PETER B. GILKEY

The eta invariant and the equivariant unitary bordism of spherical space form groups

Compositio Mathematica, tome 65, n° 1 (1988), p. 33-50
<http://www.numdam.org/item?id=CM_1988__65_1_33_0>

Compositio Mathematica 65: 33–50 (1988)

The eta invariant and the equivariant unitary bordism of spherical space form groups

PETER B. GILKEY
Math. Dept. Univ. Oregon, Eugene, OR 97403, USA

Received 26 September 1986; accepted in revised form 31 July 1987

Abstract. The eta invariant of Atiyah et al is a $\mathbb{R}/\mathbb{Z}$ valued invariant of equivariant unitary bordism completely detecting $MU*(BG)$ for spherical space form groups G. We use the eta invariant to compute the additive structure of $MU*(BQ_2)$ for $Q_2 = \{\pm 1, \pm i, \pm j, \pm k\}$.
MOS 58G12 (primary) 57R85 (secondary).

0. Introduction

Let G be a finite group with classifying space BG. G is a spherical space form group if there exists a fixed point free representation $\tau\colon G \to U(k)$ for some k. We assume henceforth G is such a group; these groups have been classified by Wolf [13]. Let $MU*(BG)$ and $M\mathrm{Spin}^c*(BG)$ be the reduced equivariant unitary and Spin^c bordism groups. If A is an Abelian group, let $A_{(p)}$ denote the p-primary torsion of A. The Anderson–Brown–Peterson splitting expresses $M\mathrm{Spin}^c*(BG)_{(2)}$ in terms of homology and in terms of connective K-theory $bu*$; see [3, 4, 7] for details:

$$\begin{aligned} M\mathrm{Spin}^c*(BG)_{(2)} &= bu*(BG)_2 \otimes \mathbb{Z}[x_4, x_8, \ldots] \\ &\quad \oplus \tilde{H}*(BG; \mathrm{Tor}\,(\tilde{\Omega}_*^{\mathrm{Spin}^c})). \end{aligned}$$

The corresponding splitting of the spectrum $\underline{MU}$ at any prime or $\underline{M\mathrm{Spin}^c}$ at odd primes is in terms of the Brown–Peterson homology $BP*$ and not $bu*$:

$$MU*(BG)_{(p)} = \{\mathbb{Z}[x_{2i} \mid i \neq p^v - 1]\} \otimes BP*(BG)$$

so [3, 4, 7] do not give $MU*(BG)_{(2)}$. We conjecture nevertheless:

CONJECTURE 0.1: *There exists an **additive** splitting*

$$MU*(BG) \cong bu*(BG) \otimes \mathbb{Z}[x_4, x_6, \ldots].$$

The Sylow subgroups of G are given as follows. Let $\mathbb{Z}_n = \{\lambda \in \mathbb{C}: \lambda^n = 1\}$ be the cyclic group of order n and let $\varrho_s(\lambda) = \lambda^s$ be the irreducible representations of $\mathbb{Z}_n$ for $0 \leqslant s < n$; $\varrho_s: \mathbb{Z}_n \to U(1)$ is fixed point free for s coprime to n so $\mathbb{Z}_n$ is a spherical space form group. Identify $SU(2)$ with the unit sphere S^3 of the quaternions. Any finite subgroup G of S^3 is fixed point free. If $m \geqslant 2$, let $Q_m \subseteq S^3$ be the group of order 2^{m+1} generated by $\{\cos(2\pi/2^m) + i \cdot \sin(2\pi/2^m), j\}$ and let $\tau_0: Q_m \mapsto SU(2)$ be the natural embedding; $Q_2 = \{\pm 1, \pm i, \pm j, \pm k\}$. Let H_p be a p-Sylow subgroup of G. H_p is cyclic if p is odd and either cyclic or one of the Q_m for $p = 2$.

There is one other group we shall need. Embed the alternating group A_4 on 4 letters as the orientation preserving isometries of the tetrahedron. The 2-fold cover of A_4 in $SU(2)$ is a group with 24 members isomorphic to the special linear group of 2×2 matrices on the field with 3 elements $SL(2, 3)$. This group may be identified with $\{\pm 1, \pm i, \pm j, \pm k, (\pm 1 \pm i \pm j \pm k)/2\} \subseteq SU(2)$.

We will use the eta invariant to study $MU*(BG)$. Let $R(G)$ be the group representation ring of G and let $R_0(G)$ be the augmentation ideal. Let $R(U)$ be the (stable) group representation ring of the unitary group. If $M \in MU*(BG)$ and if $\theta \in R_0(G) \otimes R(U)$, let $\eta(\theta, M) \in \mathbb{R}/\mathbb{Z}$ be the eta invariant of the tangential operator of the Dolbeault complex on M with coefficients in the bundle $\theta(M)$ defined by θ; see [1, 8]. The map $M \to \eta(\theta, M)$ extends to a map in bordism η: $MU*(BG) \otimes R_0(G) \otimes R(U) \mapsto \mathbb{R}/\mathbb{Z}$ taking values in $\mathbb{Q}/\mathbb{Z}$ as $MU*(BG)$ is finite in each dimension. We will prove in section 2

THEOREM 0.2: *If* $M \in MU*(BG)$ *and* $\eta(\theta, M) = 0 \,\forall\, \theta \in R_0(G) \otimes R(U)$, $M = 0$.

We also refer to a similar result by Wilson [12]. The Hattori–Stong theorem plays an essential role in the proof and Theorem 0.2 is the generalization of the Hattori–Stong theorem to equivariant unitary bordism.

The connective K-theory groups $bu*$ can be computed in terms of the representation theory. If G is cyclic,

$$bu_{2k-1}(B\mathbb{Z}_n) \cong R_0(\mathbb{Z}_n)/R_0(\mathbb{Z}_n)^{k+1} \cong \tilde{K}(S^{2k+1}/\mathbb{Z}_n).$$

If G is quaternionic, let $I = (2 - \tau_0) \cdot R(Q_m) \subseteq R_0(Q_m)$. We showed in [7]

$$bu_{4k-3}(BQ_m) = R_0(Q_m)/I^k \cong \tilde{K}(S^{4k-1}/Q_m) \text{ and } bu_{4k-5}(BQ_m) = I/I^k.$$

If $G = \mathbb{Z}_p$ for p prime, 0.1 follows from arguments of Conner–Floyd [5]. We have constructed an analytic proof for $G = \mathbb{Z}_4$ and $G = \mathbb{Z}_9$. Bendersky and Davis [2] proved 0.1 for cyclic groups which proves 0.1 at the prime p

if H_p is cyclic (which is always the case if p is odd). Mesnaoui [11] has studied $BP*(BQ_m)$ in terms of the Gysin sequence. We will prove in sections 3 and 4:

THEOREM 0.3:
(a) $bu_1(BSL(2, 3)) = 0$. *If* $k > 1$,
$bu_{4k-3}(BSL(2, 3))_{(2)} \cong bu_{4k-5}(BSL(2, 3))_{(2)} \cong \mathbb{Z}_{2^{2k-1}} \otimes \mathbb{Z}_{2^{k-2}}$
(b) $bu_1(BQ_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. *If* $k > 1$,
$bu_{4k-3}(BQ_2) \cong bu_{4k-3}(BSL(2, 3))_{(2)} \oplus \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}$ *and*
$bu_{4k-5}(BQ_2) \cong bu_{4k-5}(BSL(2, 3))_{(2)} \oplus \mathbb{Z}_{2^{k-1}} \oplus \mathbb{Z}_{2^{k-1}}$.
(c) *If* $G = SL(2, 3)$ *or* $G = Q_2$, $MU*(BG)_{(2)} = bu*(BG)_{(2)} \otimes \mathbb{Z}[x_4, x_6, \ldots]$.

REMARK: If $H_2 = Q_2$, and $X* = \tilde{H}$, $MU*$, $bu*$, or $BP*$, then

$$X*(BG)_{(2)} \cong X*(BQ_2) \text{ if } H_1(BG; \mathbb{Z}) \neq 0$$

$$X*(BG)_{(2)} \cong X*(BSL(2, 3)) \text{ if } H_1(BG; \mathbb{Z}) = 0$$

so this proves conjecture 0.1 at the prime 2 if $H_2 = Q_2$.

We believe the analytic approach we shall use to prove Theorem 0.3 is of independent interest since it has a very different flavor from the standard topological methods.*

It is a pleasant task to acknowledge the help and encouragement of *A.* Bahri at every step of the way. We also acknowledge the suggestions of the referee in shaping the final form of this paper and the help of G. Seitz.

1. Topological preliminaries

If $\varrho \in R(G)$ and $M \in MU*(BG)$, let $\varrho(M)$ be the associated vector bundle; extend this by linearity to define ring homomorphisms π: $R(G) \mapsto K(M)$ and π: $R_0(G) \mapsto \tilde{K}(M)$. If τ: $G \to U(k)$ is fixed point free, let $N(G, \tau) = S^{2k-1}/\tau(G)$. The underlying real bundle of $\pi(\tau)$ is $T(N(G, \tau)) \oplus 1$ so $N(G, \tau)$ inherits a natural stable complex structure. Since $N(G, \tau)$ is odd dimensional, it bounds in $MU*$ so $N(G, \tau)$ belongs to the reduced group $MU*(BG)$.

Let $X(-)$ be one of the functors $\tilde{H}$, MU, bu, or BP and let X be the associated coefficient ring:

$$MU* = \mathbb{Z}[x_{2i}: i = 1, 2, \ldots],$$

$$BP* = \mathbb{Z}[x_{2j}: j = p^v - 1 \quad \text{for} \quad v = 1, 2, \ldots], \; bu* = \mathbb{Z}[x_2].$$

* (added in proofs) Recently a proof by Bahri, Bendersky, and Davis has been given of 0.1 using entirely different methods.

(If $X = \tilde{H}$, let the coefficient ring be $\mathbb{Z}$). There is a slight notational difficulty, here since by $X(-)$ we mean the reduced theory while the coefficient ring is the un-reduced theory evaluated at a point, but in practice this causes no problems. If $X = MU$, BP, or bu, there is a spectral sequence for $X*(BG)$ with $E^2_{p,q}$ term $\tilde{H}_p(BG; X_q)$. By Landweber [10], all the differentials in the spectral sequence vanish. This is the crucial topological fact we shall use to derive information for these functors from information about $\tilde{H}$. For example, $\tilde{H}_{\text{even}}(BG; \mathbb{Z}) = 0$ implies $X_{\text{even}}(BG) = 0$.

Let H be a subgroup of G. Induction and transfer define maps

$$i\colon X*(BH) \to X*(BG) \quad \text{and} \quad t\colon X*(BG) \to X*(BH).$$

We need to study i and t in some detail on $MU*$. Let A, $B \subseteq G$; we wish to describe $t_A \cdot i_B\{N(B, \tau)\}$ in terms of double cosets. Let $B \mapsto S^{2k-1} \mapsto N(B, \tau)$ be the left principal B-bundle defining the B-structure on $N(B, \tau)$. Then $G \mapsto G \times_B S^{2k-1} \mapsto N(B, \tau)$ is the left-principal G-bundle defining $i_B(N(B, \tau))$ and $A \mapsto G \times_B S^{2k-1} \mapsto A\backslash\{G \times_B S^{2k-1}\}$ is the left-principal A bundle defining $t_A \cdot i_B(N(B, \tau))$. Induction changes the total space but not the base while transfer changes the base but not the total space. Decompose $G = U_i A g_i B$ into double cosets. Let $A_i = g_i \cdot B \cdot g_i^{-1} \cap A$ and let $\tau_i(a) = \tau(g_i^{-1} \cdot a \cdot g_i)\colon A_i \mapsto U(k)$. The connected components of $A\backslash\{G \times_B S^{2k-1}\}$ are $S^{2k-1}/\tau_i(A_i)$ and the total space is $A \times_{A(i)} S^{2k-1}$. This proves:

LEMMA 1.1: *Let A, $B \subseteq G$ and $\tau\colon B \mapsto U(k)$. Let $\{g_i\}$ be representatives for the double cosets $A\backslash G/B$. Let $A_i = g_i \cdot B \cdot g_i^{-1} \cap A$ and $\tau_i(a) = \tau(g_i^{-1} \cdot a \cdot g_i)$. Then $t_A \cdot i_B\{N(B, \tau)\} = \Sigma_i i_A \cdot N(A_i, \tau_i)$.*

Let $N_G(H_p) = \{g \in G\colon g \cdot H_p \cdot g^{-1} = H_p\}$ be the normalizer of the p-Sylow subgroup. Let $\operatorname{aut}(H_p)$ be the group of automorphisms and let $m\colon N_G(H_p) \mapsto \operatorname{aut}(H_p)$ by $m(g)h = ghg^{-1}$. Any automorphism of H_p induces an automorphism of $X*(BH_p)$; let $m_X\colon N_G(H_p) \mapsto \operatorname{aut}(X*(BH_p))$.

LEMMA 1.2: Let $X*(-) = \tilde{H}(-; \mathbb{Z})$, $MU*(-)$, $BP*(-)$, *or* $bu*(-)$.

(a) $|X_a(BG)| = \Pi_{b+c=a} |\tilde{H}_b(BG; \mathbb{Z})|^{\operatorname{rank}_{\mathbb{Z}}(Xc)}$.

(b) $\tilde{H}_{2k}(B\mathbb{Z}_n; \mathbb{Z}) = 0$, $\quad \tilde{H}_{2k+1}(B\mathbb{Z}_n) = \mathbb{Z}_n$,
$\tilde{H}_{2k}(BQ_m; \mathbb{Z}) = 0$, $\quad \tilde{H}_{4k+1}(BQ_m) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\tilde{H}_{4k+3}(BQ_m) = \mathbb{Z}_{2^{m+1}}$,
$\tilde{H}_{2k}(BSL(2,3); \mathbb{Z}) = 0$, $\tilde{H}_{4k+1}(BSL(2,3)) = \mathbb{Z}_3$, $\tilde{H}_{4k+3}(BSL(2,3)) = \mathbb{Z}_{24}$.

(c) *$\{i \cdot N(H, \sigma)\}$ for $H \subseteq G$ and σ a fixed point free representation of H generates $MU*(BG)$ as an $MU*$module.*

(d) *$i\colon X*(BH_p) \mapsto X*(BG)_{(p)}$ is $1-1$; $t\colon X*(BG)_{(p)} \mapsto X*(BH_p)$ is onto.*

(f) *$t\{X*(BG)_{(p)}\} \subseteq \{y \in X*(BH_p)\colon m_X(g) \cdot y = y \ \forall\, g \in N_G(H_p)\}$. If equality holds for $X* = \tilde{H}$, it follows for the other functors.*

Proof: (a) follows from Landweber [10] since the spectral sequence has trivial differentials. (b) is an easy calculation using characteristic classes and is therefore omitted. If $M \in MU*(BG)$, let $\mu(M) \in \tilde{H}(BG; \mathbb{Z})$ be the image of the fundamental class. Since $\{\mu(i \cdot N(H, \tau)\}$ generate $\tilde{H}(BG; \mathbb{Z})$, (c) follows. $\tilde{H}*(BH_p; \mathbb{Z})$ and hence $X*(BH_p)$ is p-primary. Since $i \cdot t$ is multiplication by $|G: H_p|$ on $\tilde{H}$, (d) follows for $X* = \tilde{H}*$ and hence for the other functors as the relevant spectral sequences degenerate. Since transfer commutes with group isomorphism and since conjugation by g is inner on G,

$$t(X*(BG)) \subseteq \{y \in X*(BH_p): m_X(g) \cdot y = y \;\forall\; g \in N_G(H_p)\}.$$

We now suppose equality for $X = \tilde{H}$. To prove equality for the other functors, we recall some facts from representation theory. Let A be an Abelian p-group and let $m_A: B \mapsto \operatorname{aut}(A)$. Define

$$A^0 = \{a \in A: m_A(b) \cdot a = a \;\forall\; b \in B\}$$

and

$$A^1 = \operatorname{span}_{\mathbb{Z}}\{(m_A(b) - 1) \cdot a\}_{b \in B, a \in A}.$$

If $|m_A(B)|$ is coprime to p, $A = A^0 \oplus A^1$ is a direct sum decomposition of A into the invariant and non-invariant pieces.

If $h \in H$, then $m_X(h) = 1$ since h acts by inner automorphisms. This implies $|m_X(N_G(H_p))| = |m_X(N_G(H_p)/H_p)|$ is coprime to p. We split $X*(BH_p)$ and the bordism spectral sequence for $X*(BH_p)$ as above. Since $\tilde{H}*(BG; \mathbb{Z})_{(p)} \cong \tilde{H}*(BH_p; \mathbb{Z})^0$, t is an isomorphism from $E^2_{p,q}(BH_p)^0$. Therefore t is an isomorphism from $X*(BG)_{(p)}$ to $X*(BH_p)^0$. ∎

The Smith homomorphism is used to perform induction on the dimension. Let $\tau: G \to U(k)$ be fixed point free. Embed $N(G, j \cdot \tau)$ in $N(G, (j + 1) \cdot \tau)$ by embedding C^{2kj} in $C^{2k(j+1)}$ using the first $2kj$ coordinates. The classifying space $BG = \operatorname{LIM}_{j \mapsto \infty} N(G, j \cdot \tau)$. If $M \in MU_*(G)$, let $f: M \to N(G, (j + 1) \cdot \tau)$ be the classifying space for j large. Make f transverse to $N(G, j \cdot \tau)$ and let $\Delta_\tau(M) = f^{-1}(N(G, j \cdot \tau))$; Δ_τ is a well defined $MU*$ module morphism called the Smith homomorphism. As $\Delta_\tau(i(N(H, \tau \oplus \tau_1))) = i(N(H, \tau_1))$, Δ_τ is onto by Lemma 1.2. Conner–Floyd [5] discuss the Smith homomorphism for $MSO*(BG)$ at odd primes; the situation here is similar. This proves

LEMMA 1.3: *Δ_τ extends as an $MU*$ module morphism*

$$\Delta_\tau: MU_\nu(BG) \to MU_{\nu-2k}(BG) \to 0.$$

2. The eta invariant

If $M \in MU*$ and if $\psi \in R(U)$, let index$(\psi, M) \in \mathbb{Z}$ be the index of the Dolbeault complex with coefficients in ψ; this is a bordism invariant. In particular the arithmetic genus $ag(M)$ is index$(1, M)$. If Λ^i is the ith exterior representation and if det is the determinant representation, then $R(U) = \mathbb{Z}[\Lambda^i, \det, \det^{-1}]$ modulo the obvious relations.

We study the eta invariant on Cartesian products as follows. Let $s(\Lambda^i) = \oplus_{j+k=i} \Lambda^j \otimes \Lambda^k$ define a comultiplication on $R(U)$. If $\psi \in R(U)$, decompose $s(\psi) = \Sigma_i \psi_{1,i} \otimes \psi_{2,i}$. If $M = M_1 \times M_2$ for $M_\nu \in MU*$, then $\psi(M) = \oplus_i \psi_{1,i}(M_1) \otimes \psi_{2,i}(M_2)$. We refer to [7, Lemma 4.3.6] for the proof of:

Lemma 2.1: *If $M_1 \in MU*(BG)$ and $M_2 \in MU*$, let $M = M_1 \times M_2$. If $\theta \in R_0(G) \otimes R(U)$, let $(1 \otimes s)(\theta) = \Sigma_i \theta_i \otimes \psi_i$. Then $\eta(\theta, M) = \Sigma_i \eta(\theta_i, M_1) \cdot$ index(ψ_i, M_2). If $\varrho \in R_0(G)$, $\eta(\varrho, M) = \eta(\varrho, M_1) \cdot ag(M_2)$.*

The eta invariant is closely related to $R(G)$. Embed $R(G)$ in the class functions $C(G)$ and let $(f_1, f_2)_G = |G|^{-1} \cdot \Sigma_{g \in G} f_1(g) \cdot f_2(g)$ define a non-degenerate symmetric bilinear form on $C(G)$. Restriction r: $C(G) \mapsto C(H)$ and induction ind: $C(H) \mapsto C(G)$ are dual; $(f_1, \mathrm{ind}(f_2))_G = (r(f_1), f_2)_H$ by Frobenius reciprocity if $f_1 \in C(G)$ and $f_2 \in C(H)$. If τ is fixed point free, let

$$\alpha(\tau) = \det(\tau - I)/\det(\tau) \in R_0(G) \text{ and}$$

$$\beta(\tau)(g) = \alpha(\tau)^{-1}(g) \quad \text{for} \quad g \neq 1 \quad \text{and} \quad \beta(\tau)(1) = 0; \ \beta \in C(G).$$

We note $\beta(\tau) \cdot \alpha(\tau) \cdot \varrho = \varrho$ for $\varrho \in R_0(G)$. Finally define

$$\ker(\eta, G) = \{M \in MU*(BG): \eta(\theta, M) = 0 \ \forall \ \theta \in R_0(G) \otimes R(U)\}.$$

Lemma 2.2: *Let $H \subseteq G$.*
(a) *If $\varrho \in R_0(G)$, then $\eta(\varrho, N(G, \tau)) = (\varrho, \beta(\tau))_G$.*
(b) *If $M \in MU*(BH)$ and $\theta \in R_0(G) \otimes R(U)$, $\eta(\theta, i(M)) = \eta((r \otimes 1)\theta, M)$.*
(c) *If $M \in MU*(BG)$ and $\theta \in R_0(H) \otimes R(U)$, $\eta(\theta, t(M)) = \eta((\mathrm{ind} \otimes 1)\theta, M)$.*
(d) *i: $\ker(\eta, H) \mapsto \ker(\eta, G)$ and t: $\ker(\eta, G) \mapsto \ker(\eta, H)$.*
(e) *Δ_τ: $\ker(\eta, G) \mapsto \ker(\eta, G)$.*

Proof: An analogous formula for the tangential operator of the signature compex appears in Atiyah et al. [1, II-(2.9)]; the calculations for the tangential operator of the Dolbeault complex are the same which proves (a); this expresses the eta invariant in terms of trignometric sums. The bundles $\theta(i(M))$ and $(r \otimes 1)(\theta)(M)$ agree which proves (b). To prove (c), we may suppose $M = i(N(J, \tau)) \times M_1$ for $J \subseteq G$ and $M_1 \in MU*$ by Lemma 1.2. We use Lemma 2.1 and (a, b) to deduce (c) from Frobenius reciprocity; this gives the duality of (r, i) and (ind, t) with respect to the pairing of the eta invariant. (d) follows from (b, c). Let Δ correspond to τ. The normal bundle of $\Delta(M)$ in M is given by τ. Define an algebra isomorphism u of $R_0(G) \otimes R(U)$ by $u(1 \otimes \det) = \det(\tau) \otimes \det$, $u(1 \otimes \Lambda^i) = \Sigma_{a+b=i} \Lambda^a(\tau) \otimes \Lambda^b$, $u(\varrho \otimes 1) = \varrho \otimes 1$. Then if $\psi \in R_0(G) \otimes R(U)$,

$$\psi(M)|_{\Delta(M)} = u(\psi)(\Delta(M)).$$

We use Lemmas 1.2, 2.1, and 2.2 to see

$$\eta(u(\theta), \Delta(M)) = \eta(\theta \cdot \alpha(\tau), M) \ \forall\ \theta \in R_0(G) \otimes R(U) \ \forall\ M \in MU*(BG).$$

Since u is an isomorphism, $M \in \ker(\eta, G)$ implies $\Delta(M) \in \ker(\eta, G)$. ■

We prove Theorem 0.2 one prime at a time. Let H_p be a Sylow subgroup of G. Since t: $MU*(BG)_{(p)} \mapsto MU*(BH_p)_{(p)}$ is $1 - 1$ and since $t(\ker(G, \eta)) \subseteq \ker(\eta, H_p)$, it suffices to prove Theorem 0.2 for $G = H_p$. Let τ: $H_p \mapsto U(k)$ be fixed point free and irreducible. Suppose inductively $\ker(\eta, H_p) \cap MU_v(BH_p) = \{0\}$ for $v < j$. Let $M \in \ker(\eta, H_p) \cap MU_j(BH_p)$. Then $\Delta_\tau(M) \in \ker(\eta, H_p) \cap MU_{j-2k}(BH_p) = \{0\}$ so $\Delta(M) = 0$. We complete proof of Theorem 0.2 by showing $\ker(\eta, H_p) \cap \ker(\Delta_\tau) = 0$. Suppose first H_p is cyclic.

LEMMA 2.3: *Let $M_1 = N(\mathbb{Z}_n, \varrho_1)$ and let Δ correspond to ϱ_1.*
(a) $\eta(\varrho_0 - \varrho_1, M_1) = n^{-1}$. *$MU_1(B\mathbb{Z}_n) = \mathbb{Z}_n$ is generated by M_1.*
(b) *Let $N \in MU_{2k}$. If $M_1 \times N \in \ker(\eta, \mathbb{Z}_n)$, then $N \in n \cdot MU_{2k}$ and $M_1 \times N = 0$.*
(c) $\ker(\Delta) \cap MU_{2k+1}(B\mathbb{Z}_n) = M_1 \times MU_{2k}$ *and* $\ker(\Delta) \cap \ker(\eta, \mathbb{Z}_n) = 0$.

Proof: By Lemmas 1.2 and 2.2, $\eta(\varrho_0 - \varrho_1, M_1) = n^{-1}$ and $|MU_1(B\mathbb{Z}_n)| = n$ which proves (a). Let $\psi \in R(U)$ and decompose $s(\psi) = \Sigma_i \psi_{1,i} \otimes \psi_{2,i}$. Since $T(M_1) = 1$, $\psi(M_1 \times N) = \psi'(N)$ for $\psi' = \Sigma_i \dim(\psi_{1,i})(M_1) \otimes \psi_{2,i}$. If $M_1 \times N \in \ker(\eta, \mathbb{Z}_n)$, by Lemma 2.1,

$$0 = \eta((\varrho_0 - \varrho_1) \otimes \psi, M_1 \times N) = n^{-1} \cdot \mathrm{index}(\psi', N).$$

The map $\psi \mapsto \psi'$ is an algebra isomorphism of $R(U)$, so index(ψ, N) is divisible by n $\forall$ $\psi \in R(U)$. We now come to the essence of the matter. By the Hattori–Stong theorem, $N \in n \cdot MU_{2k}$ proving (b). Consequently $M_1 \times MU_{2k} \cong \mathbb{Z}_n \otimes MU_{2k}$ has n^u elements for $u = \mathrm{rank}_{\mathbb{Z}} MU_{2k}$. Furthermore $M_1 \times MU_{2k} \cap \ker(\eta, \mathbb{Z}_n) = \{0\}$ and $M_1 \times MU_{2k} \subseteq \ker(\Delta)$. By Lemma 1.3, Δ is onto so $\ker(\Delta) \cap MU_{2k+1}(B\mathbb{Z}_n) = |MU_{2k+1}(B\mathbb{Z}_n)|/|MU_{2k-1}(B\mathbb{Z}_n)| = n^u$. ∎

If H_p is not cyclic, then $p = 2$ and $H_p = Q_m$. Let τ_0: $Q_m \mapsto SU(2)$ be the canonical representation and let $x = \cos(2\pi/2^m) + i \cdot \sin(2\pi/2^m)$ and $y = j$ generate Q_m. There are 4 linear representations of Q_m defined by:

$$\varrho_0(x) = 1 \quad \varrho_x(x) = 1 \quad \varrho_y(x) = -1 \quad \varrho_{xy}(x) = -1$$

$$\varrho_0(y) = 1 \quad \varrho_x(y) = -1 \quad \varrho_y(y) = 1 \quad \varrho_{xy}(y) = -1;$$

If $m = 2$, we will denote these by $\{\varrho_0, \varrho_i, \varrho_j, \varrho_k\}$ since $x = i, y = j, xy = k$ in that instance. If $z \in Q_m$, let H_z be the cyclic subgroup generated by z. The restriction of τ_0 to H_x or H_y is $\varrho_1 \oplus \varrho_3$. Let

$$M_x = i \cdot N(H_x, \varrho_1), M_y = i \cdot N(H_y, \varrho_1) \text{ and } M_q = i \cdot N(Q_m, \tau_0).$$

LEMMA 2.4: *Let Δ correspond to τ_0.*
(a) $MU_1(BQ_m) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *with basis* $\{M_x, M_y\}$.
(b) $MU_3(BQ_m) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2^{m+1}}$ *with basis* $\{M_x \times CP^1, M_y \times CP^1, M_q\}$.
(c) *If* $M \in MU_3(BQ_m)$ *and* $\eta(\varrho, M) = 0$ $\forall$ $\varrho \in R_0(Q_m)$, *then* $M = 0$.
(d) *If* $M = M_x \times N_x + M_y \times N_y + M_q \times N_q \in \ker(\eta, Q_m)$, *then* $N_x \in 2 \cdot MU*$, $N_y \in 2 \cdot MU*$, $N_q \in 2^m \cdot MU*$, *and* $M = 0$.
(e) $\ker(\Delta) \cap MU_{2k+1}(BQ_2) = M_x \times MU_{2k} \oplus M_y \times MU_{2k} \oplus M_q \times MU_{2k-2} \cong H_1(BQ_2; MU_{2k-2}) \oplus H_3(BQ_2; MU_{2k-4})$.

Proof: By Lemma 2.2,

$$\eta(\varrho_y - \varrho_0, M_x) = 1/2\eta(\varrho_y - \varrho_0, M_y) = 0$$

$$\eta(\varrho_x - \varrho_0, M_x) = 0 \quad \eta(\varrho_x - \varrho_0, M_y) = 1/2.$$

This gives a map from $MU_1(BQ_2) \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0$ which proves (a) since $|MU_1(BQ_2)| = 4$. Let $\alpha_0 = 2\varrho_0 - \tau_0$. By Lemmas 2.1 and 2.2,

$$\eta(\varrho_y - \varrho_0, M_x \times CP^1) = 1/2 \quad \eta(\varrho_y - \varrho_0, M_y \times CP^1) = 0 \quad \eta(\varrho_y - \varrho_0, M_q) = *$$

$$\eta(\varrho_x - \varrho_0, M_x \times CP^1) = 0 \quad \eta(\varrho_x - \varrho_0, M_y \times CP^1) = 1/2 \quad \eta(\varrho_y - \varrho_0, M_q) = *$$

$$\eta(\alpha_0, M_x \times CP^1) = 0 \quad \eta(\alpha_0, M_y \times CP^1) = 0 \quad \eta(\alpha_0, M_q) = -1/2^{m+1}$$

which proves (b, c). The rest of the proof is essentially the same as that of Lemma 2.3 and so is omitted. This completes the proof of Theorem 0.2.

The eta invariant is closely related to K-theory as well.

LEMMA 2.5: *Let τ, τ' be fixed point free and let $\varrho \in R_0(G)$.*
(a) *If* $\deg(\tau) = \deg(\tau')$, *then* $\alpha(\tau) \cdot R(G) = \alpha(\tau') \cdot R(G)$. *If* $\deg(\tau') < \deg(\tau)$, *then* $\alpha(\tau) \in \alpha(\tau') \cdot R_0(G)$. $R_0(G)/\alpha(\tau) \cdot R(G) \cong \tilde{K}(N(G, \tau))$.
(b) $\varrho \in \alpha(\tau) \cdot R(G)$ *iff* $\eta(\varrho \otimes \sigma, N(G, \tau)) = 0 \,\forall\, \sigma \in R_0(G)$. *$\eta$ is a perfect pairing* η: $\tilde{K}(N(G, \tau)) \otimes \tilde{K}(N(G, \tau)) \mapsto \mathbb{Q}/\mathbb{Z}$ *exhibiting* $\tilde{K}(N(G, \tau))$ *as its Poincare dual. If* $\varrho \in \alpha(\tau) \cdot R_0(G)$, *then* $\eta(\varrho, N(G, \tau)) = 0$.

Proof: See Gilkey [9, Theorem 3.6].

We use Lemma 2.5 to relate define a map from $MU*$ to K-theory.

LEMMA 2.6: *Let τ be fixed point free of degree k and let $v < 2k - 1$.*
(a) *If* $M \in MU_v(BG)$ *and* $\theta \in \alpha(\tau) \cdot R(G) \otimes R(U)$, *then* $\eta(\theta, M) = 0$.
(b) $\exists!\, g_\tau$: $MU_v(BG) \mapsto R_0(G)/\alpha(\tau) \cdot R(G)$ *so* $\eta(\varrho, M) = \eta(\varrho \cdot g_\tau(M), N(G, \tau))$ $\forall\, \varrho \in R_0(G)$. $\ker(g_\tau) = \{M \in MU_v(BG)\colon \eta(\varrho, M) = 0 \,\forall\, \varrho \in R_0(G)\}$

Proof: We apply Lemmas 1.2, 2.1, 2.2 and 2.5. Let $\theta = \varrho \otimes \psi \in \alpha(\tau) \cdot R(G) \otimes R(U)$. If $M = N(G, \tau')$, choose $\sigma \in R(G)$ so $\sigma(M) = \psi(M)$. Since $\varrho \cdot \sigma \in \alpha(\tau)R(G) \subseteq \alpha(\tau') \cdot R_0(G)$, $\eta(\theta, M) = \eta(\varrho \cdot \sigma, M) = 0$. If $M = i \cdot N(H, \tau')$, then $\eta(\theta, M) = \eta(r(\theta), N(H, \tau')) = 0$ since $r(\theta) \in \alpha(r(\tau)) \cdot R(H) \otimes R(U)$. Since the comultiplication s is $R(G)$ linear, $(1 \otimes s)(\theta) \in \alpha(\tau) \cdot R_0(G) \otimes R(U) \otimes R(U)$ so $\eta(\theta, i \cdot N(H, \tau') \cdot N_1) = 0$. Such manifolds generate $MU*(BG)$ which proves (a). Define f_M: $R_0(G)/\alpha(\tau) \cdot R(G) \mapsto \mathbb{Q}/\mathbb{Z}$ by $f_M(\varrho) = \eta(\varrho, M)$. $\exists!\, g_\tau(M) \in R_0(G)/\alpha(\tau) \cdot R(G)$ so $\eta(\varrho \cdot g_\tau(M), N(G, \tau)) = f_M(\varrho)$; $g_\tau(M) = 0 \Leftrightarrow \eta(\varrho, M) = 0 \,\forall\, \varrho \in R_0(G)$. ∎

3. $bu*(BQ_2)$ and $bu*(BSL(2, 3))$.

In Lemma 1.2, we showed it sufficed to study the Sylow subgroups of G. In fact, $MU*$, $BP*$, and $bu*$ are determined by the maximal cyclic subgroups.

LEMMA 3.1: *Let $X* = MU*$, $BP*$, or $bu*$. Let $\{C_v\}$ be the family of maximal cyclic prime order subgroups of G. If $M \in MU*(BG)$ and $t_v(M) = 0 \,\forall\, v$, then $M = 0$.*

Proof: If $X* = bu*$, this follows by [9, Theorem 4.1(a)]. Let $X* = MU*$. By Lemma 1.2 and the transitivity of transfer, we may assume $G = H_p$. Lemma 3.1 is trivial if H_p is cyclic so let $G = Q_m$. Let $H_z = \langle z \rangle$ and suppose $t_z(M) = 0$ for $z = x, y$, or xy. By Lemma 2.2, $\eta(\theta, M) = 0 \ \forall \ \theta \in \mathrm{ind}_z\{R_0(H_z) \otimes R(U)\}$. We showed

$$\mathrm{span}_{\mathbb{Z}}\{\mathrm{ind}_x\{R_0(H_x)\}, \mathrm{ind}_y\{R_0(H_y)\}, \mathrm{ind}_{xy}\{R_0(H_{xy})\}\} = R_0(Q_m)$$

[9, Lemma 4.6] so $\eta(\theta, M) \in \ker(\eta, G) = 0$. We use the splitting of $MU*$ in terms of $BP*$ to prove this for $BP*$. Lemma 3.1 is false for $X = \tilde{H}$ and $G = Q_m$. ■

We begin by studying $R_0(\mathbb{Z}_4)$:

LEMMA 3.2: Let $u = \alpha(\varrho_1 \oplus \varrho_3) = 2\varrho_0 - \varrho_1 - \varrho_3$, $v = \varrho_0 - \varrho_2$, *and* $w = \varrho_0 - \varrho_1$.
(a) $u^2 = 4u - 2v$, $u \cdot v = 2v$, *and* $v^2 = 2v$.
(b) $u^{n+1} = 4^n \cdot u - 2^n(2^n - 1) \cdot v$.
(c) *If* $n_1 \cdot u + n_2 \cdot v \in R_0(\mathbb{Z}_4)^{2n+2}$, *then* $n_1 \equiv 0(4^n)$ *and* $n_2 \equiv 0(2^n)$.
(d) *If* $n_2 \equiv 0(2^{n+1})$, *then* $n_2 \cdot v \in R_0(\mathbb{Z}_4)^{2n+2}$.
(e) *If* $n > 0$, *then the order of* u *in* $R_0(\mathbb{Z}_4)/R_0(\mathbb{Z}_4)^{2n+2}$ *is* 2^{2n+1}.

Proof: (a) is immediate. (b) is true for $n = 0$ so we proceed by induction.

$$\begin{aligned} u^{n+2} &= u \cdot (4^n u - 2^n(2^n - 1)v) = 4^{n+1} \cdot u - 2 \cdot 4^n \cdot v - 2^{n+1}(2^n - 1)v \\ &= 4^{n+1} \cdot u - 2^{n+1}(2^{n+1} - 1) \cdot v. \end{aligned}$$

We note $R_0(\mathbb{Z}_4)^{2n+2} = R(\mathbb{Z}_4) \cdot u^{n+1}$. Therefore $R_0(\mathbb{Z}_4)^{2n+2}$ is spanned by

$$\begin{aligned} u^{n+1} &= 4^n \cdot u - 2^n(2^n - 1) \cdot v + 0 \cdot w \\ u^{n+1} \cdot v &= 0 \cdot u + 2^{n+1} v + 0 \cdot w \\ u^{n+1} \cdot w &= * \cdot u + * \cdot v + a(n) \cdot w \end{aligned}$$

where $a(n) \neq 0$. This proves (c, d). Finally $m \cdot u \in R_0(\mathbb{Z}_4)^{2n+2}$ means

$$m \cdot u = a_1(4^n u - 2^n(2^n - 1)v) + a_2(2^{n+1})v$$

or $m = 4^n a_1$ and $a_1(2^n - 1) = 2a_2$ or equivalently m $\equiv 0 \bmod 2^{2n+1}$. ■

Let $\{\varrho_0, \varrho_i, \varrho_j, \varrho_k, \tau_0\}$ be the representations of Q_2 and let $\alpha_0 = \alpha(\tau_0)$. Define an action of $\mathbb{Z}_3$ on Q_2 by the cyclic permuation $i \mapsto j \mapsto k \mapsto i$. Decompose $X*(BQ_2) = X*(BQ_2)^{(0)} \oplus X*(BQ_2)^{(1)}$ under this action. Then

$$H_{4n-5}(BQ_2; \mathbb{Z})^{(0)} = H_{4n-5}(BSL(2, 3); \mathbb{Z})_{(2)} = \mathbb{Z}_8 \quad \text{for} \quad n > 1$$

$$H_{4n-3}(BQ_2; \mathbb{Z})^{(0)} = H_{4n-3}(BSL(2, 3); \mathbb{Z})_{(2)} = 0 \quad \text{for} \quad n > 0$$

so $X*(BQ_2)^{(0)} = X*(BSL(2, 3))_{(2)}$ for $X* = bu*$, $MU*$, or $BP*$ by Lemma 2.1. Furthermore, $X_1(BSL(2, 3))_{(2)} = 0$. If $n > 1$, then

$$bu_{4n-5}(BSL(2, 3))_{(2)} \cong bu_{4n-3}(BSL(2, 3))_{(2)} \text{ has } 8^{n-1} \text{ elements.}$$

$$|bu_{4n-5}(BQ_2)^{(1)}| = 2^{2(n-2)} \quad \text{and} \quad |bu_{4n-3}(BQ_2)^{(1)}| = 2^{2(n-1)}.$$

$R_0(Q_2)$ has 4 generators so $bu*(BQ_2)$ has 4 generators. Since no element of $bu*(BQ_2)^{(1)}$ is $\mathbb{Z}_3$ invariant, $bu*(BQ_2)^{(1)}$ has at least 2 generators so $bu*(BQ_2)^{(0)} = bu*(BSL(2, 3))_{(2)}$ has at most 2 generators. We wish to apply Lemma 3.1 and 3.2. Let $x \in bu_{4n-5}(BSL(2, 3))_{(2)}$. Then $t_z(x) \in bu_{4n-5}(BH_z) = R_0(\mathbb{Z}_4)^2/R_0(\mathbb{Z}_4)^{2n}$ for $z = i, j, k$. Since u generates $R_0(\mathbb{Z}_4)^2$, $2^{2n-1} \cdot R_0(\mathbb{Z}_4)^2 \subseteq R_0(\mathbb{Z}_4)^{2n}$ so $2^{2n-1} \cdot t_z(x) = 0$ and therefore $2^{2n-1} \cdot x = 0$. As $t_z(\alpha_0) = u$, α_0 has order 2^{2n-1}. Thus there are exactly 2 generators of $bu*(BSL(2, 3))_{(2)}$ and

$$bu_{4n-5}(BSL(2, 3)) \cong bu_{4n-3}(BSL(2, 3)) \cong \mathbb{Z}_{2^{2n-1}} \oplus \mathbb{Z}_{2^{n-2}}.$$

Consequently $bu*(BQ_2)^{(1)}$ has exactly two generators and admits a free $\mathbb{Z}_3$ action. If we can show $bu*(BQ_2)^{(1)} = \mathbb{Z}_{a(v)} \oplus \mathbb{Z}_{a(v)}$, then $a(4n - 5) = 2^{n-2}$ and $a(4n - 3) = 2^{n-1}$ which will complete the proof of Theorem 0.3(a, b).

LEMMA 3.3: *Let A be an Abelian 2-group on 2-generators with a fixed point free $\mathbb{Z}_3$ action. Then $A = \mathbb{Z}_a \oplus \mathbb{Z}_a$.*

Proof: Let $\lambda \in \mathbb{Z}_3$ be the generator and let $A = \mathbb{Z}_a \oplus \mathbb{Z}_b$ with generators x, y where $b \leqslant a$. Since the action is free, $(1 + \lambda + \lambda^2) \cdot x = 0$. If $\lambda \cdot x = cx + dy, 0 = x + \lambda x + \lambda^2 x = (1 + c + c^2) \cdot x + cd \cdot y + d \cdot \lambda(y)$ so $d(\lambda \cdot y) = (-1 - c - c^2) \cdot x + d \cdot y$. As $(1 + c + c^2)$ is odd, $b = \operatorname{ord}(y) \geqslant \operatorname{ord}(d\lambda y) \geqslant \operatorname{ord}(x) = a$. ■

The remainder of this section is devoted to the proof of two technical lemmas we will need in the next section to study $MU*(BQ_2)$ and $MU*(BSL(2, 3))_{(2)}$. The reader may wish to skip the proofs until they are needed. Let

$$\ker(\eta, R_0(G)) = \{M \in MU*(BG): \eta(\varrho, M) = 0 \; \forall \; \varrho \in \mathrm{R}_0(G)\}.$$

Then $\ker(\eta, R_0(G)) = \ker(g_\tau)$ by Lemma 2.6. Let $\tau\colon \mathbb{Z}_4 \mapsto U(n-1)$ and define

$$M_1 = N(\mathbb{Z}_4, \varrho_1 \oplus \tau), \quad M_2 = N(\mathbb{Z}_4, \varrho_3 \oplus \tau), \quad M_3 = i \cdot N(\mathbb{Z}_2, \varrho_1 \oplus \tau).$$

LEMMA 3.4: $\mathrm{Span}_{\mathbb{Z}}\{M_i\} \cap \ker(\eta, R_0(\mathbb{Z}_4)) = 0.$

Proof: We use Lemmas 2.5 and 2.6. Let $\tau' = \varrho_1 \oplus \varrho_3 \oplus \tau$ and $N = N(\mathbb{Z}_4, \tau')$. Let $g\colon MU_{2n-1}(B\mathbb{Z}_4) \mapsto R_0(\mathbb{Z}_4)/R_0(\mathbb{Z}_4)^{n+1}$ so $\eta(\varrho, M) = (\varrho \cdot g(M), N)\ \forall\ \varrho \in R_0(\mathbb{Z}_4)$. Let

$$x_1 = g(M_1) = (\varrho_0 - \varrho_1), \quad x_2 = g(M_2) = (\varrho_0 - \varrho_3),$$

$$x_3 = g(M_3) = \varrho_0 + \varrho_2 - \varrho_1 - \varrho_3.$$

Let $M = \Sigma_i n_i \cdot M_i \in \ker(\eta, R_0(\mathbb{Z}_4)) = \ker(g)$. We show $M = 0$ by showing $\eta(\theta, M) = 0\ \forall\ \theta \in R_0(\mathbb{Z}_4) \otimes R(U)$. Let $\theta = \sigma \otimes \psi$. We want to get rid of the dependence on $\psi \in R(U)$ to use Lemma 2.5. We can express $\psi(M_i)$ in terms of the representation theory;

$$\det(M_1) = \{\det(\tau) \cdot \varrho_1\}(M_1) \quad \Lambda^i(M_1) = \Lambda^i(\tau) \oplus \varrho_1 \cdot \Lambda^{i-1}(\tau)(M_1)$$

$$\det(M_2) = \{\det(\tau) \cdot \varrho_3\}(M_2) \quad \Lambda^i(M_2) = \Lambda^i(\tau) \oplus \varrho_3 \cdot \Lambda^{i-1}(\tau)(M_2)$$

$$\det(M_1) = \{\det(\tau) \cdot \varrho_1\}(M_3) \quad \Lambda^i(M_3) = \Lambda^i(\tau) \oplus \varrho_1 \cdot \Lambda^{i-1}(\tau)(M_3)$$

Thus we may assume $\theta(M_1) = (\sigma \cdot \varrho_b)(M_1)$, $\theta(M_2) = (\sigma \cdot \varrho_{3b})(M_2)$, and $\theta(M_3) = (\sigma \cdot \varrho_b)(M_3)$ for $\sigma \in R_0(\mathbb{Z}_4)$ and $b \in \mathbb{Z}$. Let

$$x_1(b) = \varrho_b \cdot x_1, \quad x_2(b) = \varrho_{3b} \cdot x_2, \quad x_3(b) = \varrho_b x_3.$$

If $x(b) = \Sigma_i n_i \cdot x_i(b)$, then $\eta(\theta, M) = \eta(\sigma \cdot x(b), N)$. If $x(b) \in R_0(\mathbb{Z}_4)^{n+1}$, $\eta(\theta, M) = 0$ which will complete the proof. As $x(b) = \varrho_b \cdot g(M) + \varrho_b \cdot (\varrho_{2b} - \varrho_0) \cdot n_2 \cdot x_2$, we must show $(\varrho_{2b} - \varrho_0) \cdot n_2 \cdot x_2 \in R_0(\mathbb{Z}_4)^{n+1}$. Let $x = g(M) \in R_0(\mathbb{Z}_4)^{n+1}$. We argue as follows. $R_0(\mathbb{Z}_4)$ is invariant under the involution $\varrho_s \to \varrho_s^* = \varrho_{-s}$ so

$$x = n_1(\varrho_0 - \varrho_1) + n_2(\varrho_0 - \varrho_3) + n_3(\varrho_0 + \varrho_2 - \varrho_1 - \varrho_3) \in R_0(\mathbb{Z}_4)^{n+1}$$

$$x^* = n_1(\varrho_0 - \varrho_3) + n_2(\varrho_0 - \varrho_1) + n_3(\varrho_0 + \varrho_2 - \varrho_1 - \varrho_3) \in R_0(\mathbb{Z}_4)^{n+1}$$

$$\varrho_1 \cdot x^* = n_1(\varrho_1 - \varrho_0) + n_2(\varrho_1 - \varrho_2) + n_3(\varrho_1 + \varrho_3 - \varrho_0 - \varrho_2) \in R_0(\mathbb{Z}_4)^{n+1}$$

$$x + \varrho_1 \cdot x^* = n_2(\varrho_0 + \varrho_1 - \varrho_2 - \varrho_3) = n_2(\varrho_0 - \varrho_2)(\varrho_0 - \varrho_3)$$

$$= n_2(\varrho_0 - \varrho_2)x_2 \in R_0(\mathbb{Z}_4)^{n+1}. \qquad \blacksquare$$

Next let $\tau = (n-1)(\varrho_1 \oplus \varrho_3)$ and let

$$M_1 = N(\mathbb{Z}_4, \varrho_1 \oplus \varrho_3 \oplus \tau) \quad M_2 = i \cdot N(\mathbb{Z}_2, \varrho_1 \oplus \varrho_1 \oplus \tau)$$

$$M_3 = N(\mathbb{Z}_4, \varrho_1 \oplus \varrho_1 \oplus \tau) \quad M_4 = N(\mathbb{Z}_4, \varrho_3 \oplus \varrho_3 \oplus \tau).$$

LEMMA 3.5: $\mathrm{Span}_{\mathbb{Z}}\{M_1, 2 \cdot M_2 + M_3 + M_4\} \cap \ker(\eta, R_0(\mathbb{Z}_4)) = 0.$

Proof: Let $N = N(\mathbb{Z}_4, (n+1) \cdot \tau)$. Let $g(M) \in R_0(\mathbb{Z}_4)/R_0(\mathbb{Z}_4)^{2n+2}$ so $\eta(\varrho, M) = \eta(\varrho \cdot g(M), N)\ \forall\ \varrho \in R_0(\mathbb{Z}_4)$. Let u and v be as in Lemma 3.2. If $x_i = g(M_i)$, then:

$$x_1 = u, \quad x_2 = (\varrho_0 + \varrho_2) \cdot u, \quad x_3 = -\varrho_1 \cdot u, \quad x_4 = -\varrho_3 \cdot u.$$

Let $M = n_1 M_1 + n_2(2M_2 + M_3 + M_4) \in \ker(\eta, R_0(\mathbb{Z}_4))$ so $g(M) \in R_0(\mathbb{Z}_4)^{2n+2}$. Let $\theta = \sigma \otimes \psi \in R_0(\mathbb{Z}_4) \otimes R(U)$. We must show $\eta(\theta, M) = 0$. As before, we must eliminate the dependence upon ψ. We compute:

$$\det(M_1) = 1 \quad \Lambda^i(M_1) = \Lambda^{i-2}(\tau) + (\varrho_1 + \varrho_3) \cdot \Lambda^{i-1}(\tau) + \Lambda^i(\tau)$$

$$\det(M_2) = 1 \quad \Lambda^i(M_2) = \Lambda^{i-2}(\tau) + (\varrho_1 + \varrho_1) \cdot \Lambda^{i-1}(\tau) + \Lambda^i(\tau)$$

$$\det(M_3) = \varrho_2 \quad \Lambda^i(M_3) = \varrho_2 \Lambda^{i-2}(\tau) + (\varrho_1 + \varrho_1) \cdot \Lambda^{i-1}(\tau) + \Lambda^i(\tau)$$

$$\det(M_4) = \varrho_2 \quad \Lambda^i(M_4) = \varrho_2 \Lambda^{i-2}(\tau) + (\varrho_3 + \varrho_3) \cdot \Lambda^{i-1}(\tau) + \Lambda^i(\tau)$$

so we may suppose $\theta(M_i)$ has the form:

$$\theta(M_1) = \sigma \cdot (\varrho_1 + \varrho_3)^a \qquad \theta(M_2) = \sigma \cdot (\varrho_1 + \varrho_1)^a$$

$$\theta(M_3) = \sigma \cdot \varrho_{2b} \cdot (\varrho_1 + \varrho_1)^a \quad \theta(M_4) = \sigma \cdot \varrho_{2b} \cdot (\varrho_3 + \varrho_3)^a.$$

for $\sigma \in R_0(\mathbb{Z}_4)$ and $a, b \in \mathbb{Z}$. Let

$$x_1(a, b) = (\varrho_1 + \varrho_3)^a \cdot x_1, \qquad x_2(a, b) = (\varrho_1 + \varrho_1)^a \cdot x_2,$$

$$x_3(a, b) = \varrho_{2b} \cdot (\varrho_1 + \varrho_1)^a \cdot x_3 \quad x_4(a, b) = \varrho_{2b} \cdot (\varrho_3 + \varrho_3)^a \cdot x_4;$$

$\eta(\theta, M_i) = \eta(\sigma \cdot x_i(a, b), N)$. Let $x(a, b) = n_1 x_1(a, b) + n_2\{2 \cdot x_2(a, b) + x_3(a, b) + x_4(a, b)\}$; we must show $x(a, b) \in R_0(\mathbb{Z}_4)^{2n+2}$.

If $a = 0$, $x_1(0, b) = x_1$, $x_2(0, b) = x_2$, $\{x_3(0, b) + x_4(0, b)\} = \{x_3 + x_4\}$ so $x(0, b) = x(0, 0) = g(M) \in R_0(\mathbb{Z}_4)^{2n+2}$. If $a > 0$, then

$$x_1(a, b) = 2^{a-1} \cdot \varrho_a \cdot (\varrho_0 + \varrho_2) \cdot x_1 \quad x_2(a, b) = 2^a \cdot \varrho_a \cdot x_2$$

$$x_3(a, b) = 2^a \cdot \varrho_{a+2b} \cdot x_3 \qquad x_4(a, b) = 2^a \cdot \varrho_{3a+2b} \cdot x_4.$$

Since $\varrho_{2b} \cdot (\varrho_0 + \varrho_2) = (\varrho_0 + \varrho_2)$ and $\varrho_{2b} \cdot x_2 = x_2$, we can multiply $x_i(a, b)$ by ϱ_{2b}:

$$\begin{aligned} \varrho_{2b} \cdot x(a, b) &= 2^a \cdot \varrho_a \cdot g(M) + 2^{a-1} n_1 (\varrho_2 - \varrho_0) \cdot x_1 \\ &\quad + 2^a \cdot n_2 \cdot (\varrho_{2a} - \varrho_0) \cdot x_4 \\ &= 2^a \cdot \varrho_a \cdot g(M) + 2^a \{n_1 + ((-1)^a + 1) n_2\} v. \end{aligned}$$

Since $g(M) = n_1 \cdot u + n_2 \cdot \{6u - 6v\} = (n_1 + 6n_2) \cdot u - 6n_2 \cdot v$, $n_2 \equiv 0(2^{n-1})$ and $n_1 \equiv 0(2^n)$ by Lemma 3.2. Thus $2^a\{n_1 + ((-1)^a + 1)n_2) \equiv 0(2^{n+1})$ so $2^a\{n_1 + ((-1)^a + 1)n_2)v \in R_0(\mathbb{Z}_4)^{2n+2}$. ∎

4. $MU*(BQ_2)$ and $MU*(BSL(2, 3))$

Let $\alpha_0 = \alpha(\tau_0)$ and $I = \alpha_0 \cdot R(Q_2)$. We recall $bu_{4n-5}(BQ_2) = I/I^n$ and $bu_{4n-3}(BQ_2) = R_0(Q_2)/I^n$. Let g_n: $MU_v(BQ_2) \mapsto bu_{4n-3}(BQ_2)$ for $v \leqslant 4n - 3$ be defined by using Lemma 2.6 so $\eta(\varrho, M) = \eta(\varrho \cdot g_n(M), N(Q_2, n \cdot \tau_0)\ \forall\ \varrho \in R_0(Q_2)$. If $v = 4n - 5$, image $(g_n) \subseteq I$ so

$$g_n\colon MU_{4n-5}(BQ_2) \mapsto bu_{4n-5}(BQ_2) \text{ and } g_n\colon MU_{4n-3}(BQ_2) \mapsto bu_{4n-3}(BQ_2).$$

We will split g_n to embed $bu*(BQ_2)$ in $MU*(BQ_2)$ equivariantly with respect to the action of $\mathbb{Z}_3$ defined previously. Let $\tau = (n - 1) \cdot \tau_0$, $\tau_1 = (n - 1) \cdot \tau_0 \oplus \varrho_1$, and $\tau_2 = (n - 2) \cdot \tau_0 \oplus 2 \cdot \varrho_1$. Let $a = 4n - 5$ and $b = 4n - 3$. For $a, b > 0$ define

$$L_1(a) = i \cdot N(H_i, \tau) - i \cdot N(H_j, \tau)$$

$$L_2(a) = i \cdot N(H_j, \tau) - i \cdot N(H_k, \tau)$$

$$L_3(a) = i \cdot N(H_k, \tau) - i \cdot N(H_i, \tau)$$

$$L_4(a) = N(Q_2, \tau)$$

$$L_5(a) = i \cdot N(H_i, \tau_2) + i \cdot N(H_j, \tau_2) + i \cdot N(H_k, \tau_2)$$

$$L_1(b) = i \cdot N(H_i, \tau_1) - i \cdot N(H_j, \tau_1)$$

$$L_2(b) = i \cdot N(H_j, \tau_1) - i \cdot N(H_k, \tau_1)$$

$$L_3(b) = i \cdot N(H_k, \tau_1) - i \cdot N(H_i, \tau_1)$$

$$L_4(b) = N(Q_2, \tau) \times CP^1$$

$$L_5(b) = L_5(4n - 5) \times CP^1;$$

$L_5(1) = 0$ as $L_5(-1)$ is undefined; $L_1 + L_2 + L_3 = 0$. Let $A_\nu = \mathrm{span}_{\mathbb{Z}} \{L_i(\nu)\} \subseteq MU_\nu(BQ_2)$; $(A_\nu)^{(1)} = \mathrm{span}_{\mathbb{Z}} \{L_1(\nu); L_2(\nu); L_3(\nu)\}$, and $(A_\nu)^{(0)} = \mathrm{span}_{\mathbb{Z}} \{L_4(\nu); L_5(\nu)\}$.

LEMMA 4.1: *Let* $\nu = 4n - 5$ *or* $\nu = 4n - 3$, *then* $g_n \colon A_\nu \cong bu_\nu(BQ_2)$ *and* $g_n\colon A_\nu^{(0)} \cong bu_\nu(BQ_2)^{(0)} \cong bu_\nu(BSL(2, 3))_{(2)}$.

Proof: We first show g is $1 - 1$. If $L \in A_\nu$ and $g(L) = 0$, $L \in \ker(\eta, R_0(Q_2))$. Decompose $L = L^{(0)} + L^{(1)}$; $\eta(\varrho, L^{(0)}) = 0$ if $\varrho \in \{R_0(Q_2)/I^n\}^{(1)}$ and $\eta(\varrho, L^{(1)}) = 0$ if $\varrho \in \{R_0(Q_2)/I^n\}^{(0)}$ so $L^{(\mu)} \in \ker(\eta, R_0(Q_2))$ for $\mu = 0, 1$. We show $L^{(\mu)} = 0$ by showing $t_z(L^{(\mu)}) = 0$. First let $\mu = 1$ and let $\tau = (n - 1) \cdot \tau_0$. For $z = i, j$, or k

$$t_z\{L_i(4n - 5)\}_{i=1,2,3} \in \mathrm{span}_{\mathbb{Z}} \{N(H_z, \tau), i \cdot N(\mathbb{Z}_2, \tau)\}$$

$$t_z\{L_i(4n - 3)\}_{i=1,2,3} \in \mathrm{span}_{\mathbb{Z}} \{N(H_z, \tau \oplus \varrho_1), N(H_z, \tau \oplus \varrho_3),$$
$$i \cdot N(\mathbb{Z}_2, \tau \oplus \varrho_1)\}$$

by Lemma 1.1. Since $t_z(L^{(1)}) \in \ker(\eta, R_0(\mathbb{Z}_4))$, $t_z(L^{(1)}) = 0$ by Lemma 3.4 so $L^{(1)} = 0$. Next let $\mu = 0$ and $\nu = 4k - 5$. Set $\tau_3 = (n - 2)\tau_0 \oplus 2\varrho_1$, $\tau_4 = (n - 2)\tau_0 \oplus 2\varrho_3$

$$t_z\{L_i(4n - 5)\}_{i=4,5} \in \mathrm{span}_{\mathbb{Z}} \{N(H_z, \tau), N(H_z, \tau_3) + N(H_z, \tau_4)$$
$$+ 2i \cdot N(\mathbb{Z}_2, \tau)\}$$

by Lemma 1.1. Since $t_z(L^{(0)}) \in \ker(\eta, R_0(\mathbb{Z}_4))$, $t_z(L^{(0)}) = 0$ by Lemma 3.5. Finally, let $\nu = 4n - 3$. If $L^{(0)} = M \times CP^1 \in \ker(\eta, R_0(Q_2))$, then $M \in A_{4n-5}^{(0)} \cap \ker(\eta, R_0(Q_2))$ so $M = 0$. This shows g is $1 - 1$.

We show g is onto as follows. Let $x_i = g(L_i(4n - 5))$ and $y_i = g(L_i(4n - 5))$. Let $\alpha_0 = \alpha(\tau_0) = 2 - \tau_0$. Then

$$x_1 = \{\mathrm{ind}_i(1) - \mathrm{ind}_j(1)\} \cdot \alpha_0 = (\varrho_i - \varrho_j) \cdot \alpha_0,$$

$$x_2 = \{\mathrm{ind}_j(1) - \mathrm{ind}_k(1)\} \cdot \alpha_0 = (\varrho_j - \varrho_k) \cdot \alpha_0,$$

$$x_3 = \{\mathrm{ind}_k(1) - \mathrm{ind}_i(1)\} \cdot \alpha_0 = (\varrho_k - \varrho_i) \cdot \alpha_0,$$

$$x_4 = \alpha_0,$$

$$x_5 = -\{\mathrm{ind}_i(\varrho_1) + \mathrm{ind}_j(\varrho_1) + \mathrm{ind}_k(\varrho_1)\} \cdot \alpha_0 = -3\tau_0 \cdot \alpha_0.$$

Since $(1 + \varrho_i + \varrho_j + \varrho_k + 2 \cdot \tau_0) \cdot \alpha_0 = 0$ and since I/I^n is a 2-group, the $\{x_i\}$ span I/I^n so g is onto in dimension $4n - 5$. Similarly

$$y_1 = \mathrm{ind}_i(\alpha(\varrho_3)) - \mathrm{ind}_j(\alpha(\varrho_3)) = \varrho_i - \varrho_j,$$

$$y_2 = \mathrm{ind}_j(\alpha(\varrho_3)) - \mathrm{ind}_k(\alpha(\varrho_3)) = \varrho_j - \varrho_k,$$

$$y_3 = \mathrm{ind}_k(\alpha(\varrho_3)) - \mathrm{ind}_i(\alpha(\varrho_3)) = \varrho_k - \varrho_i$$

$$y_4 = x_4 = \alpha_0,$$

$$y_5 = x_5 = 3\tau_0 \cdot \alpha_0 = -6\alpha_0 + -3(\varrho_i + \varrho_j + \varrho_k - 3\varrho_0)$$

so the $\{y_i\}$ span $R_0(Q_2)/I^n$; the isomorphism for $BSL(2, 3)$ follows from $\mathbb{Z}_3$ equivariance. ■

Let the generators for $MU = \mathbb{Z}[x_2, x_4 \dots]$ be normalized so $ag(x_{2i}) = 0$ for $i > 1$; these are the Hazewinkle generators. Let $P = \mathbb{Z}[x_4, x_6, \dots]$ so $MU = P[x_2]$. Let S be the P submodule of $MU*(BQ_2)$ generated by A.

LEMMA 4.2:
(a) $|A_{4n-5}| = 2^{5n-5}$ *and* $|A_{4n-3}| = 2^{5n-3}$. $A_v = MU_v(BQ_2)$ *for* $v = 1, 3$.
(b) $MU_3(BQ_2) \times (x_2)^v \subseteq S_{2v+3}$.
(c) $S* = MU*(BQ_2)$.
(d) *Cartesian product is an isomorphism*

$$A* \otimes P* \cong MU*(BQ_2) \quad \text{and} \quad (A*)^{(0)} \otimes P* \cong MU*(BSL(2, 3)).$$

Proof: (a) follows from Lemmas 1.2 and 4.1 (b) is true for $v = 0$ by (a) so we proceed by induction. If $M \in MU_3(BQ_2)$, we must show $M \times (x_2)^v \in S_{2v+3}$. Set $y = g(M)$ and let $2v + 3 = 4n - 5$ or $2v + 3 = 4n - 3$ for $n \geqslant 2$. Choose $M_1 \in A_{2v+3}$ so $g(M_1) = y \cdot \alpha_0^{n-2}$. Then for all $\varrho \in R_0(Q_2)$,

$$\eta(\varrho, M_1) = \eta(\varrho \cdot y \cdot \alpha_0^{n-2}, S^{4n-1}/Q_2) = \eta(\varrho \cdot y, S^7/Q_2) = \eta(\varrho, M).$$

Since $\Delta(L_\mu(2v + 3)) = L_\mu(2v - 1)$, $\Delta(M_1) \in A_{2v-1}$ and $g(\Delta(M_1)) = g(M_1) = y \cdot \alpha_0^{n-2} = 0$ so $\Delta(M_1) = 0$. Let $\{M_i, M_j, M_q\}$ be as in Lemma 2.4. Then

$$M_1 = M_i \times [a_i \cdot x_2^{v+1} + B_i] + M_j \times [a_j \cdot x_2^{v+1} + B_j]$$
$$+ M_q \times [a_q \cdot x_2^v + B_q]$$

where B_i, B_j, and B_q are the terms involving elements of positive degree from $P*$ so $ag(B_z) = 0$. By induction, $M_z \times B_z \in S$. If $M_2 = a_i \cdot M_i \times x_2 + a_j \cdot M_j \times x_2 + M_q$, then $M_2 \times x_2^v \in S_{2v+3}$. Furthermore

$$\eta(\varrho, M) = \eta(\varrho, M_1) = \eta(\varrho, M_2 \times x_2^v) = \eta(\varrho, M_2).$$

By Lemma 2.4(c), $M = M_2$ which proves (b). Since products of the M_z with powers of x_2 belong to S and since S is a P module, products of the M_z with MU belong to S. Since $\ker(\Delta) = \Sigma_z M_z \cdot MU$, $\ker(\Delta) \subseteq S$. Since Δ: $S* \to S*_{-4} \to 0$, we use the 5-Lemma and induction to see $S* = MU*(BQ_2)$ which proves (c).

Cartesian product gives an onto map $A* \otimes P* \to MU*(B\mathbb{Z}_4)$. We show this is an isomorphism by comparing the orders of the groups involved. By Lemma 1.2,

$$|MU_{2m-1}(BQ_2)| = |\oplus_{c \leqslant m} \tilde{H}_{2c-1}(BQ_2; MU_{2m-2c})|$$
$$= |\oplus_{c \leqslant m} \tilde{H}_{2c-1}(BQ_2; \oplus_{d \leqslant m-c} P_{2d})| = |\oplus_{c+d \leqslant m} \tilde{H}_{2c-1}(BQ_2; P_{2d})|$$
$$= |\oplus_{c \leqslant a} \tilde{H}_{2c-1}(BQ_2; \oplus P_{2m-2a})| = |\oplus_a A_{2a-1} \otimes P_{2m-2a}|$$
$$= |\{A* \otimes P*\}_{2m-1}|$$

The assertion for $BSL(2, 3)$ follows by working $\mathbb{Z}_3$ equivariantly. This completes the proof of all the assertions in this paper. ∎

Acknowledgement

This research was partially supported by NSF grant DMS-8414528.

References

1. M.F. Atiyah, V.K. Patodi and I.M. Singer, Spectral asymmetry and Riemannian geometry I. *Math. Proc. Camp. Phil. Soc.* 77 (1975) 43–69. II. *Math. Proc. Camb. Phi. Soc.* 78 (1975) 405–432. III. *Math. Proc. Camb. Phil. Soc.* 79 (1976) 71–99.
2. M. Bendersky and D. Davis, Complex bordism of cyclic 2-groups (to appear).
3. A. Bahri and P. Gilkey, The eta invariant, Pin^c bordism, and equivariant Spin^c bordism for cyclic 2-groups *Pacific J. Math* 128 (1987), 1–24.
4. A. Bahri and P. Gilkey, Pin^c bordism and equivariant Spin^c bordism of cyclic 2-groups, *Proceedings of the AMS*, 99 (1987), 380–382.
5. P.E. Conner and E.E. Floyd, *Differentiable Periodic Maps*, Springer Verlag (1964).
6. K. Fujii, T. Kobayashi, K. Shimomura, M. Sugawara, *KO* groups of lens spaces modulo powers of two. *Hiroshima Math* J.8 (1978) 469–489.
7. P. Gilkey, The eta invariant and equivariant Spin^c bordism for spherical space form groups (to appear).
8. P. Gilkey, Invariance theory, the heat equation, and the Atiyah–Singer index theorem. Publish or Perish (1984).
9. P. Gilkey, The eta invariant and the *K*-theory of spherical space forms, *Inventiones Math.* 76 (1984) 421–453.
10. P. Landweber, Complex bordism of classifying spaces, *Proceedings of AMS* V2 (1971) 175–179.
11. A. Mesneoui, Unitary cobordism and classifying spaces of quaternions (to appear).
12. G. Wilson, K-theory invariants for unitary bordism, *Quarterly J. Math.* V2 (1973) 499–526.
13. J. Wolf, *Spaces of Constant Curvature*, 5th edn, Publish or Perish (1985).