COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 63, nº 3 (1987), p. 409-417

<http://www.numdam.org/item?id=CM_1987__63_3_409_0>

Compositio Mathematica 63: 408–417 (1987)

On cohomology of graded group categories

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Received 16 March 1987; accepted 12 May 1987

In [2] Fröhlich and Wall defined cohomology groups $H^n(G, \mathscr{C})$ for G a group and $\mathscr{C}$ a strictly coherent group-like G-monoidal category. We show here a one-to-one correspondence between these categories and the categories with coherent left G-module structure; this correspondence takes $\mathscr{C}$ to its subcategory Ker $(\mathscr{C})$ of morphisms of grade 1. We prove that the groups $H^n(G, \mathscr{C})$ are isomorphic to the cohomology groups $H^n(G, \operatorname{Ker} \mathscr{C})$ as defined in [8]. Finally we give a characterization of these groups as right derived functors.

1. Let G be a group and $\mathscr{A}$ a category with a coherent (left) G-module structure as defined in [8], 1.1, and [7, II]. For $\mathscr{A}$-morphisms $g: P \to Q$ and $h: M \to N$ we write $g \otimes h: PM \to QN$ for the monoidal product, $g^0: P^0 \to Q^0$ for the monoidal inverse, and $g^\sigma: P^\sigma \to Q^\sigma$ for the image of g under the action of $\sigma \in G$ on $\mathscr{A}$. The corresponding natural transformations are:

$$a: (PQ)N \to P(QN), \quad c: PQ \to QP, \quad e: PI \to P, \quad i: PP^0 \to I,$$

$$t_\sigma: (PQ)^\sigma \to P^\sigma Q^\sigma, \quad \lambda_\sigma: I^\sigma \to I, \quad \xi: (P^\tau)^\sigma \to P^{\sigma\tau}, \quad \zeta: P^1 \to P,$$

where I denotes the neutral object of $\mathscr{A}$. We first want to show that $\mathscr{A}$ determines a strictly coherent group-like G-monoidal category $\mathscr{C} = \mathscr{A}^{gr}$ in the sense of [2], p. 257. We set

$$\operatorname{Ob}(\mathscr{C}) = \operatorname{Ob}(\mathscr{A}), \quad \text{and} \quad \operatorname{Hom}_{\mathscr{C}}(P, Q) = \bigcup_{\sigma \in G} \operatorname{Hom}_{\mathscr{A}}(P^\sigma, Q) \text{ (disjoint).}$$

Thus each $\mathscr{C}$-morphism $g: P \to Q$ determines (and is uniquely determined by) its "grade" $\sigma \in G$ and its "projection" $g_\sigma: P^\sigma \to Q$ in $\mathscr{A}$. For $h: N \to P$ of grade τ and $g: P \to Q$ of grade σ, the composite gh is defined to have grade $\sigma\tau$ and projection

$$N^{\sigma\tau} \xrightarrow{\xi^{-1}} (N^\tau)^\sigma \xrightarrow{h_\tau^\sigma} P^\sigma \xrightarrow{g_\sigma} Q.$$

Observe that $\zeta: P^1 \to P$ is the projection of id_P, and that there is a canonical embedding

$$\imath: \mathscr{A} \to \mathscr{A}^{gr} = \mathscr{C}, \quad \imath(P) = P,$$

which identifies $\mathscr{A}$ with the subcategory of morphisms of grade 1 of $\mathscr{C}$. Moreover, there is a functor

$$\mathscr{C} \times_G \mathscr{C} \to \mathscr{C}, \quad (P, Q) \mapsto PQ,$$

which maps a pair $g: P \to Q, h: M \to N$ of $\mathscr{C}$-morphisms of the same grade σ to $g \otimes h: PM \to QN$ with projection

$$(PM)^\sigma \xrightarrow{t_\sigma} P^\sigma M^\sigma \xrightarrow{g_\sigma \otimes h_\sigma} QN$$

and grade σ. Finally, we have $\varepsilon: G \to \mathscr{C}$, $\varepsilon(\sigma): I \to I$, where $\varepsilon(\sigma)$ has grade σ and projection $\lambda_\sigma: I^\sigma \to I$.

Lemma (1.1). *The above data make* $\mathscr{C} = \mathscr{A}^{gr}$, *together with the natural transformations* $\imath(a)$, $\imath(c)$ and $\imath(e)$, *into a strictly coherent group-like G-monoidal category.*

Proof: Simply check the definitions [2], pp. 237, 240, 247.

The (small) categories with coherent G-module structure form a category $\check{\mathfrak{U}}\mathfrak{b}_G$ where a morphism $\Psi: \mathscr{A} \to \mathscr{B}$ is a monoidal functor (Ψ, t_Ψ) supplied with monoidal [2], (1.10), natural transformations

$$\mu_\sigma: \Psi(P)^\sigma \to \Psi(P^\sigma), \quad P \in \mathscr{A}, \quad \sigma \in G,$$

such that

$$\mu_{\sigma\tau}\xi_{\Psi(P)} = \Psi(\xi_P)\mu_\sigma\mu_\tau^\sigma, \quad \text{and} \quad \Psi(\zeta_P)\mu_1 = \zeta_{\Psi(P)}, \tag{1}$$

for all $\sigma, \tau \in G$, and $P \in \mathscr{A}$. Notice that $t_\Psi: \Psi(PQ) \to \Psi(P)\Psi(Q)$ satisfies [2], (1.7) and (1.8), and that by [7, II], t_Ψ determines $\lambda_\Psi: \Psi(I) \to I$ such that [2], (1.9), holds. With any such $(\Psi, t_\Psi, \mu_\sigma)$ we associate a grade preserving functor

$$\Psi^{gr}: \mathscr{A}^{gr} \to \mathscr{B}^{gr}, \quad P \mapsto \Psi(P),$$

which maps $g\colon P \to Q$ of grade σ to $\Psi^{gr}(g)$ having projection

$$\Psi(P)^{\sigma} \xrightarrow{\mu_\sigma} \Psi(P^{\sigma}) \xrightarrow{\Psi(g_\sigma)} \Psi(Q).$$

Then $\iota(t_\Psi)$ and $\iota(\lambda_\Psi)$ make Ψ^{gr} a morphism of G-monoidal categories [2], p. 240. In this way we obtain a faithful functor from $\check{\mathfrak{U}}\mathrm{b}_G$ to the category $\check{\mathfrak{U}}\mathrm{b}_G^{gr}$ of strictly coherent group-like G-monoidal (small) categories. It is clear that if Ψ is an equivalence of categories, then so is Ψ^{gr}.

THEOREM (1.2): *The functor* $\check{\mathfrak{U}}\mathrm{b}_G \to \check{\mathrm{U}}\mathrm{b}_G^{gr}$, $\mathscr{A} \mapsto \mathscr{A}^{gr}$, *is an equivalence of categories.*

Proof: We first show that, given any morphism $T\colon \mathscr{A}^{gr} \to \mathscr{B}^{gr}$ in $\check{\mathfrak{U}}\mathrm{b}_G^{gr}$, T is of the form Ψ^{gr}. T determines a unique monoidal functor $\Psi\colon \mathscr{A} \to \mathscr{B}$ satisfying $\iota_{\mathscr{B}}\Psi = T\iota_{\mathscr{A}}$. In particular, $\Psi(P) = T(P)$ for objects $P \in \mathscr{A}$. We now consider the $\mathscr{A}^{gr}$-morphisms $f_{P,\sigma}\colon P \to P^{\sigma}$ having grade σ and projection id_{P^σ}. They obviously satisfy:

$$f_{Q,\sigma}\iota(g) = \iota(g^{\sigma})f_{P,\sigma}, \qquad \text{for } g\colon P \to Q \text{ in } \mathscr{A}, \tag{2}$$

$$\iota(t_\sigma)f_{PQ,\sigma} = f_{P,\sigma} \otimes f_{Q,\sigma}, \qquad P, Q \in \mathscr{A}, \tag{3}$$

$$f_{P,\sigma\tau} = \iota(\xi_P)f_{P^\tau,\sigma}f_{P,\tau}, \quad P \in \mathscr{A}, \tag{4}$$

$$f_{P,1} = \iota(\zeta_P)^{-1}, \qquad P \in \mathscr{A}. \tag{5}$$

We define $\mu_\sigma\colon T(P)^{\sigma} \to T(P^{\sigma})$ to be the projection of $T(f_{P,\sigma})$. Then (2) implies that μ_σ is natural in P, (3) implies that μ_σ is monoidal, and (4) and (5) imply condition (1). Furthermore we have $g = \iota(g_\sigma)f_{P,\sigma}$ for any morphism $g\colon P \to Q$ of grade σ in $\mathscr{A}^{gr}$. Hence $T(g) = T\iota(g_\sigma)T(f_{P,\sigma})$, proving $T = \Psi^{gr}$.

Next let $\mathscr{C}$ be a given object of $\check{\mathfrak{U}}\mathrm{b}_G^{gr}$. Let $\mathscr{A} = \mathrm{Ker}(\mathscr{C}) \subset \mathscr{C}$ be the subcategory of morphisms of grade 1. Thus $\mathrm{Ob}(\mathscr{A}) = \mathrm{Ob}(\mathscr{C})$ and $\mathscr{A}$ is a strictly coherent group-like monoidal category, [2], p. 240. We now fix for any $P \in \mathscr{A}$ and $\sigma \in G$ objects P^{σ}, P^{0} in $\mathscr{A}$ and morphisms

$$f_{P,\sigma}\colon P \to P^{\sigma} \text{ of grade } \sigma, \quad i_P\colon PP^0 \to I \text{ of grade } 1. \tag{6}$$

Note that $f_{P,\sigma}$ exists since $\mathscr{C}$ is stably graded, [2], p. 240. For any morphism $g\colon P \to Q$ in $\mathscr{A}$ we define $g^{\sigma}\colon P^{\sigma} \to Q^{\sigma}$ by

$$g^{\sigma} = f_{Q,\sigma}gf_{P,\sigma}^{-1}.$$

Moreover we define $t_\sigma: (PQ)^\sigma \to P^\sigma Q^\sigma$, $\xi: (P^\tau)^\sigma \to P^{\sigma\tau}$, and $\zeta: P^1 \to P$ by

$$t_\sigma = (f_{P,\sigma} \otimes f_{Q,\sigma}) f_{PQ,\sigma}^{-1}, \quad \xi = f_{P,\sigma\tau} f_{P,\tau}^{-1} f_{P^\tau,\sigma}^{-1}, \quad \zeta = f_{P,1}^{-1}.$$

Finally, for each morphism $g: P \to Q$ of grade 1, let $g^0: P^0 \to Q^0$ be the unique $\mathscr{A}$-morphism such that $g \otimes g^0 = i_Q^{-1} i_P$, [2], lemma 2.6. It is easy to verify that these definitions give $\mathscr{A}$ a coherent G-module structure. Now consider the functor

$$T: \mathscr{A}^{gr} \to \mathscr{C}, \quad T(P) = P, \quad T(g) = g_\sigma f_{P,\sigma},$$

where $g: P \to Q$ has grade σ and projection $g_\sigma: P^\sigma \to Q$. T is a morphism of G-monoidal categories where the corresponding natural transformations are identities. We claim that T is bijective. For any $P, Q \in \mathscr{C}$ the given grade functor $\vartheta: \mathscr{C} \to G$ induces

$$\mathrm{Hom}_{\mathscr{C}}(P, Q) = \bigcup_{\sigma \in G} \vartheta_{P,Q}^{-1}(\sigma) \quad \text{(disjoint)}$$

where $\vartheta_{P,Q}^{-1}(\sigma)$ is the set of $\mathscr{C}$-morphisms $P \to Q$ of grade σ. But

$$\vartheta_{P,Q}^{-1}(\sigma) \to \mathrm{Hom}_{\mathscr{A}}(P^\sigma, Q), \quad g \mapsto g f_{P,\sigma}^{-1}$$

is clearly a bijection. This shows that T is an isomorphism and this completes the proof.

We now fix for every $\mathscr{C} \in \check{\mathfrak{U}}\mathrm{b}_G^{gr}$ a set of data (6) thereby making $\mathrm{Ker}(\mathscr{C})$ into an object of $\check{\mathfrak{U}}\mathrm{b}_G$. Any morphism $T: \mathscr{C} \to \mathscr{D}$ in $\check{\mathfrak{U}}\mathrm{b}_G^{gr}$ induces a monoidal functor $\mathrm{Ker}(\mathscr{C}) \to \mathrm{Ker}(\mathscr{D})$, $P \mapsto T(P)$, which becomes a morphism in $\check{\mathfrak{U}}\mathrm{b}_G$ if we define $\mu_\sigma: T(P)^\sigma \to T(P^\sigma)$ by $\mu_\sigma = T(f_{P,\sigma}) f_{T(P),\sigma}^{-1}$. This gives a functor

$$\check{\mathfrak{U}}\mathrm{b}_G^{gr} \to \check{\mathfrak{U}}\mathrm{b}_G, \ \mathscr{C} \mapsto \mathrm{Ker}(\mathscr{C}),$$

which is quasi inverse to the functor that takes $\mathscr{A}$ to $\mathscr{A}^{gr}$.

2. We briefly recall the definition of the cohomology groups $H^n(G, \mathscr{A})$ for $\mathscr{A} \in \check{\mathfrak{U}}\mathrm{b}_G$, [8], 1.1. There is a "cochain complex" of the form

$$0 \to \mathscr{C}^0(G, \mathscr{A}) \xrightarrow{\delta} \mathscr{C}^1(G, \mathscr{A}) \xrightarrow{\delta} \dots \mathscr{C}^n(G, \mathscr{A}) \xrightarrow{\delta} \mathscr{C}^{n+1}(G, \mathscr{A}) \xrightarrow{\delta} \dots$$

where $\mathscr{C}^n(G, \mathscr{A})$ is the category of all maps $G^n \to \mathrm{Ob}(\mathscr{A})$ and the functors are defined by the coboundary operator of group cohomology. From this one obtains cocycle categories $\mathscr{Z}^n(G, \mathscr{A})$ as follows: the objects are the pairs (P, g) where P is an object of $\mathscr{C}^{n-1}(G, \mathscr{A})$ and $g: \delta(P) \to I$ a morphism in $\mathscr{C}^n(G, \mathscr{A})$ such that

$$\delta\delta(P) \xrightarrow{\delta(g)} \delta(I) \xrightarrow{\sim} I$$

is the canonical morphism determined by the coherent G-module structure of $\mathscr{A}$; a morphism $\alpha: (P, g) \to (Q, h)$ is a morphism $\alpha: P \to Q$ in $\mathscr{C}^{n-1}(G, \mathscr{A})$ which satisfies $h\delta(\alpha) = g$. Passing to isomorphism classes, we obtain an abelian group $Z^n(G, \mathscr{A})$ containing the subgroup $B^n(G, \mathscr{A})$ of elements represented by pairs $(\delta(M), can)$ with $M \in \mathscr{C}^{n-2}(G, \mathscr{A})$. So $B^0(G, \mathscr{A}) = B^1(G, \mathscr{A}) = 0$. By definition,

$$H^n(G, \mathscr{A}) = Z^n(G, \mathscr{A})/B^n(G, \mathscr{A}), \quad n \geqslant 0.$$

Any morphism $\Psi: \mathscr{A} \to \mathscr{B}$ in $\check{\mathfrak{U}}\mathfrak{b}_G$ induces group homomorphisms

$$\Psi_n: H^n(G, \mathscr{A}) \to H^n(G, \mathscr{B})$$

according to [8], prop. 0.1, (use [7, III], Satz 5.1). These maps are bijective if Ψ is an equivalence of categories.

LEMMA (2.1): *Let $\eta: \Psi \to \Psi'$ be a monoidal natural transformation between morphisms $\Psi, \Psi': \mathscr{A} \to \mathscr{B}$ in $\check{\mathfrak{U}}\mathfrak{b}_G$. Assume $\mu_\sigma \eta_P^\sigma = \eta_{P\sigma}\mu_\sigma$ for all $\sigma \in G$ and $P \in \mathscr{A}$. Then the maps $H^n(G, \mathscr{A}) \to H^n(G, \mathscr{B})$ induced by Ψ and Ψ' are equal.*

Proof: This follows from the fact that η induces a natural transformation between the functors $\mathscr{Z}^n(G, \mathscr{A}) \to \mathscr{Z}^n(G, \mathscr{B})$ induced by Ψ and Ψ', (see [8], proof of prop. 0.1).

Now let $H^n(G, \mathscr{C})$ for $\mathscr{C} \in \check{\mathfrak{U}}\mathfrak{b}_G^{gr}$ denote the cohomology groups of [2], prop. 7.5. According to [2], p. 262, $H^0(G, \mathscr{C})$ is the group of G-invariant automorphisms $I \to I$ of grade 1, and therefore $H^0(G, \mathscr{C})$ coincides with $H^0(G, \mathrm{Ker}\, \mathscr{C})$, [8], p. 473. We want to show for all n:

THEOREM (2.2): *The functors $H^n(G, \mathscr{C})$ of strictly coherent group-like G-monoidal categories are isomorphic to $H^n(G, \mathrm{Ker}\, \mathscr{C})$.*

Proof: First observe that $H^n(G, \mathrm{Ker}\, \mathscr{C})$ is a homotopy functor as follows easily from lemma 2.1. We may therefore assume by [2], prop. 7.5, that $\mathscr{C}$

is a G-graded group category in the sense of [2], p. 255, 257. This means that the objects of $\mathscr{A} = \operatorname{Ker} \mathscr{C}$ form an abelian group. As before, the natural transformations t_σ, ξ, and ζ are defined by the fixed morphisms

$$f = f_{P,\sigma}: P \to P^\sigma \quad \text{of grade } \sigma, \quad P \in \mathscr{C},$$

while we now can choose $P^0 = P^{-1}$ and $i_P = id_I$. The groups $H^n(G, \mathscr{C})$ are the cohomology groups of a cochain complex

$$C^0(G, \mathscr{C}) \xrightarrow{\delta} C^1(G, \mathscr{C}) \xrightarrow{\delta} \ldots C^n(G, \mathscr{C}) \xrightarrow{\delta} C^{n+1}(G, \mathscr{C}) \xrightarrow{\delta} \ldots$$

where $C^0(G, \mathscr{C})$ is the group of $\mathscr{C}$-morphisms with domain I and grade 1 and $C^{n+1}(G, \mathscr{C})$ for $n \geqslant 0$ is the group of maps $c: G^{n+1} \to \mathscr{C}$ such that

(i) the grade of $c(\sigma_0, \ldots, \sigma_n)$ is σ_0.
(ii) the domain of $c(\sigma_0, \ldots, \sigma_n)$ does not depend on σ_0.

See [2], p. 259, for the definition of δ. We first consider $H^1(G, \mathscr{C})$. If $c \in C^1(G, \mathscr{C})$ is of the form

$$c(\sigma): P \to P \quad \text{(of grade } \sigma), \quad \sigma \in G,$$

then c is a cocycle if and only if $c(\sigma\tau) = c(\sigma)c(\tau)$ for all $\sigma, \tau \in G$, [2], proof of prop. 7.6. On the other hand, consider $\mathscr{A}$-morphisms $\gamma_\sigma: P^\sigma \to P$, $\sigma \in G$, and define $g(\sigma): P^\sigma P^{-1} \to I$ by $g(\sigma) = \gamma_\sigma \otimes 1$. Then (P, g) belongs to $\mathscr{Z}^1(G, \mathscr{A})$ if and only if

$$\gamma_{\sigma\tau}\xi_P = \gamma_\sigma\gamma_\tau^\sigma, \quad \sigma, \tau \in G,$$

as is easily seen. Moreover, an $\mathscr{A}$-morphism $\alpha: I \to P$ defines a morphism $\alpha: (I, can) \to (P, g)$ if and only if the composite

$$I^\sigma I^{-1} \xrightarrow{\alpha^\sigma \otimes \alpha^0} P^\sigma P^{-1} \xrightarrow{\gamma_\sigma \otimes 1} PP^{-1} = I$$

is equal to $\lambda_\sigma \otimes 1$. Now let $\gamma_\sigma = c(\sigma) f_{P,\sigma}^{-1}$. Then c is a cocycle if and only if (P, g) is a cocycle, and there exists $\alpha: (I, can) \to (P, g)$ if and only if $\delta(\alpha) = c$. Thus we conclude $H^1(G, \mathscr{C}) \cong Z^1(G, \mathscr{A}) = H^1(G, \mathscr{A})$.

Now we want to construct homomorphisms $H^n(G, \mathscr{C}) \to H^n(G, \mathscr{A})$ for $n \geqslant 2$. Any $c \in Z^n(G, \mathscr{C})$ has the form

$$c(\sigma_0, \ldots, \sigma_{n-1}): \hat{P}(\sigma_0, \ldots, \sigma_{n-1}) \to Q(\sigma_0, \ldots, \sigma_{n-1}) \quad \text{(of grade } \sigma_0)$$

with $\hat{P}(\sigma_0, \ldots, \sigma_{n-1}) = \hat{P}(1, \sigma_1, \ldots, \sigma_{n-1})$; $\delta(c) = 1$ means each $\delta(c)(\sigma_0, \ldots, \sigma_n)$ equals $\varepsilon(\sigma_0): I \to I$ (where $\varepsilon: G \to \mathscr{C}$ is the given section of the grade functor $\mathscr{C} \to G$). In particular we have

$$I = Q(\sigma_1, \ldots, \sigma_n)\hat{P}(\sigma_0, \sigma_1\sigma_2, \ldots, \sigma_n)^{-1} \ldots \hat{P}(\sigma_0, \ldots, \sigma_{n-1})^{\pm 1}, \quad (7)$$

because this is the domain of $\delta(c)(\sigma_0, \ldots, \sigma_n)$. We let

$$P(\sigma_1, \ldots, \sigma_{n-1}) = \hat{P}(1, \sigma_1, \ldots, \sigma_{n-1})$$

and define $\hat{g}(\sigma_1, \ldots, \sigma_n): P(\sigma_2, \ldots, \sigma_n)^{\sigma_1} \to Q(\sigma_1, \ldots, \sigma_n)$ to be the composite

$$\hat{P}(\sigma_1, \ldots, \sigma_n)^{\sigma_1} \xrightarrow{f^{-1}} \hat{P}(\sigma_1, \ldots, \sigma_n) \xrightarrow{c(\sigma_1, \ldots, \sigma_n)} Q(\sigma_1, \ldots, \sigma_n).$$

Then $g(\sigma_1, \ldots, \sigma_n) = \hat{g}(\sigma_1, \ldots, \sigma_n) \otimes 1$ transforms $\delta(P)(\sigma_1, \ldots, \sigma_n)$ into (7). This means we have a morphism $g: \delta(P) \to I$ in $\mathscr{C}^n(G, \mathscr{A})$, and a routine verification shows (P, g) is an object of $\mathscr{Z}^n(G, \mathscr{A})$ as a consequence of $\delta(c)(\sigma_0, \ldots, \sigma_n) = \varepsilon(\sigma_0)$. Now suppose $c = \delta(b)$ for a $b \in C^{n-1}(G, \mathscr{C})$,

$$b(\sigma_0, \ldots, \sigma_{n-2}): \hat{M}(\sigma_0, \ldots, \sigma_{n-2}) \to N(\sigma_0, \ldots, \sigma_{n-2}).$$

Then the domain of $\delta(b)(\sigma_0, \ldots, \sigma_{n-1})$ is

$$\hat{P}(\sigma_0, \ldots, \sigma_{n-1}) = N(\sigma_1, \ldots, \sigma_{n-1})\hat{M}(\sigma_0, \sigma_1\sigma_2, \ldots, \sigma_{n-1})^{-1} \ldots \hat{M}(\sigma_0, \ldots, \sigma_{n-2})^{\pm 1}. \quad (8)$$

Let $M(\sigma_1, \ldots, \sigma_{n-2}) = \hat{M}(1, \sigma_1, \ldots, \sigma_{n-2})$ and define

$$\hat{\beta}(\sigma_1, \ldots, \sigma_{n-1}): M(\sigma_2, \ldots, \sigma_{n-1})^{\sigma_1} \to N(\sigma_1, \ldots, \sigma_{n-1})$$

to be $b(\sigma_1, \ldots, \sigma_{n-1})f_{\sigma_1}^{-1}$. Then $\beta(\sigma_1, \ldots, \sigma_{n-1}) = \hat{\beta}(\sigma_1, \ldots, \sigma_{n-1}) \otimes 1$ transforms $\delta(M)(\sigma_1, \ldots, \sigma_{n-1})$ into (8). Hence we have a morphism $\beta: \delta(M) \to P$ in $\mathscr{C}^{n-1}(G, \mathscr{A})$. But this is actually a morphism $\beta: (\delta(M), can) \to (P, g)$ in $\mathscr{Z}^n(G, \mathscr{A})$ as follows from $c = \delta(b)$. Thus (P, g) is trivial in $H^n(G, \mathscr{A})$.

We now obtain well-defined homomorphisms $H^n(G, \mathscr{C}) \to H^n(G, \mathscr{A})$. But these homomorphisms transform the exact cohomology sequence of [2], cor. 7.2, into that of [8], p. 473, (note that $U(\mathscr{C}) = \mathscr{A}^*$ and $k_G(\mathscr{C}) = \mathscr{A}^+$).

Applying then the five-lemma we conclude $H^n(G, \mathscr{C}) \xrightarrow{\sim} H^n(G, \mathscr{A})$, and the theorem is proved.

3. Let $\mathfrak{Ab}_G \subset \check{\mathfrak{Ab}}_G$ consist of the categories with strict G-module structure (i.e. the natural transformations $a, c, e, \ldots$ are identities) and of the strict morphisms Ψ(i.e., $t_\Psi = id$ and $\mu_\sigma = id$). Each $\mathscr{A} \in \check{\mathfrak{Ab}}_G$ is equivalent to an object of $\mathfrak{Ab}_G$ as follows easily from the coherence theorem for $\mathscr{A}$ along with the construction of factoring out atomic subcategories [7, II]. It is not difficult to see that $\mathfrak{Ab}_G$ is an abelian category, cf. [3], prop. 1.6.1, example f). Furthermore, we have as in [6], theorem 1.1, an exact equivalence

$$\mathbf{Ab}_G^2 \to \mathfrak{Ab}_G, \quad \varphi \mapsto \mathscr{A}_\varphi,$$

where $\mathbf{Ab}_G^2$ denotes the abelian category of all G-module homomorphisms $\varphi: A \to B$; the category $\mathscr{A}_\varphi$ is defined by

$$\mathrm{Ob}\,(\mathscr{A}_\varphi) = B, \quad \text{and} \quad \mathscr{A}_\varphi(u, v) = \{a \in A | u = \varphi(a) + v\}, \quad u, v \in B.$$

REMARK (3.1): When we define a G-graded group category $\mathscr{C}(\varphi)$ as in [2], prop. 8.1, then it is evident that $\mathscr{A}_\varphi \cong \mathrm{Ker}\,(\mathscr{C}(\varphi))$. Thus it follows from theorem 1.2 that every $\mathscr{C} \in \check{\mathfrak{Ab}}_G^{gr}$ is equivalent to a suitable $\mathscr{C}(\varphi)$.

Let **Ab** denote the category of abelian groups.

THEOREM (3.2): *The functor* $\mathfrak{Ab}_G \to \mathbf{Ab}$, $\mathscr{A} \mapsto H^n(G, \mathscr{A})$, *is the* nth *derived functor of the left exact functor* $\mathscr{A} \mapsto H^0(G, \mathscr{A}) = \mathrm{Aut}_{\mathscr{A}}(I)^G$.

Proof: By [6], theorem 1.2, we may identify $H^n(G, \mathscr{A}_\varphi)$ with the cohomology group $H^n(\varphi)$ of the mapping cone of φ. Therefore it is enough to show that

$$\mathbf{Ab}_G^2 \to \mathbf{Ab}, \quad \varphi \mapsto H^n(\varphi), \quad n \geqslant 0, \tag{9}$$

is the n-th derived functor of the left exact functor $\varphi \mapsto H^0(\varphi) = \mathrm{Ker}\,(\varphi)^G$. To show that (9) is a cohomological functor can be done by a direct, but tedious computation. Alternatively one can argue as follows. Let **C(Ab)** be the category of positive cochain complexes (A_n, d) in **Ab** (thus $A_n = 0$ if $n < 0$). The mapping cone can be regarded as a functor

$$\mathbf{C(Ab)}^2 \to \mathbf{C(Ab)}, \ \varphi \mapsto M(\varphi),$$

which sends a morphism of complexes $\varphi = (\varphi_n: A_n \to B_n)$ to the complex $M(\varphi)$ defined by

$$M(\varphi)_n = A_n \oplus B_{n-1}, \quad d(a_n, b_{n-1}) = (-d(a_n), \varphi_n(a_n) + d(b_{n-1})).$$

cf. [4], [1], § 2.6. This functor is clearly exact and therefore $\mathbf{C(Ab)}^2 \to \mathbf{Ab}$, $\varphi \mapsto H^n(M(\varphi)) = H^n(\varphi)$, $n \geqslant 0$, is a cohomological functor (it is even universal). Furthermore, since the canonical functor $C: \mathbf{Ab}_G \to \mathbf{C(Ab)}$ is exact, so is the functor $C^2: \mathbf{Ab}_G^2 \to \mathbf{C(Ab)}^2$. Thus, (9) is a cohomological functor.

Since we know by [5], lemma 5.3, that $\mathbf{Ab}_G^2$ has enough injectives, it remains to prove $H^n(\varphi) = 0$, $n \geqslant 1$, for the injective objects φ of $\mathbf{Ab}_G^2$, [3], prop. 2.2.1. But if $\varphi: A \to B$ is injective as an object of $\mathbf{Ab}_G^2$, then A, B, and Ker (φ) are injective in $\mathbf{Ab}_G$ and φ is a retraction, [5], proof of lemma 5.3. Hence by looking at the exact sequences

$$\ldots \to H^{n-1}(G, B) \to H^n(\varphi) \to H^n(G, A) \to \ldots,$$

$$0 \to H^1(G, \mathrm{Ker}\, \varphi) \to H^1(\varphi) \to H^0(G, \mathrm{Coker}\, \varphi) \to \ldots$$

[4], theorem 4, we conclude $H^n(\varphi) = 0$ for $n \geqslant 1$.

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