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KAZUYA KATO

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Duality theories for p-primary etale cohomology II

KAZUYA KATO
Department of Mathematics, Faculty of Science, University of Tokyo, Hongo, Tokyo, 113 Japan

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Introduction

In Part I [8] of this paper, we studied the "relative version" of the duality theorem in Milne [9] [10] for $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on smooth varieties in characteristic $p > 0$. In this Part II, we obtain relative duality theorems for varieties which may have singularities.

We introduce some terminologies. Let X be a scheme over $\mathbb{F}_p$. As in Part I, we call a morphism h: $T \to X$ relatively perfect if the diagram

$$\begin{array}{ccc} T & \xrightarrow{\mathfrak{F}_T} & T \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{\mathfrak{F}_X} & X \end{array}$$

is cartesian, where $\mathfrak{F}_X$ is the absolute frobenius of X defined by the map $t \mapsto t^p$ on the structure sheaf and $\mathfrak{F}_T$ is that of T. For example, in the case h is locally of finite type, h is relatively perfect if and only if h is etale. Let X_{FRP} be the category of schemes over X which are flat and relatively perfect over X. We regard X_{FRP} as a site with the etale topology. (To avoid the set theoretic difficulty that X_{FRP} is too big, we introduce universes in the definition of X_{FRP} in §2). Let $\mathcal{O}_X$ be the structure sheaf on X_{FRP} in the evident sense and assume X is locally noetherian. Then a coherent $\mathcal{O}_{X_{\mathrm{zar}}}$-module on X_{zar} is naturally extended to an $\mathcal{O}_X$-module on X_{FRP}, and we call an $\mathcal{O}_X$-module of such type a coherent $\mathcal{O}_X$-module. Fix $n \geqq 1$, and let $D(X_{\mathrm{FRP}}, \mathbb{Z}/p^n\mathbb{Z})$ be the derived category of the category of all $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X_{FRP}. We define $D_0(X)$ to be the triangulated subcategory of $D(X_{\mathrm{FRP}}, \mathbb{Z}/p^n\mathbb{Z})$ generated by coherent $\mathcal{O}_X$-modules regarded as complexes of $\mathbb{Z}/p^n\mathbb{Z}$-sheaves concentrated in degree zero. Then, $D_0(X)$ contains $\mathbb{Z}/p^i\mathbb{Z}$ for $0 \leqq i \leqq n$ as is seen from the exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{t \mapsto t^p - t} \mathcal{O}_X \longrightarrow 0,$$

and thus $D_0(X)$ contains many objects rather than coherent $\mathcal{O}_X$-modules.

Now let k be a perfect field of characteristic $p > 0$. For a k-scheme X locally of finite type, we shall define in (3.1) a complex $K_{n,X}$ of $\mathbb{Z}/p^n\mathbb{Z}$-sheaves on X_{FRP}. We shall prove the following theorems.

THEOREM (0.1). *Let X be a k-scheme locally of finite type. Then the functor $D_X = R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(\ ,K_{n,X})$ sends $D_0(X)$ into itself, and the natural homomorphism $F \to D_X \circ D_X(F)$ is an isomorphism for any object F in $D_0(X)$.*

THEOREM (0.2). *Let $f\colon X \to Y$ be a proper morphism between schemes locally of finite type over k. Then, Rf_* sends $D_0(X)$ into $D_0(Y)$, and there exists a functorial isomorphism*

$$Rf_* \circ D_X(F) \cong D_Y \circ Rf_*(F)$$

for objects F of $D_0(X)$.

As the reader will see, the definition of the dualizing complex $K_{n,X}$ and the proofs of (0.1) and (0.2) rest on the classical duality theory for coherent sheaves [4]. However we need the duality theory of Ekedahl [3] and the results in Part I concerning Breen's theory [2] and relative perfections (cf. Part I §1, §2).

In the introduction of Part I, the author predicted that this Part II would be devoted to the study of the mixed characteristic case. However he found later the above generalization of Part I and that Part I contains an insufficient point concerning the definition of the trace map (cf. (3.6)). So he is publishing this generalization and the correction as Part II. The mixed characteristic case will be studied in Part III.

The influence of the fundamental papers Milne ([9], [10]) on this paper is clear. See also Gros and Suwa [12] for duality theorems in characteristic p.

In the following, p denotes a fixed prime number. For a site S and a sheaf of rings A on S, $M(S, A)$ denotes the category of sheaves of A-modules on S, and $D(S, A)$ denotes the derived category of $M(S, A)$.

§1. Relative perfectness and flatness

The aim of this section is to prove the following result, which is necessary to show the functoriality of the topos $\tilde{X}_{\mathrm{FRP}}$ (cf. (2.1)).

PROPOSITION (1.1). *Let X be a locally noetherian scheme over $\mathbb{F}_p$ such that for any $x \in X$, $[\kappa(x): \kappa(x)^p]$ is finite. Then in the category* (Sch/X) *of all schemes over X, the full subcategory of X-schemes which are flat and relatively perfect over X is stable for finite inverse limits in* (Sch/X).

One can show easily that the full subcategory of relatively perfect X-schemes is stable for finite inverse limits in (Sch/X). However, note that the flatness is usually not stable for finite inverse limits; the equalizer of two morphisms $T' \rightrightarrows T$ between flat X-schemes is scarcely ever flat over X.

The following result (1.2) of O. Gabber is essential for the proof of (1.1).

PROPOSITION (1.2). *Let X be a locally noetherian regular scheme over $\mathbb{F}_p$. Then, a relatively perfect scheme over X is flat over X.*

For the proof, see [8] (5.2).

LEMMA (1.3). *Let X be a scheme over $\mathbb{F}_p$ and let Y be a closed subscheme of X defined locally by a nilpotent ideal on X. Then the restriction $T \mapsto T \times_X Y$ induces an equivalence between the category of relatively perfect schemes over X* (*resp. the category of flat and relatively perfect schemes over X*) *and that over Y.*

Proof. The problem is local, and hence we may assume that Y is defined by a nilpotent ideal. Then for a sufficiently large $n \geqq 1$, the iteration $\mathfrak{F}_X^n: X \to X$ of $\mathfrak{F}_X$ factors as $X \xrightarrow{g} Y \xrightarrow{\subset} X$, and the base change by g defines the quasi-inverse of the functor $T \mapsto T \times_X Y$.

(1.4) Now we prove (1.1). We may assume that $X = \operatorname{Spec}(A)$ for a local ring A. By taking the completion of A, we may assume that $A = B/I$ where $B = F[[T_1, \ldots, T_n]]$ for some field F over $\mathbb{F}_p$ such that $[F: F^p] < \infty$ and for some $n \geqq 0$, and where I is an ideal of B. It is sufficient to consider the inverse limit of a finite diagram consisting of affine schemes $\{\operatorname{Spec}(R^{(\lambda)})\}_\lambda$ over A which are flat and relatively perfect over A. For $n \geqq 1$, let $\operatorname{Spec}(R_n^{(\lambda)})$ be the flat and relatively perfect scheme over B/I^n corresponding to $\operatorname{Spec}(R^{(\lambda)})$ over A via the equivalence (1.3). Let $R_\infty^{(\lambda)} = \varprojlim_n R_n^{(\lambda)}$. Then $\operatorname{Spec}(R_\infty^{(\lambda)})$ is relatively perfect over B as is seen easily by using the fact $B \to B$; $t \mapsto t^p$ is finite flat. Furthermore by [1] Appendix B (B2.2), $\operatorname{Spec}(R^{(\lambda)}) \xrightarrow{\cong} \operatorname{Spec}(R_\infty^{(\lambda)}) \otimes_B A$. Since finite inverse limits of relatively perfect schemes are relatively perfect, the inverse limit T of the induced diagram $\{\operatorname{Spec}(R_\infty^{(\lambda)})\}_\lambda$ is relatively perfect over B. Hence by the result of Gabber (1.2), T is flat over B. So $T \otimes_B A$ is flat over A, but this scheme is the inverse limit of $\{\operatorname{Spec}(R^{(\lambda)})\}_\lambda$.

Question (1.5). The author does not know whether every relatively perfect morphism $T \to X$ with X locally noetherian is flat.

§2. The site X_{FRP}

We fix universes $\mathbb{U}$ and $\mathbb{V}$ such that $\mathbb{N} \in \mathbb{U} \in \mathbb{V}$ ([11] Ex. I). Let X be a $\mathbb{U}$-scheme over $\mathbb{F}_p$, and assume that X satisfies the following condition (S).

(S) X is locally noetherian and $[\kappa(x): \kappa(x)^p] < \infty$ for any $x \in X$.

We denote by X_{FRP} the category of all $\mathbb{U}$-schemes over X which are flat and relatively perfect over X. We regard X_{FRP} as a site endowed with the etale topology. That is, a covering in X_{FRP} is a family of etale morphisms $\{f_\lambda: T_\lambda \to T\}_\lambda$ such that $T = \bigcup_\lambda f_\lambda(T_\lambda)$. Let $\tilde{X}_{\mathrm{FRP}}$ be the topos of all $\mathbb{V}$-sheaves on X_{FRP}.

In the rest of this paper, schemes, rings, and fields are assumed to be elements of $\mathbb{U}$ unless the contrary is explicitly stated.

We give some lemmas concerning X_{FRP} and $\tilde{X}_{\mathrm{FRP}}$.

LEMMA (2.1). *Let X and Y be schemes over $\mathbb{F}_p$ satisfying the above condition (S), and let $f: X \to Y$ be a morphism. Then, the functor $f_*: \tilde{X}_{\mathrm{FRP}} \to \tilde{Y}_{\mathrm{FRP}}$ defined by $f_*(F)(T) = F(T \times_Y X)$ for $T \in \mathrm{Ob}(Y_{\mathrm{FRP}})$ has an exact left adjoint f^*, and defines a morphism of topoi $(f^*, f_*): \tilde{X}_{\mathrm{FRP}} \to \tilde{Y}_{\mathrm{FRP}}$. (Here we call a functor exact if it commutes with finite inverse limits and finite direct limits.)*

This is a consequence of (1.1). The point is that the exactness of the adjoint functor f^* follows from (1.1) ([11] IV, 4.7).

We omit the proof of the following easy lemma.

LEMMA (2.2). *Let X, Y and f be as in (2.1) and assume that f is a closed immersion. Then,*

(1) *$f_*: \tilde{X}_{\mathrm{FRP}} \to \tilde{Y}_{\mathrm{FRP}}$ is exact.*

(2) *For any object F of $\tilde{X}_{\mathrm{FRP}}$, the canonical morphism $F \to f^* f_* F$ is an isomorphism.*

(3) *For any ring Λ and for any objects F, G of $D(X_{\mathrm{FRP}}, \Lambda)$, the canonical morphism*

$$f_* R\,\mathcal{H}om_\Lambda(F, G) \to R\,\mathcal{H}om_\Lambda(f_* F, f_* G)$$

is an isomorphism.

REMARK (2.3). For X and Y as in (2.2), (2.2) shows that $\tilde{X}_{\mathrm{FRP}}$ is identified with a subtopos ([11] Ex. IV, 9) of $\tilde{Y}_{\mathrm{FRP}}$. However it is not true in general that this subtopos is the complement ([11] Ex. IV, 9) of the open subtopos $\tilde{U}_{\mathrm{FRP}}$ of $\tilde{Y}_{\mathrm{FRP}}$ where $U = Y - X$.

(2.4) For a scheme X over $\mathbb{F}_p$ satisfying (S) and for $n \geqq 1$, we denote by $W_n(\mathcal{O}_X)$ the sheaf on X_{FRP} defined by

$$\Gamma(T, W_n(\mathcal{O}_X)) = \Gamma(T_{\mathrm{zar}}, W_n(\mathcal{O}_{T_{\mathrm{zar}}})) \quad (T \in \mathrm{Ob}\,(X_{\mathrm{FRP}}))$$

where $W_n(\mathcal{O}_{T_{\mathrm{zar}}})$ is the sheaf of Witt vectors of length n on T_{zar}. We prove two lemmas concerning $W_n(\mathcal{O}_X)$-modules.

LEMMA (2.5). *The module theoretic inverse image functor*

$$M(X_{\mathrm{zar}}, W_n(\mathcal{O}_{X_{\mathrm{zar}}})) \to M(X_{\mathrm{FRP}}, W_n(\mathcal{O}_X))$$

is exact.

This follows from the flatness of $W_n(\mathcal{O}_X)$ over $W_n(\mathcal{O}_{X_{\mathrm{zar}}})$ which is proved in [7] Lemma 2.

We denote the above functor and the induced functor

$$D(X_{\mathrm{zar}}, W_n(\mathcal{O}_{X_{\mathrm{zar}}})) \to D(X_{\mathrm{FRP}}, W_n(\mathcal{O}_X))$$

as $M \mapsto M^{\mathrm{FRP}}$.

LEMMA (2.6). *Let X and Y be schemes over $\mathbb{F}_p$ satisfying* (S), *and let $f\colon X \to Y$ be a morphism. Then for $n \geqq 1$ and for a quasi-coherent $W_n(\mathcal{O}_{X_{\mathrm{zar}}})$-module M on X_{zar}, the canonical morphisms*

$$(g_*(M))^{\mathrm{FRP}} \to g_*(M^{\mathrm{FRP}})$$

$$(Rg_*(M))^{\mathrm{FRP}} \to Rg_*(M^{\mathrm{FRP}})$$

are isomorphisms. Here g_* on the left (resp. right) side denotes the direct image functor $\tilde{X}_{\mathrm{zar}} \to \tilde{Y}_{\mathrm{zar}}$ (resp. $\tilde{X}_{\mathrm{FRP}} \to \tilde{Y}_{\mathrm{FRP}}$).

Proof. For a scheme T over $\mathbb{F}_p$, let $W_n(T)$ be the scheme $(T_{\mathrm{zar}}, W_n(\mathcal{O}_{T_{\mathrm{zar}}}))$. Then, for any relatively perfect scheme T over Y, the canonical morphism

$$W_n(T \times_Y X) \to W_n(T) \times_{W_n(Y)} W_n(X)$$

is an isomorphism by [7] Lemma 2. By using this fact, (2.6) follows from [4] II (5.6).

§3. The dualizing functor

Let k be a perfect field of characteristic $p > 0$, and fix $n \geqslant 1$. For a k-scheme X locally of finite type, we define and study the "dualizing complex" $K_{n,X}$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules on X_{FRP}.

(3.1) Let g_n: $W_n(X) \to \operatorname{Spec} W_n(k)$ be the canonical morphism, where $W_n(X)$ is the scheme $(X_{\mathrm{zar}}, W_n(\mathcal{O}_{X_{\mathrm{zar}}}))$, and let $g_n^\Delta(W_n(k))$ be the dualizing complex on $W_n(X)$ in the theory of coherent sheaves ([4] Ch. VI §3). Here it is important that $g_n^\Delta(W_n(k))$ is a complex, not merely an object of the derived category. Let $R_{n,X} = (g_n^\Delta(W_n(k)))^{\mathrm{FRP}}$. The absolute frobenius $\mathfrak{F}_X$: $X \to X$ is finite and hence induces the trace morphism

$$Tr_{\mathfrak{F}_X}\colon (\mathfrak{F}_X)_* g_n^\Delta(W_n(k)) \to g_n^\Delta(W_n(k))$$

([4] VI §4, VII (2.1)). By (2.6), this induces

$$Tr_{\mathfrak{F}_X}\colon (\mathfrak{F}_X)_* R_{n,X} \to R_{n,X}.$$

On the other hand, for any sheaf F on X_{FRP}, we have a canonical isomorphism τ: $(\mathfrak{F}_X)_* F \xrightarrow{\cong} F$ by

$$((\mathfrak{F}_X)_* F)(T) \;=\; F(T \times_{X_{\mathfrak{F}_X}\swarrow} X \xrightarrow{pr_2} X) \xrightarrow{\cong} F(T)$$

($T \in \mathrm{Ob}\,(X_{\mathrm{FRP}})$) where the last isomorphism is given by the isomorphism $T \xrightarrow{\cong} T \times_{X_{\mathfrak{F}_X}\swarrow} X$ induced by $\mathfrak{F}_T$. Let

$$C\colon R_{n,X} \to R_{n,X}$$

be the composite $Tr_{\mathfrak{F}_X} \circ \tau^{-1}$. Note that τ does not preserve the $W_n(\mathcal{O}_X)$-module structures and we have $C(F(a)t) = aC(t)$ for a local section a of $W_n(\mathcal{O}_X)$ and for a local section t of $R_{n,X}$, where F denotes the homomorphism $W_n(\mathcal{O}_X) \to W_n(\mathcal{O}_X)$; $(a_0, \ldots, a_{n-1}) \mapsto (a_0^p, \ldots, a_{n-1}^p)$.

Now we define $K_{n,X}$ to be the "mapping fiber" of the $\mathbb{Z}/p^n\mathbb{Z}$-homomorphism

$$C - 1\colon R_{n,X} \to R_{n,X}.$$

So the degree i part $(K_{n,X})^{(i)}$ of $K_{n,X}$ is $(R_{n,X})^{(i)} \oplus (R_{n,X})^{(i-1)}$, and the differential of this complex is given by

$$(R_{n,X})^{(i)} \oplus (R_{n,X})^{(i-1)} \to (R_{n,X})^{(i+1)} \oplus (R_{n,X})^{(i)}\ (x, y) \mapsto (\delta x, (C-1)(x) - \delta y)$$

where δ denotes the differential of the complex $R_{n,X}$.

We denote the functor

$$R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(\ \ , K_{n,X}) : D^-(X_{\mathrm{FRP}}, \mathbb{Z}/p^n\mathbb{Z}) \to D(X_{\mathrm{FRP}}, \mathbb{Z}/p^n\mathbb{Z})$$

by D_X. On the other hand, we denote by D'_X the dualizing functor for coherent sheaves

$$R\,\mathcal{H}om_{W_n(\mathcal{O}_{X_{\mathrm{zar}}})}(\ \ , g_n^{\Delta}(W_n(k))) : D^-(X_{\mathrm{zar}}, W_n(\mathcal{O}_{X_{\mathrm{zar}}})) \to D(X_{\mathrm{zar}}, W_n(\mathcal{O}_{X_{\mathrm{zar}}})).$$

The triangle

$$K_{n,X} \longrightarrow R_{n,X} \xrightarrow{C-1} R_{n,X}$$

induces a natural morphism

$$(D'_X(M))^{\mathrm{FRP}} \to D_X(M^{\mathrm{FRP}})[1] \qquad (3.1.1)$$

for a bounded complex of coherent $\mathcal{O}_{X_{\mathrm{zar}}}$-modules M.

(3.2) Let $f\colon X \to Y$ be a proper morphism between schemes locally of finite type over k. Then we obtain a canonical trace morphism

$$Rf_* K_{n,X} \to K_{n,Y} \qquad (3.2.1)$$

as follows. The trace map of [4] VI §4 for the proper morphism $W_n(X) \to W_n(Y)$ gives a commutative diagram

$$\begin{array}{ccc} f_* R_{n,X} & \xrightarrow{Tr} & R_{n,Y} \\ {\scriptstyle C-1}\downarrow & & \downarrow{\scriptstyle C-1} \\ f_* R_{n,X} & \xrightarrow{Tr} & R_{n,Y}. \end{array} \qquad (3.2.2)$$

This defines

$$Tr\colon f_* K_{n,X} \to K_{n,Y}. \qquad (3.2.3)$$

Since $f_* K_{n,X} \xrightarrow{\cong} Rf_* K_{n,X}$ in the derived category, (3.2.3) induces the morphism (3.2.1).

(3.3) For a smooth scheme X over k, let $W_n\Omega_X^{\cdot}$ be the De Rham-Witt complex on X_{FRP}, and let $\nu_n(r)_X$ ($r \geqq 0$) be the subsheaf of $W_n\Omega_X^r$ generated by local sections of the form $d\log(a_1)\ldots d\log(a_r)$ ($a_1, \ldots, a_r \in (\mathcal{O}_X)^\times$). (See [8] §4.)

PROPOSITION (3.4). *Let X be a smooth scheme over k purely of dimension r. Then, there exists a canonical quasi-isomorphism of complexes*

$$\nu_n(r)_X[r] \to K_{n,X}.$$

Proof. This is deduced from the duality theory of Ekedahl ([3]) who showed that there exists a canonical quasi-isomorphism of complexes of $W_n(\mathcal{O}_{X_{\mathrm{zar}}})$-modules

$$W_n\Omega^r_{X_{\mathrm{zar}}}[r] \to g_n^\Delta(W_n(k)). \tag{3.4.1}$$

Since there exists an exact sequence

$$0 \to \nu_n(r) \to W_n\Omega_X^r \xrightarrow{C-1} W_n\Omega_X^r \to 0$$

where C is the Cartier operator ([8] §4, [6] Ch. III), (3.4) follows from the quasi-isomorphism (3.4.1) if we prove that (3.4.1) is compatible with the actions of C. To see this, the problem being etale local, we may assume that X is the projective r-space $\mathbf{P}_k^r$. Then, both the Cartier operator C and the operator on $W_n\Omega^r_{X_{\mathrm{zar}}}$ induced by C of $g_n^\Delta(W_n(k))$ are elements of

$$\begin{aligned}\mathrm{Hom}_{W_n(\mathcal{O}_{X_{\mathrm{zar}}})}((\mathfrak{F}_X)_*(W_n\Omega^r_{X_{\mathrm{zar}}}), W_n\Omega^r_{X_{\mathrm{zar}}}) &= \mathrm{Hom}_{W_n(\mathcal{O}_{X_{\mathrm{zar}}})}(W_n\Omega^r_{X_{\mathrm{zar}}}, W_n\Omega^r_{X_{\mathrm{zar}}}) \\ &= \Gamma(X, W_n(\mathcal{O}_{X_{\mathrm{zar}}})) = W_n(k),\end{aligned}$$

where the equalities follow from the fact that $W_n\Omega^r_{X_{\mathrm{zar}}}$ is a dualizing complex. Hence, it suffices to show that the isomorphism between free $W(k)$-modules of rank one

$$\iota\colon H^r(X, W\Omega^r_{X_{\mathrm{zar}}}) \to \varprojlim_n H^0(X, g_n^\Delta(W_n(k)))$$

is compatible with the actions of C. On $H^r(X, W\Omega^r_{X_{\mathrm{zar}}})$ the Cartier operator C is inverse to the operator F ([5] I, 2) as is seen from the definition of it. On the other hand, it is easily seen that via the trace map ι': $\varprojlim_n H^0(X, g_n^{\Delta}(W_n(k))) \cong W(k)$, the operator C on $g_n^{\Delta}(W_n(k))$ corresponds to the inverse of F: $W(k) \to W(k)$. Since the isomorphism $\iota' \circ \iota$ is compatible with the actions of F, this completes the proof of (3.4).

For a smooth k-scheme X purely of dimension r, the functor $R\,\mathcal{H}om_{\mathbf{Z}/p^n\mathbf{Z}}(\ ,\ \nu_n(r)_X)$ was studied in Part I. In particular, we have by Part I (6.1),

PROPOSITION (3.5). *Let X be a smooth k-scheme. Then, the homomorphism $D'_X(M)^{\mathrm{FRP}} \to D_X(M^{\mathrm{FRP}})$ [1] (3.1.1) is an isomorphism for any coherent $\mathcal{O}_{X_{\mathrm{zar}}}$-module M.*

In (4.3), we shall see that this proposition is generalized to singular varieties.

REMARK (3.6).* In Part I §5, for a proper morphism f: $X \to Y$ between smooth schemes X and Y over k purely of dimension r and s, respectively, I defined the trace map

$$Rf_*\nu_n(r)_X[r] \to \nu_n(s)_Y[s]. \tag{3.6.1}$$

However the argument there is not sufficient. The argument was to induce (3.6.1) from a commutative diagram

$$\begin{array}{ccc} Rf_*W_n\Omega^r_X[r] & \to & W_n\Omega^s_Y[s] \\ C-1 & & C-1 \\ Rf_*W_n\Omega^r_X[r] & \to & W_n\Omega^s_Y[s]. \end{array}$$

However in the derived category, a square does not determine a morphism between mapping fibers. By the definition of the trace map of this Part II, the right definition of (3.6.1) is given as

$$Rf_*\nu_n(r)_X[r] \cong f_*K_{n,X} \xrightarrow[(3.2.3)]{} K_{n,Y} \cong \nu_n(s)_Y[s]$$

by using complexes. The results of Part I survive by this correction. For example Part I (6.1) quoted above survives, but in fact, this result is proved in Part I without using the trace map.

* See: Note added in proof (p. 270).

§4. Proofs of the duality theorems

In this section, we prove Theorems (0.1) and (0.2) in the Introduction. Let k be a perfect field of characteristic $p > 0$.

LEMMA (4.1). *Let Z be a smooth scheme over k and let $i\colon X \to Z$ be a closed immersion. Then the homomorphism*

$$i_* K_{n,X} \to R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_*(\mathbb{Z}/p^n\mathbb{Z}), K_{n,Z})$$

in the derived category defined by the trace map $i_* K_{n,X} \to K_{n,Z}$ (3.2) is an isomorphism.

Proof. In $D(Z_{\mathrm{FRP}}, \mathbb{Z}/p^n\mathbb{Z})$, we have isomorphisms

$$\begin{aligned} i_* R_{n,X} &\cong i_*(D'_X(W_n(\mathcal{O}_{X_{\mathrm{zar}}})^{\mathrm{FRP}}) \underset{(2.6)}{\cong} (i_* D'_X(W_n(\mathcal{O}_{X_{\mathrm{zar}}})))^{\mathrm{FRP}} \\ &\underset{[4]}{\cong} (D'_Z i_*(W_n(\mathcal{O}_{X_{\mathrm{zar}}})))^{\mathrm{FRP}} \\ &\underset{(3.5)}{\cong} R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_* W_n(\mathcal{O}_{X_{\mathrm{zar}}}), K_{n,Z})\,[1]. \end{aligned}$$

From this we obtain a commutative diagram

$$\begin{array}{ccc} i_* R_{n,X} & \xrightarrow{\cong} & R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_* W_n(\mathcal{O}_X), K_{n,Z})[1] \\ {\scriptstyle C-1}\downarrow & & \downarrow{\scriptstyle {}^tF-1} \\ i_* R_{n,X} & \xrightarrow{\cong} & R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_* W_n(\mathcal{O}_X), K_{n,Z})[1] \end{array} \qquad (4.1.1)$$

where ${}^tF$ is the transpose of $F\colon i_* W_n(\mathcal{O}_X) \to i_* W_n(\mathcal{O}_X)$; $(a_0, \ldots a_{n-1}) \mapsto (a_0^p, \ldots, a_{n-1}^p)$. Now (4.1) follows from (4.1.1) and the distinguished triangles

$$i_* K_{n,X} \to i_* R_{n,X} \xrightarrow{C-1} i_* R_{n,X}$$

$$i_* \mathbb{Z}/p^n\mathbb{Z} \to i_* W_n(\mathcal{O}_X) \xrightarrow{F-1} i_* W_n(\mathcal{O}_X).$$

PROPOSITION (4.2). *Let X be a k-scheme locally of finite type, and let X_{red} be the reduced part of X. Then the canonical homomorphism $K_{n,X} \to K_{n,X_{\mathrm{red}}}$ is an isomorphism in $D(X_{\mathrm{FRP}}, \mathbb{Z}/p^n\mathbb{Z})$. Here we identify $(X_{\mathrm{red}})_{\mathrm{FRP}}$ with X_{FRP} by (1.3).*

Proof. The problem is local and hence we may assume that there is a closed immersion $i: X \to Z$ for a smooth k-scheme Z. Then, (4.1) shows $i_* K_{n,X} \xrightarrow{\cong} i_* K_{n,X_{\text{red}}}$. This proves (4.2) by (2.2)(2).

The theorems (0.1) and (0.2) are reduced to the classical duality theory for coherent sheaves [4] by the following (4.3). (We use the trace map (3.2.1) to define $Rf_* \circ D_X(F) \to D_Y \circ Rf_*(F)$ of (0.2).)

PROPOSITION (4.3). *The statement of (3.5) holds if we drop the smooth assumption on X and assume only that X is locally of finite type over k.*

Proof. The problem being local, we may assume that there is a closed immersion $i: X \to Z$ for Z smooth. By (2.2)(2), it is sufficient to prove

$$i_*(D'_X(M)^{\text{FRP}}) \xrightarrow{\cong} i_* D_X(M^{\text{FRP}})\,[1]$$

for any coherent $W_n(\mathcal{O}_{X_{\text{zar}}})$-module M. But we have

$$\begin{aligned}
i_* D_X(M^{\text{FRP}}) &\cong R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_* M^{\text{FRP}}, i_* K_{n,X}) \quad \text{by (2.2)(3)}\\
&\cong R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_* M^{\text{FRP}}, R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_*\mathbb{Z}/p^n\mathbb{Z}, K_{n,Z})) \quad \text{by (4.1)}\\
&\cong R\,\mathcal{H}om_{\mathbb{Z}/p^n\mathbb{Z}}(i_* M^{\text{FRP}}, K_{n,Z}) \quad (\text{for } i_*\mathbb{Z}/p^n\mathbb{Z} \text{ is flat over } (\mathbb{Z}/p^n\mathbb{Z})_Z\\
&\qquad \text{and } i_* M^{\text{FRP}} \otimes_{\mathbb{Z}/p^n\mathbb{Z}} i_*\mathbb{Z}/p^n\mathbb{Z} \cong i_* M^{\text{FRP}})\\
&= D_Z(i_* M^{\text{FRP}})\\
&\cong D'_Z(i_* M)^{\text{FRP}}[-1] \quad \text{by (3.5)}\\
&\cong i_* D'_X(M)^{\text{FRP}}[-1] \quad \text{by [4].}
\end{aligned}$$

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Note added in proof (see p. 267): A right definition of the trace map is already given in M. Gros "Classes de Chern et classes de cycles en cohomologie de Hodge–Witt logarithmique" for smooth varieties (for the usual etale site, not for X_{FRP}) (*Bull. Soc. Math. France* 1985).

Correction to Part I (added in proof): In Lemma (5.3.2), add the assumption that a is a non-zero-divisor of A.

The author thanks Professor W. Messing for pointing out the mistake. This Lemma (5.3.2) is in the proof of a result of O. Gabber, but this mistake is due to the author.