \varphi) \;=\; (1/b_N(s)) \left\{ \Sigma_0^m \int_{\Delta\sigma(\varepsilon)} R^{s+i+1} U(N) \mathscr{P}_i^*(\psi E_\sigma)\, \mathrm{d}x \right.$$

$$\left. + \int_{\partial\Delta\sigma(\varepsilon)} \Omega_\sigma(\sigma, x) \right\}. \tag{4.6}$$

Theorem 1 then follows easily from the

LEMMA 2: *There exists $B'(N) \geqslant B(N)$, $B'(N)$ independent of ψE_σ, such that for* Re $(s) > B'(N)$

$$\lim_{\varepsilon \to 0} \sum_{\sigma \in \mathscr{T}} \int_{\partial\Delta\sigma(\varepsilon)} \Omega_\sigma(s, x) \;=\; 0. \tag{4.7}$$

REMARK: The independence assertion in the lemma is needed when one wants to perform the meromorphic continuation to **C** starting in the half-plane Re $(s) > B'(N)$.

Proof: The basic idea is to use the growth condition on R (2.5). However, to prove the assertion (4.7), one needs to be precise about the integration by parts.

First, the complex of differentials needs to be defined. One wants to consider ψE_σ as a "test function", so it is necessary to define the differentials as follows.

Let $\tilde{\mathscr{F}} = \{\psi(x)$: if $\varphi \in \mathscr{F}_N$, then $\varphi(1/x)(x_1 \ldots x_n)^N = \psi(x)$ for some $N\}$. Set $\tilde{\Omega}^j = \Omega^j \otimes_{\mathbf{C}[x]} \tilde{\mathscr{F}}$. Next, define the ring

$$\mathscr{E} = \mathbf{C}(s)[x_1, \ldots, x_n, 1/x_1 \ldots x_n, \mathrm{e}(1/x_1), \ldots, \mathrm{e}(1/x_n)].$$

Set $\mathscr{Q} = 1/(\mathrm{e}(1/x_1) - 1) \ldots (\mathrm{e}(1/x_n) - 1)$ and define

$$\mathscr{E}_{\mathscr{Q}} = \mathscr{E}[\mathscr{Q}].$$

Then $\mathscr{E}_{\mathscr{Q}}$ admits an action by the ring $\mathscr{D}_n(\mathbf{C}(s))$. Set

$$\mathscr{W} = \mathscr{D}_n(\mathbf{C}(s))\mathscr{Q}$$

considered as a $\mathscr{D}_n(\mathbf{C}(s))$ submodule of $\mathscr{E}_{\mathscr{Q}}$. Now let

$$\hat{\Omega}^j = \tilde{\Omega}^j \otimes_{\mathbf{C}[x_1, \ldots, x_n]} \mathscr{W}.$$

REMARK: When one works over a particular chain Δ_σ, the complex $\hat{\Omega}^{\cdot}$ is denoted $\Omega_\sigma^{\cdot}$ and $\mathscr{Q}$ is the function E_σ.

Now form the de Rham complex (with tensor product over $\mathbf{C}(s)[x_1, \ldots, x_n, 1/x_1 \ldots x_n]$)

$$\hat{\Omega}^{\cdot} \otimes \mathscr{M}_{N+2}$$

where $\mathscr{M}_{N+2}$ is the module studied in section 3 with $T = x_1 \ldots x_n$. As above, the generator is denoted $R^s U(N)$.

This complex admits an exterior derivative by defining

$$\mathrm{d}(\omega \otimes \eta) = \mathrm{d}\omega \otimes \eta + \Sigma_1^n (\mathrm{d}x_i \wedge \omega) \otimes (D_{x_i}\eta).$$

Integration by parts will then be an identity inside this complex derived from this formula in degree n

$$a(x)\,\mathrm{d}x \otimes D_{x_k}(R^s U(N)) = D_{x_k}(a)\,\mathrm{d}x \otimes R^s U(N)$$

$$+ \mathrm{d}[(-1)^{k-1} a(x)\,\widehat{\mathrm{d}x_k} \otimes R^s U(N)]$$

where $\widehat{\mathrm{d}x_k} = \mathrm{d}x_1 \ldots \mathrm{d}x_{k-1}\mathrm{d}x_{k+1} \ldots \mathrm{d}x_n$.

Let $I = (i_1, \ldots, i_n)$ be a vector of nonnegative integers. Let $a(x)\,\mathrm{d}x$ be an element of Ω^n_σ. Define for each $k = 1, \ldots, n$

$$D_x^{I_l(k)} = D_{x_1}^{i_1} \ldots . D_{x_{k-1}}^{i_{k-1}}$$

$$D_x^{I_r(k)} = D_{x_{k+1}}^{i_{k+1}} \ldots . D_{x_n}^{i_n}.$$

If $k = 1$ resp. n then $D_x^{I_l(k)} = 1$ resp. $D_x^{I_r(k)} = 1$.

Also, set for each $k = 1, \ldots, n$, $\alpha \in \Omega^n_\sigma$ and $\beta \in \mathscr{M}_{N+2}$

$$\mathscr{A}_k(D_x^{I_l(k)}\alpha, D_x^{I_r(k)}\beta) = (-1)^{k-1} \sum^{i_k} (-1)^{i_k - v}(D_x^{I_l(k)} D_{x_k}^{i_k - v}\alpha)\,\widehat{\mathrm{d}x_k}$$

$$\otimes\ (D_{x_k}^{v-1} D_x^{I_r(k)}\beta).$$

Then integration by parts for the differential monomial D_x^I reads as follows

$$a(x)\,\mathrm{d}x \otimes D_x^I(R^s U(N)) = (-1)^{|I|}(D_x^I a(x))\,\mathrm{d}x \otimes R^s U(N)$$

$$+ \Sigma_1^n (-1)^{i_1 + \cdots + i_{k-1}}\,\mathrm{d}\mathscr{A}_k(D_x^{I_l(k)} a, D_x^{I_r(k)}(R^s U(N))).$$

If one writes

$$\mathscr{P}_j = \sum_{i=0}^{m} s^i P_{i,j}(x, D_x),$$

where

$$P_{i,j} = \sum_{|I|=0}^{L} p^I_{i,j}(x) D_x^I,$$

and sets

$$a^I_{i,j}(\sigma, x) = p^I_{i,j}(x)\psi E_\sigma,$$

then one sees that $\Omega_\sigma(s, x)$ is a sum of differentials of the form

$$\mathscr{A}_v(D_x^{I_l(v)} a_{i,j}^I(\sigma, x), D_x^{I_r(v)}(R^{s+j+1} U(N))). \tag{4.8}$$

The important points to realize are that the set of monomials $D_x^{I_r(v)}$ consists of a finite set with weight of $I_r(v)$ for any r, v at most L. Moreover, the factor $D_x^{I_l(v)} a_{i,j}^I(\sigma, x)$ is an integrable function over Δ_σ. Thus, to prove (4.7), one needs to understand at first the behavior of each term $D_x^{I_r(v)}(R^{s+j+1} U(N))$ over the components $B_{1,v}(\varepsilon, \sigma)$ and $B_{2,v}(\varepsilon, \sigma)$ of $\partial\Delta_\sigma(\varepsilon)$ as $\varepsilon \to 0$ and when Re (s) is sufficiently large. This evidently depends only upon the operators $\{\mathscr{P}_i\}$ and not upon any $a_{i,j}^I(\sigma, x)$.

By linearity, (4.7) will follow by showing

(4.9). There exists $B'(N) \geqslant B(N)$ such that Re $(s) > B'(N)$ implies

$$\lim_{\varepsilon\to 0} \sum_{\sigma\in\mathscr{T}} \sum_{v\in\mathscr{I}_\sigma} (-1)^v \int_{B_{1,v}(\varepsilon,\sigma) - B_{2,v}(\varepsilon,\sigma)} \mathscr{A}_v(D^{I_l(v)} a_{i,j}^I(\sigma, x), D^{I_r(v)}(R^{s+j+1} U(N))) = 0,$$

for all v, i, j, I.

Here, the convergence is uniform on compact subsets of this halfplane.

Note: In the following, we denote this integrand by $\mathscr{A}_v(i, j, I, \sigma)$.

Proof of (4.9): Let $\mathscr{T}_i = \{\sigma \in \mathscr{T} : |\mathscr{I}_\sigma| = i\}$. We prove (4.9) for $\mathscr{T}_n$ and leave the simple modifications of the arguments below for the other $\mathscr{T}_i$ to the reader.

We first consider the behavior of the integrals in (4.9) over the component of the vth side of $\partial\Delta_\sigma(\varepsilon)$ (i.e., $B_{1,v} - B_{2,v}$) for which the point $b_{1,v}(\sigma)$ or $b_{2,v}(\sigma)$ (cf. (4.3)) lies on S_ε. Without loss of generality, we may assume this component is $B_{1,v}(\varepsilon, \sigma)$. One now first proves the following.

(4.10) There exists $B'(N) \geqslant B(N)$ such that for any function $a(x)$, integrable over $\Delta_\sigma(\varepsilon)$ and uniformly bounded in ε, Re $(s) > B'(N)$ implies

$$\lim_{\varepsilon\to 0} \int_{B_{1,v}(\varepsilon,\sigma)} a(x) D_x^J(R^s U(N)) \widehat{\mathrm{d}x_v} = 0$$

for each $v = 1, \ldots, n$, $\sigma \in \mathscr{T}_n$, and index J with $|J| \leqslant L$. Here, the convergence is uniform on compact subsets of the halfplane Re $(s) > B'(N)$.

The proof of (4.10) is by induction on $|J|$. Let $|J| = 1$. Then, $D^J = D_i$ for some i. Let $\Delta_\sigma(z) = \Delta_\sigma \cap \mathbf{C}^n(z)$.

One has

$$D^J(R^s U(N)) = \left[\frac{sD_i R}{R} - \frac{N+2}{x_i}\right] R^s U(N). \tag{4.11}$$

Because $-z_i^2 D_{z_i}(1/P)(z) = D_i R(1/z)$, one has for a point $(x_1, \ldots, x_n) = (1/z_1, \ldots, 1/z_n)$ in $\Delta_\sigma(z)$

$$\left|\frac{D_i R(1/z)}{R(1/z)}\right| = \left|\frac{z_i^2 D_{z_i}(P)(z)}{P(z)}\right|.$$

By Proposition 1, there exist (independent of σ) $C, \gamma > 0$ such that for all points $(z_1, \ldots, z_n) \in \Delta_\sigma(z)$

$$|1/P(z_1, \ldots, z_n)| < C|z_1 \ldots z_n|^{-\gamma}.$$

Moreover, there exists a positive integer M_0 (independent of σ) such that

$$(z_1 \ldots z_n)^{-M_0} z_i^2 D_{z_i} P(z)$$

is a bounded rational function in $\Delta_\sigma(z)$. Now choose a nonnegative integer M_1 so that $M_1 \leqslant \inf\{\gamma, M_0\}$. Set $M(J) = M_0 - M_1$. One concludes that

$$(z_1 \ldots z_n)^{-M(J)}(z_i^2 D_{z_i} P/P)(z)$$

and therefore

$$\psi_J(x) = (x_1 \ldots x_n)^{M(J)}(D_i R/R)(x)$$

are bounded rational functions on $\Delta_\sigma(z)$.

So, one can write

$$D_i R(x)/R(x) = \psi_J(x)/(x_1 \ldots x_n)^{M(J)} \tag{4.12}$$

with $\psi_J(x)$ bounded in particular on $|B_{1,v}(e, \sigma) \cup B_{2,v}(\varepsilon, \sigma)|$, uniformly in ε, for each v, σ.

Substituting (4.12) into (4.11), setting

$$g(J)(s, x) = s\psi_J(x)x_i - (N+2)(x_1 \ldots x_n)^{M(J)},$$

and $N(J) = 1 + M(J)$, one has

$$D^J(R^s U(N)) = \left[\prod_{j\neq i} x_j g(J)(s, x)/(x_1 \ldots x_n)^{N(J)}\right] R^s U(N). \tag{4.13}$$

Let $\mathscr{R} = \mathbf{C}(s)[x_1, \ldots, x_n, 1/TQ(x_1 \ldots x_n)]$ and $\mathscr{R}_N = \mathscr{R}R^s U(N)$. Then,

$$h(J)(s, x) = \left[\prod_{j\neq i} x_j g(J)(s, x)/(x_1 \ldots x_n)^{N(J)}\right]$$

is an element of $\mathscr{R}$ which is a polynomial in s (of degree $|J|$) whose coefficients are bounded over $\partial\Delta_\sigma(\varepsilon)$, uniformly in ε.

Moreover, in $\mathscr{R}_N$ we may identify

$$(1/x_1 \ldots x_n)^{N(J)} R^s U(N) = R^s U(N + N(J)).$$

Thus, for $x \in \Delta_\sigma(\varepsilon)$, for each sufficiently small and positive ε,

$$D^J(R^s U(N))(x) = h(J)(s, x) R^s U(N + N(J))(x). \tag{4.14}$$

Using the bound (2.5) and the uniform (in ε) boundedness over each $B_{1,v}(\varepsilon, \sigma)$ of the coefficients (of s) of $a(x)h(J)(s, x)$, one concludes that there exist constants C_1, $C_2 > 0$ such that for each v, σ, index J with $|J| = 1$, and ε with $0 < \varepsilon \ll 1$, if $C(s) = C_1 s + C_2$, then

$$|a(x)h(J)(s, x)|\, |R^s U(N + N(J))(x)|\,|_{B_{1,v}(\varepsilon,\sigma)} < |C(s)|\varepsilon^{\theta(s)}\, e^{\operatorname{Im}(s)\arg(R(x))} \tag{4.15}$$

for each $x \in B_{1,v}(\varepsilon, \sigma)$, where $\theta(s) = (\alpha \operatorname{Re}(s)/n) - (N + N(J))$.

Setting $B(N_1) = [\![\max_J\{n(N + N(J))/\alpha]\!] + 1$, one sees, for s restricted to arbitrary compact subsets of the halfplane $\operatorname{Re}(s) > B(N_1)$, that the integral over $B_{1,v}(\varepsilon, \sigma)$ of the $n - 1$ differential $a(x)D^J(R^s U(N))\, \widehat{dx_v}$ converges uniformly to 0 as $\varepsilon \to 0$. This proves (4.10) for $|J| = 1$.

Proceeding by induction one assumes that for any index J with $|J| < k$, there exist integers $M(J)$, $N(J)$, a polynomial in s of degree $|J|$ with rational coefficients in x, denoted $h(J)(s, x)$, and a rational function $\psi_J(x)$, such that each of the coefficients of any power of s and ψ_J are bounded uniformly in ε over $\partial\Delta_\sigma(\varepsilon)$ and so that for all $x \in \Delta_\sigma(\varepsilon)$, for each ε and σ, one has

$$\frac{D^J R(x)}{R(x)} = \frac{\psi_J(x)}{(x_1 \ldots x_n)^{M(J)}} \tag{4.16}$$

and

$$\begin{aligned} D^J(R^s U(N))(x) &= [h(J)(s, x)/(x_1 \ldots x_n)^{N(J)}]R^s U(N) \\ &= h(J)(s, x) R^s U(N + N(J)) \end{aligned}$$

hold for all s in $\mathbf{C}$.

Now let J be an index with $|J| = k$. Writing $J = \mathbf{e}_i + J_i$, where J_i has weight $k - 1$ and $\mathbf{e}_i$ is the unit basis vector for the ith coordinate plane, it is straightforward to extend the above argument to show the existence of a polynomial of degree k in s with coefficients rational functions in x, denoted $h(J)(s, x)$, and a rational function $\psi_J(x)$, satisfying the equality (4.13) for certain integers $M(J)$ and $N(J)$. In addition, the above arguments also show the uniform in ε boundedness of these rational functions in x over the chains $\Delta_\sigma(\varepsilon)$.

Defining $B(N(J)) = [\![n(N + N(J))/\alpha]\!] + 1$ for each J with $|J| \leqslant L$, and setting $B'(N) = \max_J B(N(J))$ one sees that if ω $(= \mathrm{Re}\,(s)) > B'(N)$, then for each such J, the exponent of ε in the inequality ($\theta(\omega)$ defined in (4.15))

$$|R^s U(N + N(J))(x)|\,|_{B_{1,v}(\varepsilon,\sigma)} < |C(s)|\varepsilon^{\theta(\omega)}\, \mathrm{e}^{\tau \arg(R(x))} \tag{4.17}$$

is positive. By the compactness of each Δ_σ, one concludes that (4.10) holds.

To verify (4.9), one proceeds as follows. Let α, β denote distinct values in $\{1, 2\}$. For each v and σ, if $b_{\alpha,v}(\sigma) = 1/D$, then there evidently exists a unique $\sigma' \neq \sigma$ such that $b_{\beta,v}(\sigma') = 1/D$ and $\sigma(i) = \sigma'(i)$ for $i \neq v$. Thus, one can order the 2^n possible σ

$$\sigma(1, v) < \sigma(2, v) < \cdots < \sigma(2^n, v)$$

so that each pair $\sigma(2k - 1, v)$ and $\sigma(2k, v)$ satisfies the properties

i) $\sigma(2k - 1, v)(i) = \sigma(2k, v)(i)$ for $i \neq v$
ii) $b_{\alpha,v}(\sigma(2k - 1, v)) = b_{\beta,v}(\sigma(2k, v)) = 1/D$.

Thus, $B_{\alpha,v}(\sigma(2k - 1, v)) = B_{\beta,v}(\sigma(2k, v))$. Denote this common $n - 1$ chain by $B(k, v)$.

One can then rearrange the summation in (4.9) by writing it as $I_1 + I_2$ where

$$\begin{aligned} I_1 = \sum_{v=1}^{n} (-1)^v \sum_{k=1}^{2^{n-1}} \varepsilon(\alpha, \beta) \int_{B(k,v)} [&\mathscr{A}_v(i, j, I, \sigma(2k - 1, v)) \\ &- \mathscr{A}_v(i, j, I, \sigma(2k, v))] \end{aligned}$$

and

$$I_2 = \sum_{v=1}^{n} (-1)^v \sum_{k=1}^{2^{n-1}} (-\varepsilon(\alpha, \beta)) \left[\int_{B_{\beta,v}(\varepsilon,\sigma(2k-1,v))} \mathscr{A}_v(i, j, I, \sigma(2k - 1, v)) \right.$$

$$\left. - \int_{B_{\alpha,v}(\varepsilon,\sigma(2k,v))} \mathscr{A}_v(i, j, I, \sigma(2k, v)) \right]$$

where

$$\varepsilon(\alpha, \beta) = \begin{cases} -1 & \text{if } \alpha < \beta \\ 1 & \text{if } \alpha > \beta. \end{cases}$$

One can apply (4.10) to each of the terms in I_2. Thus, there exists $B'(N)$, independent of v, σ, i, j, I, ε such that ω $(= \mathrm{Re}\,(s)) > B'(N)$ implies

$$\lim_{\varepsilon \to 0} I_2(s) = 0$$

where the convergence is uniform on compact subsets of this halfplane. (4.9) will now follow by showing that for each $\varepsilon > 0$, $I_1 = 0$. By expanding out the integrand over $B(k, v)$, one sees that for $\omega > B(N)$

$$\mathscr{A}_v(i, j, I, \sigma(2k - 1, v)) - \mathscr{A}_v(i, j, I, \sigma(2k, v))$$

$$= \sum_{\{(J_1, J_2): J_1 + J_2 = I - \mathbf{e}_v\}} \{D_1^J(p_{i,j}^I \psi(E_{\sigma(2k-1,v)}$$

$$- E_{\sigma(2k,v)}))D_2^J(R^{s+j+1}U(N))\}\,|_{B(k,v)}\, \widehat{\mathrm{d}x_v}.$$

Now one notes that for any differential monomial D_x^A, the definitions of $E_{\sigma(2k-1,v)}$, $E_{\sigma(2k,v)}$ imply immediately that for any choice of $\varepsilon > 0$ which determines $B(k, v)$,

$$D_x^A(E_{\sigma(2k-1,v)} - E_{\sigma(2k,v)})\,|_{B(k,v)} = 0.$$

Thus, $I_1 = 0$. This completes the proof of (4.7). ∎

One can now finish the

Proof of Theorem 1: For Re $(s) > B'(N)$ one has by Lemma 1

$$\mathscr{D}'_P(s, \varphi) = \sum_{\sigma \in \mathscr{T}} I_\sigma(s, \varphi) = \sum_{\sigma \in \mathscr{T}} \lim_{\varepsilon \to 0} I_\sigma(\varepsilon, s, \varphi)$$

$$= (1/b_N(s))\, \Sigma_0^m \sum_{\sigma \in \mathscr{T}} \int_{\Delta_\sigma} R^{s+i+1} U(N) \mathscr{P}_i^* (\psi E_\sigma)\, \mathrm{d}x. \tag{4.18}$$

This implies that a meromorphic continuation into the halfplane Re $(s) \leqslant B'(N)$ for $\mathscr{D}'_P(s, \varphi)$ can be accomplished by using (4.18) to continue into vertical strips

$$B'(N) - i - 1 < \mathrm{Re}\,(s) \leqslant B'(N) - i, \quad i \geqslant 1,$$

by induction on i. One obtains that the poles of $\mathscr{D}'_P(s, \varphi)$ (and therefore of $\mathscr{D}_P(s, \varphi)$) must then lie in the set $\mathscr{B}_N$, as defined in the Introduction. This completes the proof of Theorem 1.

Section 5

In this section a more concrete complement to the general point of view in Sections 2–4 is presented. The aim is to characterize the largest pole of $\mathscr{D}_P(s, \varphi)$ when P is a real polynomial which satisfies the $(+)$ condition (cf. section 1) as well as (5.9). As a result the notation from section 1 is used here without explicit reference. We consider first an example analyzed in [3, pg. 28].

Let $P(z_1, z_2) = z_2 + z_1^4 + z_1^2 z_1^3$. Then $R(x_1, x_2) = x_1^4 x_2^3/(x_1^2 + x_2^3 + x_1^4 x_2^2)$.

Let Y denote $C^2(x_1, x_2)$. One can obtain arithmetic progressions containing the candidates for the poles of each $I_\sigma(s, \varphi)$, σ a choice of sign function on $\{1, 2\}$, by constructing a proper birational map $\pi\colon X \to Y$ with the property that $x_1^4 x_2^3 \circ \pi$ and $(x_1^2 + x_2^3 + x_1^4 x_2^2) \circ \pi$ are locally in normal crossing form on $\mathscr{X} = \pi^{-1}(\{x_1^2 + x_2^3 + x_1^4 x_2^2 = 0\} \cup \{x_1^4 x_2^3 = 0\})$.

In general, if $\{p_i\}_1^T$ are points in $\mathscr{X}$ for which there are coordinates (u_i, v_i) centered at p_i and defined in a neighborhood $\mathscr{U}_i$ such that $\mathscr{X} \subset \bigcup_{i=1}^T \mathscr{U}_i$ then the arithmetic progressions are obtained as follows.

Let

$$R \circ \pi(u_i, v_i) = u_i^{A_i} v_i^{B_i} H(u_i, v_i),\ H|_{\mathscr{U}_i} \neq 0,$$

and

$$\pi^*\left(\frac{1}{(x_1x_2)^2}\,\mathrm{d}x_1\,\mathrm{d}x_2\right) = u_i^{a_i}v_i^{b_i}h(u_i, v_i),\ h|_{\mathscr{U}_i} \neq 0,$$

$A_i, B_i, a_i, b_i \in Z$, for each $i = 1, 2, \ldots, T$. Then, if $\varphi \in \mathscr{F}_0$ (cf. section (2)), the progressions are defined by

$$\text{i)}\quad A_is + a_i = -1, -2, \ldots,$$

$$\text{ii)}\quad B_is + b_i = -1, -2, \ldots$$

If $\varphi \in \mathscr{F}_N$, $N \neq 0$, then this must be modified. If $\Phi(x_1, x_2) = \varphi(1/x_1, 1/x_2)$ and $\Phi \circ \pi(u_i, v_i) = u_i^{e_i}v_i^{f_i}K$, $K|_{\mathscr{U}_i} \neq 0$, then the progressions are

$$\text{i)}\quad A_is + a_i + e_i = -1, -2, \ldots$$

$$\text{ii)}\quad B_is + b_i + f_i = -1, -2, \ldots \quad i = 1, 2, \ldots, T.$$

That these assertions are true follows from the discussion in [5].

In particular, the largest possible pole of each $I_\sigma(s, \varphi)$, $\varphi \in \mathscr{F}_N$, is described as

$$\beta_\varphi = \max_{i=1,\ldots,T}\left\{\frac{-(1 + a_i + e_i)}{A_i}, \frac{-(1 + b_i + f_i)}{B_i}\right\}.$$

For $P(z)$ in the example above and $\varphi \in \mathscr{F}_0$, the progressions are given by the equations

$$\text{i)}\quad \begin{aligned} 7s - 3 &= -1, -2, \ldots \\ 3s - 2 &= -1, -2, \ldots \end{aligned}$$

$$\text{ii)}\quad \begin{aligned} 4s - 2 &= -1, -2, \ldots \\ 8s - 4 &= -1, -2, \ldots \end{aligned}$$

$$\text{iii)}\quad \begin{aligned} 12s - 6 &= -1, -2, \ldots \\ 5s - 3 &= -1, -2, \ldots \end{aligned}$$

$$\text{iv)}\quad \begin{aligned} 8s - 4 &= -1, -2, \ldots \\ 12s - 6 &= -1, -2, \ldots \end{aligned}$$

Note that the largest possible ratio is $\beta_\varphi = 5/12$. This agrees with [3]. If φ is not a unit at infinity, more work is required to determine the progressions and thus a closed expression for β_φ. However, this has essentially been done for the situation in which $x = (x_1, x_2)$, and so that when one writes $R(x) = 1/P(1/x) = x_1^{M_1}x_2^{M_2}/Q(x_1, x_2)$ then $Q(x)$ is analytically irreducible at $(x_1, x_2) = (0, 0)$ [7].

As discussed in Remark (5.15), the polynomials $P(z)$ of interest in diophantine problems of an asymptotic nature presumably consist of those for which the largest pole of $\mathscr{D}_P(s, -)$ is at least 1. It is therefore important to be able to characterize exactly the largest pole. This is done in terms of a combinatorial object, the Newton Polyhedron $\Gamma_\infty(P)$ of P at infinity defined as follows.

$$\text{Let } P(z_1, \ldots, z_n) = \Sigma_I a_I z^I. \tag{5.1}$$

Then $\Gamma_\infty(P)$ is the boundary of the convex hull with respect to infinity of the set

$$\bigcup_{\{I: a_I \neq 0\}} (I - \mathbf{R}^n_+).$$

Let $M = (m(1), \ldots, m(n))$ be the integral vector defined so that $m(j) = \max\{i_j\colon i_j \text{ is the } j\text{th component of some index } I \text{ for which } a_I \neq 0\}$. As in section 4, define

$$R(x_1, \ldots, x_n) = 1/P(1/x_1, \ldots, 1/x_n) = \Pi_j x_j^{m(j)}/Q(x_1, \ldots, x_n). \tag{5.2}$$

Thus, $Q(x) = \Sigma_I a_I x^{M-I}$. Evidently, Q also satisfies the $(+)$-condition. For future reference, also define

$$\begin{aligned} \varphi(z_1, \ldots, z_n) &= z_1^{d(1)} \ldots z_n^{d(n)} \qquad d(i) \in \mathbf{N} \cup \{0\} \text{ fo each } i, \text{ and} \\ u &= [1/(x_1, \ldots, x_n)^2]\, \mathrm{d}x_1 \ldots \mathrm{d}x_n. \quad \text{Set } \vec{d} = (d(1), \ldots, d(n)). \end{aligned} \tag{5.3}$$

(5.4) REMARKS:

1) When $Q(0) \neq 0$, it is simple to show that $\beta_\varphi = \max_i\{d(i) + 1)/m(i)\}$. Thus, we assume in the following that $Q(0) = 0$.

2) Let Γ be the Newton polyhedron of Q with respect to the origin. It can easily be described in terms of $\Gamma_\infty(P)$. One can construct [12] a smooth algebraic variety X_Γ and proper birational transformation $\pi\colon X_\Gamma \to \mathbf{C}^n$ satisfying the following property. To each cone α appearing in a "small

partition" of $(\mathbf{R}^n_+)^*$ dual to Γ (cf. [12, sec. 9] for the notion of small) one assigns a chart $\mathbf{C}^n(\alpha)$ so that $\pi_\alpha =_{\text{def.}} \pi|_{\mathbf{C}^n(\alpha)}$ is a monomial. One can cover X_Γ by a finite union of charts.

Indeed, each cone α is spanned by n independent covectors $\alpha_1 = (a_{11}, \ldots, a_{n1}), \ldots, \alpha_n = (a_{1n}, \ldots, a_{nn})$ (each $a_{ij} \in \mathbf{N} \cup \{0\}$), each of which is primitive (i.e., gcd, $(a_{1j}, \ldots, a_{nj}) = 1$ for each j). To α there corresponds a chart $\mathbf{C}^n(\alpha)(u_1, \ldots, u_n)$ of X_Γ so that

$$\begin{aligned} x_i \circ \pi_\alpha(u_1, \ldots, u_n) &= \Pi_j u_j^{aij} \quad \text{for each} \quad i = 1, \ldots, n. \\ x^M \circ \pi_\alpha(u_1, \ldots, u_n) &= \Pi_j u_j^{Aj} \\ \Phi \circ \pi_\alpha(u_1, \ldots, u_n) &= \Pi_j u^{\vec{d} \cdot \alpha_j}, \ \Phi \text{ defined in terms of } \varphi \text{ by (4.1)}, \\ Q \circ \pi_\alpha(u_1, \ldots, u_n) &= \Pi_j u_j^{Bj} Q_\alpha(u_1, \ldots, u_n), \text{ with } Q_\alpha(0) \neq 0, \\ \pi_\alpha^*(u) &= \Pi_j u_j^{C_j}\, du_1 \ldots du_n. \end{aligned} \tag{5.5}$$

Q_α defines the strict transform of Q in $\mathbf{C}^n(\alpha)$. One thus observes that if P satisfies the $(+)$ condition then each Q_α satisfies the $(+)$ condition in the chart $\mathbf{C}^n(\alpha)(u)$.

3) Set $|\alpha_j| = \Sigma a_{ij}$ for each j. Then each C_j is given by the expression

$$|\alpha_j| - 1 - 2|\alpha_j| = -|\alpha_j| - 1.$$

Moreover, if $a_I x^{M-I}$ is a monomial in Q, the corresponding monomial in $Q \circ \pi_\alpha$ is $a_I u^{\alpha 1 \cdot (M-I)} \ldots u_n^{\alpha n \cdot (M-I)}$. Thus, for each j

$$B_j = \min_j\{\alpha_j \cdot (M - I)\colon a_I \neq 0\} = \alpha_j \cdot M - \max\{\alpha_j \cdot I\colon a_I \neq 0\},$$

while $A_j = \alpha_j \cdot M$. Hence, $A_j - B_j = M(\alpha_j) = \max\{\alpha_j \cdot I\colon a_I \neq 0\}$, $j = 1, \ldots, n$.

4) In the chart $\mathbf{C}^n(x)$ set

$$\Phi_\theta(\delta) = \{ (x_1, \ldots, x_n)\colon |\mathrm{Arg}\,(x_i)| \leqslant \theta \quad \text{and} \quad |x_i| \leqslant \delta \text{ for each } i\}.$$

Then the proof of Lemma 1 evidently applies to show that the following inequality holds for all x in the interior of $\Phi_{\theta'}(1/a')$ for some positive θ', $\theta' \leqslant \theta$ and $1 > a' \geqslant a$, where 'a' is the number used in the definition of Γ_θ (cf. Sec. 1):

$$|Q(x_1, \ldots, x_n)| \geqslant c|x_1|^{m_1'} \ldots |x_n|^{m_n'} \tag{5.6}$$

where c and each m_i' is positive. However, (5.6) need not extend to the boundary of $\Phi_{\theta'}(1/a')$. In particular, Q may vanish identically over a part of the boundary of the form $\Phi_{\theta'}(1/a') \cap \{x_i = 0\}$, for some i.

5) One can estimate the preimage of $\Phi_{\theta'}(1/a')$ under the map π as follows. Let α be a cone as in 2). Let $A(\alpha)$ be the matrix

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{pmatrix}$$

with rows ordered so that $\det A(\alpha) = 1$. Set $B(\alpha) = A(\alpha)^{-1}$. If $b^1, \ldots, b^n$ are the row vectors of $B(\alpha)$, $b^j = (b_{1j}, \ldots, b_{nj})$, then one checks that π_α^{-1} is defined by

$$u_k = \Pi_i x_i^{b_{ki}}, \quad k = 1, \ldots, n$$

where this makes sense.

In particular, if $x \in \Phi_{\theta'}(1/a')$, then for each k

$$u_k = \Pi_i |x_i|^{b_{ki}} \, \mathrm{e}^{i\beta_k},$$

where the argument vector

$$(\beta_1, \ldots, \beta_k) = (\mathrm{Arg}\,(x_1), \ldots, \mathrm{Arg}\,(x_n)) \circ B(\alpha).$$

Thus, if $x = (x_1, \ldots, x_n) \in \Phi_{\theta'}(1/a')$, then there exists $\mu(B(\alpha), \theta') > 0$ such that if $\pi_\alpha^{-1}(x)$ is defined then it lies in the (unbounded) sector $\Phi_{\mu(B(\alpha),\theta')}(\infty)$ in $\mathbf{C}^n(\alpha)$. Moreover, given $\varepsilon > 0$, there exists $\theta(B(\alpha), \varepsilon)$ so that if $\theta' \leqslant \theta(B(\alpha), \varepsilon)$ then $|\mathrm{Arg}\,(x_i)| < \theta'$ for each i implies $\mu(B(\alpha), \theta') < \varepsilon$. Since there exist only finitely many cones α in the partition, one can find $\theta(\varepsilon) > 0$ such that $\theta' \leqslant \theta(\varepsilon)$ and $|\mathrm{Arg}(x_i)| < \theta'$ for each i implies $\mu(B(\alpha), \theta') < \varepsilon$ for any matrix $B(\alpha)$.

6) If one writes for each α

$$Q_\alpha = c_\alpha + h_\alpha(u),$$

then not only is c_α a positive constant but h_α is a real polynomial satisfying the (+) condition in $\mathbf{C}^n(\alpha)$. Thus, by (1.7)(*i*), (4), and (5), it follows that for sufficiently small θ' one has

$$|Q_\alpha(u)| > 0$$

for all points u in $\Phi_{\mu(B(\alpha),\theta')}(\infty)$ for each α.

7) One notes that an expression for $\mathscr{D}_P(s, -)$, rather than some tail $\mathscr{D}'_P(s, -)$, is given by (2.10), where the chains Δ_σ are taken to equal the chains Γ_σ, defined in section 1. One then defines the $I_\sigma(s, -)$ using the Γ_σ.

8) For a covector $a = (a_1, \ldots, a_n)$ set

$$M(a) = \sup \{\gamma\text{: the plane } a \cdot x = \gamma \text{ intersects } \Gamma_\infty(P)\}.$$

Then $a \cdot x = M(a)$ is the equation of the support plane for $\Gamma_\infty(P)$ in the direction a. Moreover,

$$\tau(a) = |a|/M(a) \tag{5.7}$$

is the reciprocal of the value t for which the diagonal $y_1 = \cdots = y_n = t$ meets the support plane for $\Gamma_\infty(P)$ in the direction determined by a.

For φ defined in (5.3) define

$$\tau_\varphi(a) = (|a| + a \cdot \vec{d})/M(a)$$

and

$$\beta_\varphi = \inf \{\tau_\varphi(a)\text{: } a \text{ is a primitive covector in } (\mathbf{Z}^n_+)^*\}. \tag{5.8}$$

A little thought allows one to conclude that β_φ is the evident candidate for the leading pole for each $I_\sigma(s, \varphi)$ and so, for $\mathscr{D}_P(s, \varphi)$. ■

Before stating Theorem 2, a useful lemma is noted. It allows one to reduce the evaluation of the residue of $\mathscr{D}_P(s, \varphi)$ at β_φ to exactly one of the currents $I_\sigma(s, \varphi)$ when (5.9) is satisfied.

In the following, let $l_i(s) = A_i s + a_i$, $i = 1, \ldots, n$, be linear functions with integral coefficients, each $A_i \geqslant 0$, in the complex variable s. Let $(x_1, \ldots, x_n)$ be complex coordinates in a neighborhood $\mathscr{W}$ of $0 \in \mathbf{C}^n$. Set Δ to be a compact analytic n-chain in $\mathscr{W}$. If φ: $[0, 1]^n \to \mathscr{W}$ parameterizes Δ, assume $0 \in \partial\Delta$. Assume that $\varphi|_{\{t_i=0\}}$ maps into a coordinate plane $x_{j(i)} = 0$ and that variation $(\arg (x_i \circ \varphi))$ is at most 2π for each i.

Now let E: $\mathbf{R} \to \mathbf{C}$ be a flat function at 0 with $E(0) = 0$. Define the "generalized current"

$$I(s, \psi) = \int_\Delta x_1^{l_1(s)} \ldots x_n^{l_n(s)} \psi(x) E(x_1) \ldots E(x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

where $\psi \in C^\infty(\mathscr{W}, \mathbf{C})$. Then one has

LEMMA 3: *For any such ψ, $I(s, \psi)$ is an entire function in s. That is, $I(s, \psi)$ is analytic in some halfplane* Re $(s) \gg 0$ *and admits an analytic continuation into* $\mathbf{C}$ *with no poles.*

Proof: This is a straightforward extension to the case $n > 1$ of the result which is easily deduced from the classical regularization method of [5] when $n = 1$.

We are able to give a fairly simple proof of Theorem 2 for real polynomials P satisfying the $(+)$ condition as well as the following condition.
(5.9) The polynomial Q, defined by the expression (5.2), is a "convenient" polynomial, in the sense of Kushnirenko, such that $Q(0) = 0$. Thus, for each $i = 1, \ldots, n$, there is a positive integer $n(i)$ so that $x_i^{n(i)}$ appears in (5.2) with a positive coefficient.

REMARK: The basis for this ability is that (5.9) implies the very strong property that the support of the residue current $\mathrm{Pol}_{s=\varrho} I_\sigma(s, -)$ (cf. (5.10)) is the point $0 \in \mathbf{C}^n$ for each σ and any candidate pole ϱ. If (5.9) is not satisfied then this need not hold and the property stated in (5.11) will fail.

We can now state

THEOREM 2: *Assume that the real polynomial $P(z_1, \ldots, z_n)$ satisfies the $(+)$ condition and the polynomial Q, defined by (5.2), satisfies (5.9). Let $\varphi(z_1, \ldots, z_n) = z_1^{d(1)} \ldots z_n^{d(n)}$. Then β_φ (cf. (5.8)) is the largest pole of $\mathscr{D}_P(s, \varphi)$.*

Observe that from this, one has an immediate

COROLLARY: *The largest pole of $\mathscr{D}_P(s, 1)$ is the value $1/\tau$ where τ is that value of t at which the diagonal meets $\Gamma_\infty(P)$ in the sense of (5.7).*

Proof of Theorem 2: For each σ and any possible pole ϱ, set
(5.10) $\mathrm{Pol}_{s=\varrho} I_\sigma(s, -)$ = the polar part of the Laurent series for $I_\sigma(s, -)$ at

$$s = \varrho.$$

The point of assuming (5.9) is that it implies the following
(5.11) For all but one choice function σ, one has that $\mathrm{Pol}_{s=\varrho} I_\sigma(\sigma, \varphi) = 0$.

This can be seen as follows. First, one notes that the support $\mathscr{S}(\sigma, \varrho)$ of the current $\mathrm{Pol}_{s=\varrho} I_\sigma(s, -)$ is contained in the locus

$$\{Q = 0\} \cap \Phi_{\theta'}(1/a').$$

Pulling back each I_σ to X_Γ one sees that

$$\mathscr{S}(\sigma, \varrho) \subseteq \bigcup_\alpha \pi_\alpha[\{Q \circ \pi_\alpha = 0\} \cap \Phi_{\mu(B(\alpha), \theta')}(\infty)].$$

One has

$$\{Q \circ \pi_\alpha = 0\} \;=\; \{Q_\alpha = 0\} \cup (\cup D_i), \tag{5.12}$$

where each D_i is defined by an equation of the form $u_{j(i)} = 0$, when $(u_1, \ldots, u_n)$ are the coordinates in $\mathbf{C}^n(\alpha)$ with respect to which (5.5) holds. Moreover, choosing θ' sufficiently small, one sees from (5.4), Remarks (5) and (6), that

$$\{Q_\alpha = 0\} \cap \Phi_{\mu(B(\alpha), \theta')}(\infty) \;=\; \varnothing.$$

Now consider a component D_i from (5.12). Evidently, there exist an integer r, $1 \leqslant r \leqslant n$, and indices $k(1) < k(2) < \cdots < k(r)$, so that its image under π_α can be described as the locus

$$\mathscr{H}_i \;=\; \{x_{k(1)} \;=\; \cdots \;=\; x_{k(r)} \;=\; 0\},$$

in $\mathbf{C}^n(x)$. Thus, Q restricted to this $n - k(r)$ dimensional coordinate plane is identically zero. On the other hand, if $r < n$, let

$$\{j(1) < \cdots < j(n - r)\} \;=\; \{1, \ldots, n\} \backslash \{k(1), \ldots, k(r)\}.$$

Then, for each $l = 1, \ldots n - r$, there exists $n(l) > 0$ so that $x_{j(l)}^{n(l)}$ appears in the expression (5.2) for Q. Thus, the polynomial

$$Q|_{\mathscr{H}_i}(x_{j(1)}, \ldots, x_{j(n-r)})$$

satisfies the $(+)$-condition and is a "convenient" polynomial, as well. It can therefore be identically zero only if $r = n$. Hence, each divisor D_i must blow down to $\{0\}$ in $\mathbf{C}^n(x)$. So, the image under π_α of $D_i \cap \Phi_{\mu(B(\alpha), \theta')}(\infty)$ is the origin.

Now, if σ is not the choice function σ_+, defined by

$$\sigma_+(j) \;=\; + \quad \text{for all} \quad j = 1, \ldots, n,$$

then for at least one j, $\sigma(j) = -$. Thus, at least one factor $E_{\sigma(j)}(1/x_j)$ is flat with (limiting) value 0 at $x_j = 0$ when restricted to the chain $\gamma_{\sigma(j)} = \gamma_-$. Applying Lemma 3 to the pullback of $I_\sigma(s, -)$, one concludes that (5.11) holds for each $\sigma \neq \sigma_+$.

So, to prove the theorem, it suffices to show

$$\operatorname{Pol}_{s=\beta_\varphi} I_{\sigma_+}(s, \varphi) \neq 0. \tag{5.13}$$

To evaluate $\operatorname{Pol}_{s=\beta_\varphi} I_{\sigma_+}(s, \varphi)$, return to the original definition of I_{σ_+}. From (1.10)

$$\begin{aligned} I_{\sigma_+} &= (-1)^n \int_{\gamma_+} \cdots \int_{\gamma_+} \left(\frac{1}{P}\right)^s \varphi(z) \prod_{j=1}^n \left(\frac{\mathrm{e}(z_j)}{\mathrm{e}(z_j) - 1} - 1\right) \mathrm{d}z_1 \ldots \mathrm{d}z_n \\ &= (-1)^n \Bigg[\sum_{d=1}^n (-1)^{n-d} \sum_{\{I \subset \{1,\ldots,n\}:\, \# I = d\}} \int_{\gamma_+} \cdots \\ &\qquad \int_{\gamma_+} \left(\frac{1}{P}\right)^s \varphi \prod_{u \in I} \left(\frac{\mathrm{e}(z_u)}{\mathrm{e}(z_u) - 1}\right) \mathrm{d}z_1 \ldots \mathrm{d}z_n \\ &\qquad + (-1)^n \int_{\gamma_+} \cdots \int_{\gamma_+} \left(\frac{1}{P}\right)^s \varphi \, \mathrm{d}z_1 \ldots \mathrm{d}z_n \Bigg]. \end{aligned}$$

Define

$$J_d(s, \varphi) = (-1)^{n-d} \sum_{\{I:\, \# I = d\}} \int_{\Gamma_{\sigma_+}} \left(\frac{1}{P}\right)^s \varphi \cdot \prod_{u \in I} \frac{\mathrm{e}(z_u)}{\mathrm{e}(z_u) - 1} \mathrm{d}z_1 \ldots \mathrm{d}z_n$$

for $d = 1, 2, \ldots, n$ and

$$J_0(s, \varphi) = (-1)^n \int_{\Gamma_{\sigma_+}} \left(\frac{1}{P}\right)^s \varphi \, \mathrm{d}z_1 \ldots \mathrm{d}z_n.$$

Since $I_{\sigma_+}(s, \varphi) = (-1)^n \Sigma_{i=0}^n J_i(s, \varphi)$, it suffices to understand $\operatorname{Pol}_{s=\beta_\varphi} J_i(s, \varphi)$ for each $i = 0, 1, \ldots, n$.

It is clear from Lemma 2 that one also has

$$\operatorname{Pol}_{s=\beta_\varphi} J_i(s, \varphi) = 0, \quad i = 1, \ldots, n.$$

This is because the function $\mathrm{e}(z)/(\mathrm{e}(z) - 1)$ satisfies the conditions in the lemma along the arc γ_+.

Thus,

$$\mathrm{Pol}_{s=\beta_\varphi}\mathscr{D}_P(s, \varphi) = \mathrm{Pol}_{s=\beta_\varphi}J_0(s, \varphi). \tag{5.14}$$

Indeed, the same conclusion is seen to hold for any possible pole ϱ of any $I_\sigma(s, -)$, by (5.11).

Now, to define $J_0(s, \varphi)$ as an integral over the chain $\gamma_+ \times \cdots \times \gamma_+$ makes the residue calculation difficult to complete. Fortunately, this is not necessary. There is an evident homotopy deforming γ_+ into $[a, \infty)$. thus, for $\mathrm{Re}\,(s) \gg 0$

$$\begin{aligned} J_0(s, \varphi) &= \int_a^\infty \cdots \int_a^\infty \left(\frac{1}{P}\right)^s z_1^{d_1} \ldots z_n^{d_n}\, \mathrm{d}z_1 \ldots \mathrm{d}z_n \\ &= \int_0^{1/a} \cdots \int_0^{1/a} R^s \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_n}{x_1^{d_1+2} \ldots x_n^{d_n+2}}. \end{aligned}$$

using the definition of φ.

This can be effectively used to show $\mathrm{Pol}_{s=\beta_\varphi}J_0(s, \varphi) \neq 0$ as follows.

For each $\varepsilon > 0$, set $\Delta(\varepsilon) = \pi^{-1}([\varepsilon, 1/a]^n)$. By (5.4)(5), one has $\Delta(\varepsilon) \cap \mathbf{C}^n(\alpha) \subset \mathbf{R}_+^n(\alpha)$. Moreover, defining

$$\Delta = \overline{\bigcup_{0<\varepsilon<1/2a} \Delta(\varepsilon)},$$

one sees that

$$\Delta \cap C^n(\alpha) \subseteq [0, \infty)^n, \quad \text{for each } \alpha.$$

Now, a divisor D in $\pi^{-1}(\{0\})$ is said to contribute effectively to β_φ if the multiplicity pair (cf. (5.5))

$$(M_D, m_D) = (\mathrm{ord}_D(R \circ \pi), \mathrm{ord}_D\pi^*(\Phi u))$$

satisfies

$$\beta_\varphi = -(1 + m_D)/M_D.$$

Let $D_1, \ldots, D_r$ be the divisors contributing effectively to β_φ. *A priori*, one knows that each of the coefficients of $\mathrm{Pol}_{s=\beta_\varphi}J_0(s, \varphi)$ is a sum of integrals concentrated along

$$\mathscr{E} = \cup\,(D_i \cap \Delta) \cup \{(\text{strict transform of } Q) \cap \Delta\}.$$

However, because of (5.4)(6), one can eliminate {strict transform of Q} from $\mathscr{E}$. Thus,

$$\mathscr{E} = \cup (D_i \cap \Delta).$$

One now observes that for any α for which $\mathscr{E} \cap \mathbf{C}^n(\alpha)$ is not empty and at any point $p \in \mathscr{E} \cap \mathbf{C}^n(\alpha)$, one can find coordinates $(Y_1, \ldots, Y_n)$ centered at p and defined in a neighborhood $\mathscr{W}$ so that

$$R \circ \pi(Y_1, \ldots, Y_n) = \prod_{i=1}^{t} Y_i^{Mi} \mathscr{V}_1$$

$$\pi^*(\Phi u) = \prod_{i=1}^{t} Y_i^{mi} \mathscr{V}_2 \, \mathrm{d}Y_1 \ldots \mathrm{d}Y_n.$$

Here the important property to observe is that

$$\mathscr{V}_i|_{\mathscr{W} \cap \Delta} > 0 \quad \text{for} \quad i = 1, 2.$$

Thus, one sees by a partition of unity that there exists a coefficient of $\mathrm{Pol}_{s=\beta_\varphi} J_0(s, \varphi)$ (in particular the highest order pole term) which equals a finite sum of integrals, each of which has the form

$$\int_K \omega,$$

where K is a compact subset of $[0, \infty)^m$, for some $m \leqslant n - 1$, and ω is a differential which is nonnegative and locally integrable over K. Thus, the value of this coefficient is positive. This completes the proof of the theorem.

(5.15) REMARKS:
1) Vasileev [13] has also given a geometric characterization of the order of the pole of $\mathscr{D}_P(s, \varphi)$.
2) One can use a Tauberian theorem of Hardy–Littlewood ("Tauberian theorems concerning power series and Dirichlet series whose coefficients are positive", cf. collected works of Hardy, vol. 6) to understand the asymptotic behavior of a certain weighted sum defined as follow. Let $P(z_1, \ldots, z_n)$ be a polynomial with positive integral coefficients so that the polynomial Q, obtained via (5.2), satisfies (5.9). Define

$$c_k = \mathrm{card}\,\{(m_1, \ldots, m_n) \in \mathbf{N}^n : P(m_1, \ldots, m_n) = k\}.$$

Clearly, for Re $(s) \gg 0$

$$\Sigma_{k \geqslant 1} c_k / k^s = \Sigma_{m \in \mathbf{N}^n} 1/P(m)^s.$$

The result of Hardy–Littlewood applies to the sequence $\{c_k\}$. Let J be the order of the pole β_1.

Set

$$s_n = \sum_{k=1}^{n} \frac{c_k}{k^{\beta_1}}.$$

Then the theorem shows that

$$s_n \sim \frac{A}{\Gamma(J+1)} (\log n)^J \quad (\sim \ = \text{asymptotic equivalence}),$$

where A is the coefficient of the term $1/(s - \beta_1)^J$ in the Laurent series for the above Dirichlet series at $s = \beta_1$. The constant A is an effectively computable constant using an explicit toroidal resolution and the analysis of the proof of Theorem (2).

(3) It follows from (2) that if $\beta_1 < 1$ then

$$(1/x) \sum_{k<x} x_k \sim 0 \quad \text{as} \quad x \to \infty.$$

On the other hand, if $\beta_1 \geqslant 1$, an application of Perron's formula [9] shows that for P as in Theorem 2 but with integral coefficients and for example, a simple pole at β_1, one has in the notation of (2), and with deg $P = d$,

$$(1/x) \sum_{k<x} c_k = Cx^{\beta_1 - 1} + 0_\varepsilon(x^{\beta_1 - (d+1)/d + \varepsilon}) \quad \text{as} \quad x \to \infty,$$

where $C > 0$. So, from the point of view of investigating when and how often such a polynomial assumes arbitrarily large integral values, it makes sense to restrict one's attention to those whose polyhedra force $\beta_1 \geqslant 1$.

Concluding remarks

Using the functional equations from Section 3, one can also show the existence of analytic continuations for Dirichlet series with several polynomials and in several variables. This in turn should allow one to obtain asymptotic information on the sizes of the sets

$$\{P_1 \leqslant k_1, \ldots, P_m \leqslant k_m\} \quad \text{as} \quad k_1, \ldots, k_m \to \infty .$$

It would be interesting to improve the error estimates in [10] using the identity (4.18) and information on the degree of the b-function $b_2(s)$ but it is not yet clear how to do this.

It would be interesting to prove Theorem 1 for polynomials P satisfying only the growth condition

$$\lim_{\substack{\|x\| \to \infty \\ x \in [B, \infty)^n}} |P(x)| = \infty$$

However, it is not yet clear how to do this.

These and other issues we hope to discuss in subsequent work.

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