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## On two types of complete discrete valuation fields

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### Introduction

In this paper, we divide complete discrete valuation fields of mixed characteristics  $(0, p)$  whose residue fields are not necessarily perfect, into two types (type-I and type-II; for the definition, see §1 (1.3)), and show that there is much difference between the structures of Milnor  $K$ -groups and between the abelian extensions for these two types.

Let  $K$  be a complete discrete valuation field and  $A$  be the valuation ring of  $K$  with the maximal ideal  $m_A$  and the residue field  $F$ . We do not have a satisfactory ramification theory of  $K$  unless  $F$  is perfect. One way to study the ramification (especially wild ramification) in the imperfect residue field case, is to use local class field theory of Kato and Parsin and to study Milnor  $K$ -groups  $K_q^M(K)$  for  $q \geq 1$  (cf. [10] or §3; for the definition of Milnor  $K$ -groups, cf. Conventions). For  $i \geq 1$ , we denote by  $U_q^i$  (resp.  $V_q^i$ ) the subgroup of the  $q$ -th Milnor  $K$ -group  $K_q^M(K)$  generated by the elements of the form  $\{1 + a, b_1, \dots, b_{q-1}\}$  such that  $a \in m_A^i$  and  $b_j \in A^*$  (resp.  $b_j \in K^*$ ) for  $1 \leq j \leq q - 1$ . Then, we have  $V_q^{i+1} \subset U_q^i \subset V_q^i$  for  $i \geq 1$  (cf. §2). When  $F$  is perfect, the group we consider is  $K_1^M(K) = K^*$ , in which case,  $U_1^i (= V_1^i)$  corresponds to the  $i$ -th upper ramification group of the maximal abelian extension of  $K$  (cf. [17], [18]) and it is clear that  $U_1^i/U_1^{i+1} \simeq F$ . By analogy with the perfect residue case, it seems fundamental to know the structures of  $V_q^i/V_q^{i+1}$  for  $i \geq 1$ . If  $K$  is of equal characteristic, the structures of  $V_q^i/V_q^{i+1}$  are determined and described in terms of the residue field  $F$  (cf. [1]). But in the case of mixed characteristics, the appearance of them is quite different, it depends on whether  $K$  is of type-I or type-II for large  $i$ .

In the following, we always assume that  $K$  is of mixed characteristics with residue field  $F$ . In §2, we will have,

**THEOREM 1.** *Suppose that  $K$  is of type-I (resp. type-II and  $F$  has a finite  $p$ -base of order  $q - 1$ ). Let  $U_q^i$  and  $V_q^i$  be the subgroup of  $K_q^M(K)$  as above. Then, we*

have  $U_q^i = V_q^i$  (resp.  $U_q^i = V_q^{i+1}$ ) for sufficiently large  $i$ . In particular, in resp. case, for sufficiently large  $i$  such that  $(i, p) = 1$ , we have  $V_q^i/V_q^{i+1} = 0$ .

We remark that if  $K$  is of equal characteristic  $p > 0$  and  $F$  has a finite  $p$ -base of order  $q - 1 > 0$ , we have  $V_q^i \neq U_q^i$  if  $p \mid i$ , and  $U_q^i \neq V_q^{i+1}$  if  $i$  is prime to  $p$ , hence  $V_q^i/V_q^{i+1} \neq 0$  for all  $i$  (cf. [1]).

By using the above theorem, we will see phenomena about abelian extensions of  $K$  that we do not encounter in the perfect residue field case. For example (cf. Th. (3.1)),

**COROLLARY 2.** *Let  $K$  be of type-II. Then, there is no totally ramified cyclic extension of degree  $p^n$  if  $n$  is sufficiently large.*

Recall that this never happens if the residue field  $F$  is perfect. In fact, in that case, there is a totally ramified  $\mathbb{Z}_p$ -extension of  $K$ .

The above Theorem 1 has the following (semi-)global application. Let  $X$  be a proper smooth variety over a local field  $k$  which has good reduction. (Here, a local field means a finite extension of the  $p$ -adic number field  $\mathbb{Q}_p$ .) In §4, we define a kind of idele class group  $SK_1^{\text{top}}(X)$  which has an explicit presentation by Milnor  $K$ -groups. This group approximates the abelianized fundamental group  $\pi_1^{ab}(X)$  by class field theory of  $X$ . We have a commutative diagram

$$\begin{array}{ccc} SK_1^{\text{top}}(X) & \xrightarrow{N} & k^* \\ \downarrow & & \downarrow \\ \pi_1^{ab}(X) & \longrightarrow & \text{Gal}(k^{ab}/k), \end{array}$$

where the map labeled  $N$  is the norm map (which will be defined in §4) and the vertical arrows are the reciprocity maps which have dense images. The next theorem is considered as the local field version of the finiteness of  $CH_0(Y)^{\deg 0}$  (the degree zero part of the group of zero cycles modulo rational equivalence) for a proper smooth variety  $Y$  over a finite field.

**THEOREM 3** (cf. Th. (4.2.2)). *The norm map:  $SK_1^{\text{top}}(X) \rightarrow k^*$  has finite kernel. Further, the natural map:  $\pi_1^{ab}(X) \rightarrow \text{Gal}(k^{ab}/k)$  has finite kernel.*

The second statement can also be proved by using the Albanese variety (cf. [2] Prop. (2.4)), but here, we give a purely class field theoretic proof of this result. In the case of  $\dim X = 1$ , Coombes [5] studied the étale coverings of

$X$  by using a different definition of  $SK_1^{\text{top}}(X)$ . It is due to him to analyze the idele class groups by using the “ $\pi$ -adic” filtration.

## Conventions

For a ring  $R$ ,  $R^*$  denotes the group of all invertible elements in  $R$ . For  $q \geq 0$ , the  $q$ -th Milnor  $K$ -group  $K_q^M(R)$  is defined to be  $(R^* \otimes \overset{q\text{-times}}{\dots} \otimes R^*)/J$  where  $J$  is the subgroup generated by the elements of the form  $a_1 \otimes \dots \otimes a_q$  such that  $a_i + a_j = 1$  for some  $i \neq j$ . The class of  $a_1 \otimes \dots \otimes a_q$  is denoted by  $\{a_1, \dots, a_q\}$ .

For a complete discrete valuation field  $K$  of mixed characteristics  $(0, p)$ ,  $v_K$  denotes the discrete valuation and  $e_K$  denotes the absolute ramification index ( $= v_K(p)$ ). A finite extension  $L/K$  is called *fiercely ramified* if  $[L:K] = [M:F]_{\text{insep}}$  where  $M$  and  $F$  are the residue fields of  $L$  and  $K$ , respectively.

For a scheme  $T$ ,  $T_i$  (resp.  $T^i$ ) denotes the set of all points of dimension  $i$  (resp. codimension  $i$ ) in  $T$ .

## §1. Definition of type-I and type-II

Let  $K$  be a complete discrete valuation field of mixed characteristics  $(0, p)$  and  $A$ ,  $m_A$  and  $F$  be the valuation ring, the maximal ideal and the residue field, respectively. We denote by  $\Omega_A^1$  the absolute differential module ( $= \Omega_{A/\mathbb{Z}}^1$ ) and  $\hat{\Omega}_A^1$  is defined to be the completion of  $\Omega_A^1$  with respect to the  $m_A$ -adic topology. We shall describe the structure of this  $A$ -module in order to define the notion of type-I and type-II.

An  $A$ -module  $M$  is called pro-free if it is the  $m_A$ -adic completion of a free  $A$ -module.

LEMMA (1.1). *Let  $I$  be a  $p$ -base of  $F$ . For any lifting  $\tilde{I}$  of  $I$  and any prime element  $\pi$  of  $A$ ,  $\hat{\Omega}_A^1$  is topologically generated by the elements  $d\pi$  and  $dt$  ( $t \in \tilde{I}$ ). More precisely,  $\hat{\Omega}_A^1$  is a quotient of a pro-free  $A$ -module with a base  $(d\pi, dt$  ( $t \in \tilde{I}$ )) by an  $A$ -submodule generated by one element of the form  $a \cdot d\pi + \sum_{t \in \tilde{I}} b_t \cdot dt$  where  $a, b_t \in A$  and  $a \neq 0$ . Put  $m = \inf \{v_K(a)\} \cup \{v_K(b_t) \mid t \in \tilde{I}\}$ . Then, as an  $A$ -module, the torsion part  $(\hat{\Omega}_A^1)_{\text{tors}}$  of  $\hat{\Omega}_A^1$  is of finite length  $m$ , and generated by the element  $(\pi^{-m}a)d\pi + \sum_{t \in \tilde{I}} (\pi^{-m}b_t)dt$ .*

*Proof.* By [4] IX §2 Th. 1, there exists a Cohen subring  $A_0$  of  $A$  containing  $\tilde{I}$  (this means that  $A_0$  is a complete discrete valuation subring containing  $\tilde{I}$

such that  $p$  is a prime element of  $A_0$  and  $A_0/pA_0 \simeq A/m_A = F$ ). Since  $A_0$  is formally smooth over  $\mathbb{Z}$  ([7] Chap. 0 Th. (19.8.2)),  $\Omega_{A_0}^1 \otimes \mathbb{Z}/p^n$  is a free  $A_0/p^n$ -module (loc. cit. Cor. (20.4.1)), and we can take  $dt$  ( $t \in \tilde{I}$ ) as its base, namely,

$$\Omega_{A_0}^1 \otimes \mathbb{Z}/p^n \simeq \bigoplus_{t \in \tilde{I}} (A_0/p^n) \cdot dt.$$

Let  $f(T)$  be a monic minimal polynomial over  $A_0$  of some prime element  $\pi$  of  $A$ . Note that  $A \simeq A_0[T]/(f(T))$ . The exact sequence

$$(f(T))/(f(T))^2 \otimes \mathbb{Z}/p^n \xrightarrow{d} \Omega_{A_0[T]}^1/(f(t)) \otimes \mathbb{Z}/p^n \rightarrow \Omega_A^1 \otimes \mathbb{Z}/p^n \rightarrow 0$$

satisfies the Mittag-Leffler condition, so the inverse limit of this sequence is still exact. Thus,  $\hat{\Omega}_A^1$  is topologically generated by the elements  $d\pi$  and  $dt$  ( $t \in \tilde{I}$ ) having the one relation which comes from  $d(f(t)) = 0$ . We write this relation of the form  $a \cdot d\pi + \sum_{t \in \tilde{I}} b_t \cdot dt = 0$  in  $\hat{\Omega}_A^1$ . Note that  $a = f'(\pi)$ , hence  $a \neq 0$ . The assertions of Lemma (1.1) follows from these facts.

**COROLLARY (1.2).** *The following conditions are equivalent.*

- (i) *The natural homomorphism  $(\hat{\Omega}_A^1)_{\text{tors}} \rightarrow \Omega_F^1$  is the 0-map.*
- (ii) *For a lifting  $\tilde{I}$  of a  $p$ -base of  $F$  and a prime element  $\pi$  of  $A$ , we have a relation in  $\hat{\Omega}_A^1$  of the form  $a \cdot d\pi + \sum_{t \in \tilde{I}} b_t \cdot dt = 0$  such that  $v_K(a) < v_K(b_t)$  for all  $t \in \tilde{I}$ .*

**DEFINITION (1.3).** We call  $K$  of type-I if one of the equivalent conditions in Corollary (1.2) is satisfied. Otherwise, we call  $K$  of type-II.

**EXAMPLE (1.3.1).** If  $F$  is perfect, we have  $\Omega_F^1 = 0$ , therefore,  $K$  is of type-I by (1.2) (i).

**EXAMPLE (1.3.2).** Suppose that we have a relation in  $\hat{\Omega}_A^1$   $a \cdot d\pi + \sum_{t \in \tilde{I}} b_t \cdot dt = 0$  ( $a \neq 0$ ) for some prime element  $\pi$  and for some lifting  $\tilde{I}$  of a  $p$ -base. We put  $m = \inf_{t \in \tilde{I}} (v_K(b_t) - v_K(a))$ . Then,  $L = K(\sqrt[p]{\pi})$  is of type-I if and only if  $m \geq e_K + 1$ . For example, let  $K = \text{Frac}(\mathbb{Z}_p[T]_{(p)})$  be the fraction field of the completion of the local ring of  $\mathbb{Z}_p[T]$  at the prime ideal  $(p)$  where  $T$  is a variable. Then  $K(\sqrt[p]{pT})$  is of type-II and  $K(\sqrt[p]{p(1 + pT)})$  is of type-I.

**PROPOSITION (1.4).** *Let  $K/k$  be an extension of complete discrete valuation fields of mixed characteristics whose valuation rings we denote by  $A$  and  $R$ , respectively. If  $A$  is formally smooth over  $R$ ,  $K$  and  $k$  are of the same type.*

*Proof.* Since  $A$  is formally smooth over  $R$ ,  $\hat{\Omega}_{A/R}^1$  is a pro-free  $A$ -module ([7 Cor. (20.4.11)]) and the following sequence is exact and splits (loc. cit. Prop. (20.7.18)).

$$0 \rightarrow (\hat{\Omega}_R^1 \otimes A)^\wedge \rightarrow \hat{\Omega}_A^1 \rightarrow \hat{\Omega}_{A/R}^1 \rightarrow 0$$

Thus, we have  $(\hat{\Omega}_A^1)_{\text{tors}} = (\hat{\Omega}_R^1 \otimes A)_{\text{tors}}^\wedge$ . This shows that a generator of  $(\hat{\Omega}_R^1)_{\text{tors}}$  is still a generator of  $(\hat{\Omega}_A^1)_{\text{tors}}$ . On the other hand, let  $F$  (resp.  $F_0$ ) be the residue field of  $A$  (resp.  $R$ ). Since the extension  $F/F_0$  is separable,  $\Omega_{F_0}^1 \rightarrow \Omega_F^1$  is injective. So, we obtain the proposition from Corollary (1.2) (i).

**COROLLARY (1.5).** *Suppose that there exists a discrete valuation subfield  $k$  of  $K$  such that a prime element of the valuation ring  $R$  of  $k$  is still a prime element of  $A$  and the residue field of  $k$  is perfect. Then,  $K$  is of type-I.*

*Proof.* This follows from Proposition (1.4) and Example (1.3.1).

**COROLLARY (1.6).** *If  $L/K$  is tamely ramified,  $K$  and  $L$  are of the same type.*

*Proof.* By Proposition (1.4), we may assume that the residue field  $F$  of  $K$  is separably closed. In this case,  $L$  is defined by the equation  $X^i = \pi_K$  where  $i$  is an integer prime to  $p$  and  $\pi_K$  is a prime element of  $A$ . Then, a solution  $\pi_L$  of this equation is a prime element of the valuation ring  $B$  of  $L$ . If in  $\hat{\Omega}_A^1$  we have a relation  $a \cdot d\pi_K + \sum_{t \in \tilde{I}} b_t \cdot dt = 0$ , then in  $\hat{\Omega}_B^1$  we have  $a i \pi_L^{i-1} \cdot d\pi_L + \sum_{t \in \tilde{I}} b_t \cdot dt = 0$ . This corollary follows from the fact that  $0 \leq v_L(i \pi_L^{i-1}) < i$ , and that  $v_L(a)$  and  $v_L(b_t)$  ( $t \in \tilde{I}$ ) are divisible by  $i$ .

**PROPOSITION (1.7).** *Let  $L/K$  be a totally ramified (resp. fiercely ramified cf. Conventions) extension. If  $K$  is of type-II (resp. of type-I),  $L$  is of type-II (resp. of type-I).*

*Proof.* Let  $B$  be the valuation ring of  $L$ . This proposition is immediate from the following claim.

(1.7.1) In each case, the natural homomorphism:  $(\hat{\Omega}_A^1)_{\text{tors}} \otimes B \rightarrow (\hat{\Omega}_B^1)_{\text{tors}}$  is surjective.

For the proof of (1.7.1), we only consider the case when  $L/K$  is totally ramified. The other case can be proved by the same method. Let  $a \cdot d\pi + \sum_{t \in \tilde{I}} b_t \cdot dt$  be a generator of  $(\hat{\Omega}_A^1)_{\text{tors}}$ . Since  $K$  is of type-II, at least one of  $b_t$

is a unit. Note that since  $L/K$  is totally ramified,  $\hat{\Omega}_B^1$  is topologically generated by  $d\pi_L$  and  $dt$  ( $t \in \tilde{I}$ ) where  $\pi_L$  is a prime element of  $B$  (cf. Lemma (1.1)). Put  $d\pi = \alpha \cdot d\pi_L + \sum_{t \in \tilde{I}} \beta_t \cdot dt$  in  $\hat{\Omega}_B^1$ . The elements  $\alpha$  and  $\beta_t$  of  $B$  are divisible by  $\pi_L$ . In fact,  $v_L(\alpha) = \text{length}_B(\Omega_{B/A}^1)$  and by considering the above relation in  $\Omega_F^1$ , we have  $\beta_t \equiv 0 \pmod{\pi_L}$ . Now, the element  $a \cdot d\pi + \sum_{t \in \tilde{I}} b_t \cdot dt = a\alpha \cdot d\pi_L + \sum_{t \in \tilde{I}} (a\beta_t + b_t) \cdot dt$  belongs to  $(\hat{\Omega}_B^1)_{\text{tors}}$  and generates this group because  $a\beta_t + b_t$  is a unit when  $b_t$  is a unit. Thus, we obtain (1.7.1).

Q.E.D.

## §2. Milnor $K$ -groups

In this section, we study the structures of Milnor  $K$ -groups of complete discrete valuation fields of mixed characteristics.

Let  $K, A, F, m_A$  be as in §1. First, we define two filtrations on the  $q$ -th Milnor  $K$ -group  $K_q^M(K)$ . For  $i \geq 1$ ,  $U^i K_q^M(K)$  (resp.  $V^i K_q^M(K)$ ) is defined to be the subgroup generated by the elements of the form  $\{a_1, \dots, a_q\}$  such that  $a_1 - 1 \in m_A^i$ ,  $a_2, \dots, a_q \in A^*$  (resp.  $a_1 - 1 \in m_A^i$ ,  $a_2, \dots, a_q \in K^*$ ). For an element  $a$  ( $\neq 1$ ) of  $A$  and a prime element  $\pi$ , we can verify the formula

$$\{1 - a\pi, \pi\} = \{(1 - a\pi)(1 - a)^{-1}, -(1 - \pi)(1 - a)^{-1}\}.$$

It follows that

$$V^1 = U^1 \supset V^2 \supset U^2 \supset V^3 \supset \dots$$

On the subquotients of these groups, for a fixed prime element  $\pi$ , there are surjective homomorphisms (cf. [10] §1.3 Lemma 6).

$$\Omega_F^{q-2} \rightarrow V^i K_q^M(K) / U^i K_q^M(K) \quad (2.1.1)$$

$$\Omega_F^{q-1} \rightarrow U^i K_q^M(K) / V^{i+1} K_q^M(K) \quad (2.1.2)$$

Here, (2.1.1) (resp. (2.1.2)) is defined by

$$\begin{aligned} a \cdot \frac{db_1}{b_1} \wedge \dots \wedge \frac{db_{q-2}}{b_{q-2}} &\mapsto \text{the class of } \{1 + \tilde{a}\pi^i, \tilde{b}_1, \dots, \tilde{b}_{q-2}, \pi\} \\ \left( \text{resp. } a \cdot \frac{db_1}{b_1} \wedge \dots \wedge \frac{db_{q-1}}{b_{q-1}} \right) &\mapsto \text{the class of } \{1 + \tilde{a}\pi^i, \tilde{b}_1, \dots, \tilde{b}_{q-1}\} \end{aligned}$$

where  $\sim$  means a lifting to  $A$ . In [1] and [3], kernels of these homomorphisms are determined in equal characteristic case and for  $i < e_K p/(p-1)$  in mixed characteristic case. In this paper, we cannot determine their kernels for  $i > e_K p/(p-1)$ , but will see that the appearance of the subquotients for sufficiently large  $i$  is much different from the above known cases.

THEOREM (2.2). We put  $\text{length}_A((\hat{\Omega}_A^1)_{\text{tors}}) = m$ .

- (i) For  $q < 0$ , if  $K$  is of type-I,  $V^i K_q^M(K) = U^i K_q^M(K)$  for  $i > \inf(2(m+1), m+1 + e_K p/(p-1))$ .
- (ii) Suppose that  $F$  has a finite  $p$ -base of order  $q-1$ . If  $K$  is of type-II we have  $U^i K_q^M(K) = V^{i+1} K_q^M(K)$  for  $i > \inf(2m, m + e_K p/(p-1))$ .

REMARK (2.3). In the case  $q = 1$  (namely, the case of  $K_1^M(K) = K^*$ ), we have always  $U^i = V^i$  and  $U^i/U^{i+1} = F$  for  $i \geq 1$ . Compared with this, the group  $K_q^M(K)$  for  $q \geq 2$  is much more complicated and interesting. For example, assume that  $K$  is of type-II and  $F$  has a finite  $p$ -base of order  $q-1$ . Then, for  $i \geq 0$  and  $(i, p) = 1$ , we have  $gr_V^i = V^i/V^{i+1} = 0$ ! In fact, if  $i$  is prime to  $p$ ,  $V^i/U^i$  is generated by the classes of the elements of the form  $\{1 + u_1 \pi^i, u_2, \dots, u_{q-1}, \pi\}$  with  $u_1, \dots, u_{q-1} \in A^*$ , which is equal to  $-i^{-1}\{1 + u_1 \pi^i, u_2, \dots, u_{q-1}, -u_1\}$ . This shows that  $V^i/U^i = 0$  ( $i^{-1}$  makes sense because  $V^i/U^i$  is a  $p$ -torsion group). On the other hand, we have  $U^i = V^{i+1}$  by the theorem. Hence,  $V^i/V^{i+1} = 0$ .

For the proof of the theorem, we need the following two lemmas. We denote by  $\hat{\Omega}_A^{q-1}$  the completion of  $\Omega_A^{q-1} (= (q-1)\text{-th exterior power of } \Omega_A^1 \text{ over } A)$ .

LEMMA (2.4). For a prime element  $\pi$  of  $A$  and  $j \geq 1$ , there is a surjective homomorphism  $\varrho_j: \hat{\Omega}_A^{q-1} \rightarrow U^j K_q^M(K)/V^{2j} K_q^M(K)$  which sends  $a \cdot db_1 \wedge \dots \wedge db_{q-1}$  to  $\{1 + ab_1 \dots b_{q-1} \pi^j, b_1, \dots, b_{q-1}\}$  for  $a \in A$  and  $b_1, \dots, b_{q-1} \in A \setminus \{0\}$ .

LEMMA (2.5). For an element  $c$  of  $A$  such that  $v_K(c) > e_K/(p-1)$ , there exists a homomorphism

$$\exp_c: \hat{\Omega}_A^{q-1} \rightarrow U^1 K_q^M(K)^\wedge$$

such that  $\exp_c(a \cdot db_1 \wedge \dots \wedge db_{q-1}) = \{\exp(pcab_1 \dots b_{q-1}), b_1, \dots, b_{q-1}\}$  for  $a \in A$  and  $b_1, \dots, b_{q-1} \in A \setminus \{0\}$ . Here,  $U^1 K_q^M(K)^\wedge$  means the completion with respect to the topology defined by the filtration  $U^i K_q^M(K)$ ,  $\exp$  is the exponential homomorphism  $T \mapsto \sum_{n=0}^{\infty} T^n/n!$ . (Recall that for  $j > e_K/(p-1)$ ,  $\exp: m_A^j \rightarrow A^*$  is well-defined.)



*Proof of lemmas.* We use an exact sequence:  $0 \rightarrow N \rightarrow A^{\otimes q} \xrightarrow{\psi} \Omega_A^{q-1} \rightarrow 0$  where  $\psi(a \otimes b_1 \otimes \cdots \otimes b_{q-1}) = adb_1 \wedge \cdots \wedge db_{q-1}$  and  $N$  is the subgroup of  $A^{\otimes q}$  generated by the elements of the form

- (i)  $a \otimes b_1 b_2 \otimes c_1 \otimes \cdots \otimes c_{q-2} - ab_1 \otimes b_2 \otimes c_1 \otimes \cdots \otimes c_{q-2} - ab_2 \otimes b_1 \otimes c_1 \otimes \cdots \otimes c_{q-2}$  and  
 (ii)  $a \otimes b_1 \otimes \cdots \otimes b_{q-1}$  where  $b_i = b_j$  for some  $i \neq j$ .

First, we shall prove Lemma (2.4). This presentation of  $\Omega_A^{q-1}$  implies the existence of a homomorphism  $\Omega_A^{q-1} \rightarrow U^j K_q^M(K)/V^{2j} K_q^M(K)$  as in (2.4). In fact, we have  $\{1 + ab\pi^j, b\} = -\{1 + ab\pi^j, -a\pi^j\}$ , therefore, the application  $A^{\otimes q} \rightarrow U^j K_q^M(K)/V^{2j} K_q^M(K)$  which sends  $a \otimes b_1 \otimes \cdots \otimes b_{q-1}$  to  $\{1 + ab_1 \cdots b_{q-1} \pi^j, b_1, \dots, b_{q-1}\}$ , is a homomorphism. It is trivial that its kernel contains  $N$ . Its surjectivity is clear by the definition of the homomorphism. Finally, since the target group is annihilated by some power of  $p$ , we may replace  $\Omega_A^{q-1}$  by  $\hat{\Omega}_A^{q-1}$ . This completes the proof of Lemma (2.4).

By the same method as above, for the proof of Lemma (2.5), it suffices to show that the application  $A \rightarrow U^1 K_2^M(K)^\wedge$  which is defined by  $x \mapsto \{\exp(pcx), x\}$  is a homomorphism, namely, additive.

We use the formula  $\exp(T) = \prod_{m \geq 1} (1 - T^m)^{-\mu(m)/m}$  (cf. [6] Chap. III §1) where  $\mu$  is the Möbius function. Let  $P$  be the set of all natural numbers prime to  $p$ . We put  $E(T) = \prod_{m \in P} (1 - T^m)^{-\mu(m)/m} \in \mathbb{Z}_{(p)}[[T]]$ . Then, by the formula above, we have

$$\exp(pcx) = \exp(cx)^p = E(cx)^p \cdot E((cx)^p)^{-1}.$$

Therefore,

$$\begin{aligned} \{\exp(pcx), x\} &= p\{E(cx), x\} - \{E((cx)^p), x\} \\ &= - \sum_{m \in P} (p\mu(m)/m) \{1 - (cx)^m, x\} + \sum_{m \in P} (\mu(m)/m) \{1 - (cx)^{pm}, x\} \end{aligned}$$

Here, the multiplication by  $m^{-1}$  makes sense since  $U^1 K_q^M(K)^\wedge$  is naturally considered as a  $\mathbb{Z}_p$ -module. We may assume  $K$  contains a primitive  $p$ -th root of unity  $\zeta$ . Indeed, this follows from the fact that there is a norm homomorphism  $N: K_q^M(K(\zeta)) \rightarrow K_q^M(K)$  such that  $N \circ i = [K(\zeta): K]$  where  $i: K_q^M(K) \rightarrow K_q^M(K(\zeta))$  is the natural map. We shall calculate the second term. From  $1 - (cx)^{pm} = \prod_{i=0}^{p-1} (1 - (cx\zeta^i)^m)$ , we obtain

$$\begin{aligned} \{1 - (cx)^{pm}, x\} &= \sum_{i=0}^{p-1} (1/m) \{1 - (cx\zeta^i)^m, x^m\} = - \sum_{i=0}^{p-1} \{1 - (cx\zeta^i)^m, c\zeta^i\} \\ &= -\{1 - (cx)^{pm}, c\} - \sum_{i=0}^{p-1} \{1 - (cx\zeta^i)^m, \zeta^i\}. \end{aligned}$$

It follows that

$$\begin{aligned}
 \{\exp(px), x\} &= \sum_{m \in P} (p\mu(m)/m) \{1 - (cx)^m, c\} \\
 &\quad - \sum_{m \in P} (\mu(m)/m) \{1 - (cx)^{pm}, c\} \\
 &\quad - \sum_{m \in P} (\mu(m)/m) \sum_{i=0}^{p-1} \{1 - (cx\zeta^i)^m, \zeta^i\} \\
 &= -p\{\exp(cx), c\} + \sum_{i=0}^{p-1} \{E(cx\zeta^i), \zeta^i\}
 \end{aligned}$$

Thus, we have only to show the following lemma.

LEMMA (2.5.1). *The application  $A \rightarrow A^*/(A^*)^p$  which sends  $x$  to the class of  $E(cx)$  is a homomorphism.*

*Proof.* In  $\mathbb{Q}[[T]]$ , we have  $\exp(T) \equiv E(T) \pmod{\deg p}$ , hence, the total degree of  $E(T_1 + T_2) - E(T_1)E(T_2)$  in  $\mathbb{Q}[[T_1, T_2]]$ , a posteriori, in  $\mathbb{Z}_{(p)}[[T_1, T_2]]$  is not less than  $p$ . This shows that  $E(c(x + y)) \equiv E(cx)E(cy) \pmod{c^p}$ . Since  $1 + m_A^j \subset (A^*)^p$  for  $j > e_K p/(p - 1)$ , this completes the proof of Lemma (2.5.1). Q.E.D.

We now proceed to the proof of Theorem (2.2). Let  $K$  be of type-I. We shall show that  $V^i K_q^M(K) = U^i K_q^M(K)$  for  $i > 2(m + 1)$ . By Lemma (1.1), we have in  $\hat{\Omega}_A^1$   $a \cdot d\pi + \sum_{t \in T} b_t \cdot dt = 0$  such that  $m = v_K(a) < v_K(b_t)$ . By the homomorphism  $\varrho_j$  where  $j = i - m - 1$ , this relation induces in  $K_q^M(K)/V^{2j} K_q^M(K)$   $\{1 + ac_1 \cdots c_{q-2} \pi^{i-m}, c_1, \dots, c_{q-2}, \pi\} = -\sum_{t \in T} \{1 + tb_t c_1 \cdots c_{q-2} \pi^{i-m-1}, c_1, \dots, c_{q-2}, t\}$  with  $c_1, \dots, c_{q-2} \in A \setminus \{0\}$ . This shows that  $V^i K_q^M(K) \subset U^i K_q^M(K) + V^{2j} K_q^M(K)$ . But  $V^{2j} K_q^M(K)$  is contained in  $U^i K_q^M(K)$  because  $2j = 2(i - m - 1) > i$ . Thus, we have  $V^i K_q^M(K) = U^i K_q^M(K)$ . For  $i > m + 1 + e_K p/(p - 1)$ , using the homomorphism  $\exp_c$  (cf. Lemma (2.5)) where  $v_K(c) = i - e_K - m - 1$  instead of  $\varrho_j$ , we have  $V^i K_q^M(K) = U^i K_q^M(K)$ . Thus (i) has been proved.

Next, we proceed to the proof of (ii). Let  $K$  be of type-II and  $(t_i)_{1 \leq i \leq q-1}$  be a lifting of a  $p$ -base of  $F$  and  $i > 2m$ . By the same method as above, the relation on  $\pi$  and  $t_i$  in  $\hat{\Omega}_A^1$  implies by  $\varrho_{i-m}$  in Lemma (2.4) that the elements of the form  $\{1 + c\pi^i, t_1, \dots, t_{q-1}\}$  belong to  $V^{i+1} K_q^M(K) + V^{2(i-m)} K_q^M(K) = V^{i+1} K_q^M(K)$ . On the other hand, by (2.1.2),  $U^i K_q^M(K)/V^{i+1} K_q^M(K)$  is generated by the elements of this form. Hence, we have  $U^i K_q^M(K) = V^{i+1} K_q^M(K)$ .

Using  $\exp_c$  in Lemma (2.5), we have  $U^i K_q^M(K) = V^{i+1} K_q^M(K)$  for  $i > m + e_K p / (p - 1)$ . Q.E.D.

### §3. Cyclic extensions of complete discrete valuation fields

We use the same notation as in §1 and §2. In this section, we prove the following two theorems by using Theorem (2.2) in §2. We put  $\delta_1 = \inf(2(m + 1), m + 1 + e_K p / (p - 1))$  and  $\delta_2 = \inf(2m, m + e_K p / (p - 1))$  where  $m = \text{length}_A(\hat{\Omega}_A^1)_{\text{tors}}$ . Recall that for  $i > \delta_1$ ,  $V^i K_q^M(K) = U^i K_q^M(K)$  (resp. for  $i > \delta_2$ ,  $U^i K_q^M(K) = V^{i+1} K_q^M(K)$ ) when  $K$  is of type-I (resp. type-II) and  $F$  has a finite  $p$ -base of order  $q - 1$ .

**THEOREM (3.1).** *Let  $L/K$  be a cyclic extension of degree  $p^n$ . If  $K$  is of type-II (resp. type-I) and  $L/K$  is totally (resp. fiercely) ramified, we have  $n \leq \delta_2$  ( $n \leq \delta_1$ ).*

**REMARK (3.1.1).** The assertion for type-I in Theorem (3.1) is already known. In [16] and [8], for any cyclic extension  $L$  of  $K$  with residue field  $M$ , it was proved that  $[M:F]_{\text{insep}}$  is smaller than some constant dependent only on  $K$ . But the proofs of their theorems are completely different from ours, and our proof of type-I case and type-II case proceeds in parallel, so here, we treat type-I case together with type-II case to emphasize a symmetry between them.

**REMARK (3.1.2).** Let  $K$  be a complete discrete valuation field of equal characteristic. In this case, we fix a discrete valuation sub-field  $k \subset K$  such that  $K/k$  is separable (not necessarily algebraic) and the residue field of  $k$  is perfect. Then, we can define the notion of type-I and type-II over  $k$ . Namely,  $K$  is called of type-I over  $k$  if the natural homomorphism:  $(\hat{\Omega}_{A/R}^1)_{\text{tors}} \rightarrow \Omega_F^1$  is the 0-map where  $A$  and  $R$  are the valuation rings of  $K$  and  $k$ , respectively. Otherwise,  $K$  is called of type-II. Theorem (3.1) still holds if we make an additional assumption. Let  $L/K$  be a totally (resp. fiercely) ramified cyclic extension and assume that  $K$  is of type-II (resp. type-I) and that there exists an element  $g$  of  $R$  such that  $dg$  generates  $\hat{\Omega}_R^1$  as an  $\hat{R}$ -module and  $g \in N_{L/K}(L^*)$ . Then, the degree of  $L/K$  cannot be greater than some constant dependent only on  $K$  and  $k$ .

Next theorem is similar to (3.1), but says that it is difficult to extend a fiercely (resp. totally) ramified cyclic extension of a field of type-I (resp. type-II) to any cyclic extension of larger degree. For a cyclic extension  $L/K$  of degree  $p$  with Galois group  $G$ , we define the ramification number  $s(L/K)$  to be

$\inf_{x \in L^*} v_L(s(x)x^{-1} - 1)$  where  $s$  is a generator of  $G$ . In the terminology of [18], if  $L/K$  is totally (resp. fiercely) ramified,  $s(L/K) = i - 1$  (resp.  $i$ ) where  $i = \inf \{j \mid G_j = 0\}$ .

**THEOREM (3.2).** *Let  $M/K$  be a cyclic extension of degree  $p$  such that  $s(M/K) > e_M/p(p-1)$ . If  $K$  is of type-I (resp. type-II) and  $M/K$  is fiercely ramified (resp. totally ramified), there is no cyclic extension  $L/K$  of degree  $p^n$  such that  $L/K$  contains  $M/K$  and  $n > [\delta_1/e_K] + 1$  (resp.  $[\delta_2/e_K] + 1$ ).*

**REMARK (3.2.1).** Let  $K$  be a complete discrete valuation field containing a primitive  $p$ -th root of unity and  $M/K$  be an extension defined by the equation  $X^p = a$ . Let  $U^i(K^*/p)$  be the filtration on  $K^*/p$  induced from  $U^i K_1^M(K)$ . Then,  $s(M/K) > e_M/p(p-1)$  if and only if  $a \notin U^{e_K}(K^*/p)$ .

**REMARK (3.2.2).** Let  $L/K$  be a cyclic extension of degree a power of  $p$  which does not contain unramified extensions. If  $[L : K]$  is sufficiently large, there exist intermediate fields  $K'$  and  $K'_1$  such that  $[K'_1 : K'] = p$  and  $s(K'_1/K') > e_{K'_1}/p(p-1)$  (cf. [8] Lemma (4-1)). So, Theorem (3.2) also contains the fact that there is no fiercely (resp. totally) ramified cyclic extension of a field of type-I (resp. type-II) having a large degree.

For the proof of the theorems, we do not use local class field theory of Kato and Parsin directly, but the idea of the proof deeply depends on their theory. We will use the notation in [10] and [11], which we explain rapidly. For a field  $k$  of characteristic  $\neq p$ , we denote by  $H_{p^n}^q(k)$  the Galois cohomology group  $H^q(k, \mathbb{Z}/p^n(q-1))$  where  $(q-1)$  means the Tate twist. If  $k$  is of characteristic  $p > 0$ , we define

$$H_{p^n}^q(k) = \text{Coker} (W_n \Omega_k^{q-1} \xrightarrow{F-1} W_n \Omega_k^{q-1} / dW_n \Omega_k^{q-2})$$

where  $W_n \Omega_k^*$  is the De Rham-Witt complex [9] and  $F$  is the operator of De Rham-Witt complex as usual (cf. [9] p. 569). Note that there is an isomorphism  $H_{p^n}^1(k) = H^1(k, \mathbb{Z}/p^n)$  (cf. [18] p. 163).

When  $\text{char}(k) \neq p$  (resp.  $\text{char}(k) = p$ ), by composing the cohomological symbol:  $K_q^M(k) \rightarrow H^q(k, \mathbb{Z}/p^n(q))$  (resp. the differential symbol:  $K_q^M(k) \rightarrow W_n \Omega_k^q$ ) (cf. [11], [3]), the product structure of Galois cohomology (resp. De Rham-Witt complex) defines a pairing

$$H_{p^n}^s(k) \otimes K_q^M(k) \rightarrow H_{p^n}^{s+q}(k).$$

Now, we return to the situation of theorems. Since  $A$  is a direct limit of complete discrete valuation rings  $A_i$  whose residue fields are of finite transcendence degree over  $\mathbb{F}_p$ , a relation in  $\hat{\Omega}_A^1$  comes from one of  $\hat{\Omega}_{A_i}^1$  for some

*i.* So we may assume that  $A$  and  $A_i$  are of the same type and  $\text{length}_A(\hat{\Omega}_A^1)_{\text{tors}} = \text{length}_{A_i}(\hat{\Omega}_{A_i}^1)_{\text{tors}}$ . Thus, we may assume  $F$  has a finite  $p$ -base. Put  $[F:F^p] = p^{q-1} < \infty$ . Consider the pairing

$$\psi: H_{p^n}^1(K) \otimes K_q^M(K) \rightarrow H_{p^n}^{q+1}(K). \quad (3.3.1)$$

This group  $H_{p^n}^{q+1}(K)$  is rather “small”. Let  $i$  be the composite of two canonical homomorphisms  $W_n(F) \rightarrow H_{p^n}^1(F) \rightarrow H_{p^n}^1(K)$ . We fix a prime element  $\pi$  and define a homomorphism  $i_q: W_n \Omega_F^{q-1} \rightarrow H_{p^n}^{q+1}(K)$  such that  $i_q(a \cdot d \log b_1 \wedge \cdots \wedge d \log b_{q-1}) = \psi(i(a) \otimes \{\tilde{b}_1, \dots, \tilde{b}_{q-1}, \pi\})$  where  $\tilde{b}_i$  is a lifting to  $A$ . This homomorphism is independent of the choice of  $\pi$  and induces an isomorphism  $H_{p^n}^q(F) \xrightarrow{\sim} H_{p^n}^{q+1}(K)$  ([11] Th. 3). Therefore, we have:

LEMMA (3.3.2). *For any prime element  $\pi$  and any lifting of a  $p$ -base  $(t_i)_{1 \leq i \leq q-1}$ , there exists a surjective homomorphism  $W_n(F) \rightarrow H_{p^n}^{q+1}(K)$  which sends  $a$  to  $\psi(i(a) \otimes \{t_1, \dots, t_{q-1}, \pi\})$ .*

Concerning  $H_{p^n}^{q+1}(K)$ , we also use the fact that the exact sequence  $0 \rightarrow \mathbb{Z}/p^{n-1}(q) \rightarrow \mathbb{Z}/p^n(q) \rightarrow \mathbb{Z}/p(q) \rightarrow 0$  induces an exact sequence ([11] Lemma 8)

$$0 \rightarrow H_{p^{n-1}}^{q+1}(K) \rightarrow H_{p^n}^{q+1}(K) \rightarrow H_p^{q+1}(K) \rightarrow 0. \quad (3.3.3)$$

Let  $\chi \in H_p^1(K)$  be a character of order  $p^n$ . By the pairing (3.3.1),  $\chi$  defines a homomorphism  $\varphi_\chi: K_q^M(K) \rightarrow H_{p^n}^{q+1}(K)$ .

For a finite extension  $L/K$ , there is a norm map  $N_{L/K}: K_q^M(L) \rightarrow K_q^M(K)$ . We will use the following properties:

- (i)  $N_{L/K}$  is compatible with the corestriction map in Galois cohomology by the cohomological symbol.
- (ii)  $N_{L/K}(U^1 K_q^M(L)) \subset U^1 K_q^M(K)$  and  $N_{L/K}(V^e K_q^M(L)) \subset V^e K_q^M(K)$  for  $i > 0$  where  $e$  is the ramification index of  $L$  over  $K$  (for the proof, cf. [10] §1.2 Lemma 3, [12] Prop. 2).

By the property (i) above, we have  $N_{L/K} K_q^M(L) \subset \text{Ker } \varphi_\chi$  where  $L/K$  is the extension corresponding to  $\chi$ .

We will need two more lemmas. A finite extension  $L/K$  is called purely wildly ramified if any subextension of  $L/K$  is wildly ramified.

LEMMA (3.3.4). *Let  $\chi \in H_p^1(K)$  be a character such that the extension attached to  $\chi$  is purely wildly ramified. Then, the homomorphism  $\varphi_\chi: K_q^M(K) \rightarrow H_{p^n}^{q+1}(K)$  defined by  $\chi$  as above is surjective. More precisely,  $\varphi_\chi(U^1 K_q^M(K)) = H_{p^n}^{q+1}(K)$ .*

*Proof of the lemma.* Let  $M/K$  be the subextension of degree  $p$ . The following diagram is commutative.

$$\begin{array}{ccccccc}
 K_q^M(M) & \xrightarrow{N_{M/K}} & K_q^M(K) & \longrightarrow & K_q^M(K)/N_{M/K}K_q^M(K) & \longrightarrow & 0 \\
 \downarrow \varphi_{\chi|M} & & \downarrow \varphi_\chi & & \downarrow \varphi_{p^{n-1}\chi} & & \\
 H_{p^{n-1}}^{q+1}(M) & & & & & & \\
 \downarrow \text{Cor}_{M/K} & & & & & & \\
 0 \longrightarrow H_{p^{n-1}}^{q+1}(K) & \longrightarrow & H_p^{q+1}(K) & \longrightarrow & H_p^{q+1}(K) & \longrightarrow & 0
 \end{array}$$

Here, the bottom row is an exact sequence (3.3.3) and the corestriction map  $\text{Cor}_{M/K}: H_{p^{n-1}}^{q+1}(M) \rightarrow H_{p^{n-1}}^{q+1}(K)$  is surjective because the cohomological dimension of  $K \leq q + 1$  ([11] Th. 2). Considering  $N_{M/K}(U^1 K_q^M(M)) \subset U^1 K_q^M(K)$ , this lemma is reduced to the case  $n = 1$ . In this case, the lemma follows from Lemma (3.3.2) and [10] §3 Lemma 15. Q.E.D.

LEMMA (3.3.5). Let  $\chi$  and  $\varphi_\chi$  be as above. If the extension attached to  $\chi$  is totally (resp. fiercely) ramified, the images of  $V^i K_q^M(K)$  and  $U^i K_q^M(K)$  (resp.  $U^i K_q^M(K)$  and  $V^{i+1} K_q^M(K)$ ) by the homomorphism  $\varphi_\chi$  coincide for  $i \geq 1$ .

*Proof of the lemma.* Let  $L/K$  be the extension corresponding to  $\chi$ . It suffices to show the following claim.

$$\begin{aligned}
 & V^i K_q^M(K) \subset U^i K_q^M(K) + N_{L/K} K_q^M(L) \\
 & (\text{resp. } U^i K_q^M(K) \subset V^{i+1} K_q^M(K) + N_{L/K} K_q^M(L)).
 \end{aligned} \tag{3.3.6}$$

If  $\chi$  is totally ramified, the above is clear because we can take a prime element  $\pi$  which is a norm from  $L$ , and  $V^i K_q^M(K)/U^i K_q^M(K)$  is generated by the elements of the form  $\{1 + a\pi^i, c_1, \dots, c_{q-2}, \pi\}$  with  $a, c_1, \dots, c_{q-2} \in A^*$  (cf. (2.1.1)). In the fiercely ramified case, we prove (3.3.6) by induction on the degree of  $L/K$ . Let  $M/K$  be the subextension of degree  $p$ . Then, we can take a lifting of a  $p$ -base  $(t_i)_{1 \leq i \leq q-1}$  such that  $t_1 = N_{M/K}(t')$  where  $t' \in M^*$ . As above,  $U^i K_q^M(K)/V^{i+1} K_q^M(K)$  is generated by the elements of the form  $\{1 + a\pi^i, t_1, \dots, t_{q-1}\}$  with  $a \in A$  (cf. (2.1.2)). Since  $\{1 + a\pi^i, t', t_2, \dots, t_{q-1}\}$  is in  $V^{i+1} K_q^M(M) + N_{L/M} K_q^M(L)$  by induction on  $[L:K]$ , the claim follows from the fact that  $N_{M/K}(V^{i+1} K_q^M(M)) \subset V^{i+1} K_q^M(K)$ .

For the proof of the theorems, we may replace  $F$  by a separable extension which preserves a  $p$ -base. So, we may assume that  $H_p^q(F) = H_p^{q+1}(K) \neq 0$

(For example, replace  $F$  by  $F(\bigcup_{n \geq 1} \{T^{p^{-n}}\})$  where  $T$  is a variable.). The exact sequence (3.3.3) shows that  $H_{p^n}^{q+1}(K)$  has an element of order  $p^n$ .

*Proof of Theorem (3.1).* Let  $L/K$  be as in Theorem (3.1) and  $\chi$  be the character corresponding to  $L/K$ . Since the argument proceeds in parallel, we only consider the case when  $L/K$  is totally ramified and  $K$  is of type-II. By Lemma (3.3.4), we obtain a surjective homomorphism  $\varphi_\chi: U^1 K_q^M(K) \rightarrow H_{p^n}^{q+1}(K)$ .

On the other hand, by Theorem (2.2) and Lemma (3.3.5), we know that the images of  $U^i K_q^M(K)$  for  $i > \delta_2$  by  $\varphi_\chi$  all coincide. Since  $U^i K_q^M(K) \subset p^n K_q^M(K)$  for sufficiently large  $i$ , this implies that  $\varphi_\chi$  induces a surjective homomorphism  $U^1 K_q^M(K)/U^{\delta_2+1} K_q^M(K) \rightarrow H_{p^n}^{q+1}(K)$ . The fact that  $U^i K_q^M(K)/U^{i+1} K_q^M(K)$  is annihilated by  $p$  for all  $i \geq 1$  and the target group has an element of order  $p^n$ , implies the inequality  $\delta_2 \geq n$ . Q.E.D.

*Proof of Theorem (3.2).* As above, we may assume that  $[F: F^p] = p^{q-1} < \infty$  and  $H_p^{q+1}(K) \neq 0$ . Let  $\psi$  be a character corresponding to  $M/K$  and consider the homomorphism  $\varphi_\psi: K_q^M(K)/N_{M/K} K_q^M(M) \rightarrow H_p^{q+1}(K)$ . We put  $t = f_{M/K} \cdot s(M/K)$  where  $f_{M/K}$  is the degree of the residue extension of  $M/K$ . If  $M/K$  is fiercely ramified (resp. totally ramified), we have  $\varphi_\psi(V^t K_q^M(K)) = H_p^{q+1}(K)$  and  $\varphi_\psi(U^t K_q^M(K)) = 0$  (resp.  $\varphi_\psi(U^t K_q^M(K)) = H_p^{q+1}(K)$  and  $\varphi_\psi(V^{t+1} K_q^M(K)) = 0$ ) (cf. [10] §3.3 Proofs of Prop. 2 and Th. 1).

Let  $\chi \in H_{p^n}^{q+1}(K)$  be a character such that the cyclic extension  $L$  attached to  $\chi$ , contains  $M$ . The commutative diagram

$$\begin{array}{ccc} K_q^M(K)/N_{L/K} K_q^M(L) & \longrightarrow & K_q^M(K)/N_{M/K} K_q^M(M) \\ \varphi_\chi \downarrow & & \downarrow \varphi_\psi \\ H_{p^n}^{q+1}(K) & \xrightarrow{p^{n-1}} & H_p^{q+1}(K) \subset H_{p^n}^{q+1}(K) \end{array}$$

implies that  $p^{n-1} \varphi_\chi(V^t) \neq 0$  and  $p^{n-1} \varphi_\chi(U^t) = 0$  (resp.  $p^{n-1} \varphi_\chi(U^t) \neq 0$  and  $p^{n-1} \varphi_\chi(V^{t+1}) = 0$ ). Since the assumption on  $s(M/K)$  implies  $t > e_K/(p-1)$ , we have  $p^{n-1} U^t = U^{t+e_K(n-1)}$  and  $p^{n-1} V^t = V^{t+e_K(n-1)}$ .

Let  $K$  be of type-I (resp. type-II). By Theorem (2.2), we have  $V^i = U^i$  (resp.  $U^i = V^{i+1}$ ) for  $i > \delta_1$  (resp.  $\delta_2$ ). Therefore,  $t + e_K(n-1)$  cannot be greater than  $\delta_1$  (resp.  $\delta_2$ ). Hence,  $e_K(n-1) < \delta_1$  (resp.  $\delta_2$ ), which implies  $n \leq [\delta_1/e_K] + 1$  (resp.  $[\delta_2/e_K] + 1$ ).

#### §4. A global application

(4.1) In this section, we prove Theorem 3 in Introduction as a (semi-)global application of Theorem (2.2) (i) which asserts  $U^i = V^i$  for sufficiently

large  $i$  in the case of type-I. We will use the class field theory of varieties over a local field. We cannot find an adequate reference for it, but it is not our aim to explain the construction of class field theory, so we omit the details of the proof concerning the construction, for example, of the reciprocity laws and the residue theorems etc. The proofs are essentially the same as in the case of the class field theory of schemes of finite type over  $\mathbb{Z}$  in [14].

Let  $k$  be a complete discrete valuation field of mixed characteristics  $(0, p)$ . We denote by  $R$  the valuation ring of  $k$  and assume that its residue field  $F$  is perfect. Let  $\mathfrak{X}$  be a proper smooth scheme over  $R$  which is geometrically connected and purely of relative dimension  $d$ , and let  $X = \mathfrak{X} \otimes_R k$  and  $Y = \mathfrak{X} \otimes_R F$ . In the following,  $\pi$  always denotes a prime element of  $R$ .

Let  $K_q^M(\mathcal{O}_{\mathfrak{X}})$  be the sheaf of Milnor  $K$ -groups for  $q \geq 1$ . (For a scheme  $T$ , the sheaf of Milnor  $K$ -groups  $K_q^M(\mathcal{O}_T)$  is defined to be the sheaf associated to the presheaf  $U \mapsto K_q^M(\Gamma(U, \mathcal{O}_T))$ .) For  $i \geq 0$ , a sheaf  $K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi^i)$  on  $\mathfrak{X}$  is defined to be  $\text{Ker}(K_q^M(\mathcal{O}_{\mathfrak{X}}) \rightarrow K_q^M(\mathcal{O}_{\mathfrak{X}}/\pi^i))$ . The objects we consider in this section are the following groups. For  $q \geq 1$ , we define  $I_q^{\text{top}}(\mathfrak{X}, X)$  or simply  $I_q^{\text{top}}(X)$  by

$$I_q^{\text{top}}(X) = \varinjlim_i H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi^i)) \quad (4.1.1)$$

where the cohomology group at the right hand side is the Zariski cohomology with support on  $Y$ . There exists a natural filtration on  $I_q^{\text{top}}(X)$  induced by the definition. Namely, we define  $U^i I_q^{\text{top}}$  for  $i \geq 0$  as follows.

$$U^i I_q^{\text{top}} = \text{Ker}(I_q^{\text{top}}(X) \rightarrow H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi^i))). \quad (4.1.2)$$

This group has an explicit presentation by Milnor  $K$ -groups, and can be considered as an idele class group, which we explain briefly.

Let  $\mathcal{F}$  be a sheaf on  $\mathfrak{X}_{\text{Zar}}$ . Then, the spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in Y^{p-1}} H_x^{p+q}(\mathfrak{X}, \mathcal{F}) \Rightarrow H_Y^{p+q}(\mathfrak{X}, \mathcal{F})$$

yields the following exact sequence

$$\bigoplus_{y \in Y_1} H_y^d(\mathfrak{X}, \mathcal{F}) \rightarrow \bigoplus_{x \in Y_0} H_x^{d+1}(\mathfrak{X}, \mathcal{F}) \rightarrow H_Y^{d+1}(\mathfrak{X}, \mathcal{F}) \rightarrow 0 \quad (4.1.3)$$

because  $H_x^{d+1}(\mathfrak{X}, \mathcal{F}) = 0$  for  $x \in Y^p$  with  $p < d$ . For a scheme  $T$  of dimension  $r$ , we define a  $P$ -chain to be a sequence  $P = (p_0, \dots, p_r)$  of points of  $T$  such that  $p_i \in T_i$  and  $\overline{\{p_0\}} \subset \dots \subset \overline{\{p_r\}}$ . Applying the localizing sequences



repeatedly, we have a surjective homomorphism  $\bigoplus_P (\mathcal{F})_{p_{d+1}}/(\mathcal{F})_{p_d} \rightarrow H_x^{d+1}(\mathfrak{X}, \mathcal{F})$  where  $P$  runs over the  $P$ -chains of  $\mathfrak{X}$  of the form  $P = (x, p_1, \dots, p_{d+1})$ . Let  $K$  be the function field of  $\mathfrak{X}$  and for a  $P$ -chain  $P = (p_0, \dots, p_{d+1})$  of  $\mathfrak{X}$ , we denote by  $K_P$  the field  $K$  which is regarded as a discrete valuation field by the discrete valuation defined by  $p_d$ . Let  $\eta$  be the generic point of  $Y$ . By the fact described above, we have a surjective homomorphism

$$\bigoplus_{P'} K_q^M(K_{P'})/U^0 K_q^M(K_{P'}) \oplus \bigoplus_P K_q^M(K_P) \rightarrow H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi^i)). \quad (4.1.4)$$

Here,  $P'$  (resp.  $P$ ) runs over the  $P$ -chains such that  $p'_d \neq \eta$  (resp.  $p_d = \eta$ ), and for a discrete valuation field  $L$  with valuation ring  $A$ ,  $U^i K_q^M(L)$  for  $i \geq 1$  is defined as in §2 and  $U^0 K_q^M(L)$  is the image of  $K_q^M(A) \rightarrow K_q^M(L)$ .  $P$ -chains with  $p_d = \eta$  are regarded as  $P$ -chains of  $Y$ , so we sometimes identify them. The kernel of (4.1.4) can be described using the term  $Q$ -chain as in [14]. (In [14], the henselian topology was used but the argument proceeds similarly in the case of Zariski topology.). We can define another filtration  $V^i I_q^{\text{top}}$  on  $I_q^{\text{top}}(X)$  which is induced by (4.1.4) from  $\bigoplus_P V^i K_q^M(K_P)$  where  $P$  runs over all  $P$ -chains of  $Y$ .

For a  $P$ -chain  $P$  of  $Y$ , the fixed prime element  $\pi$  and  $i \geq 1$ , there are two surjective homomorphisms (cf. (2.1.1) and (2.1.2))

$$\Omega_{F_P}^{q-1} \rightarrow U^i K_q^M(K_P)/V^{i+1} K_q^M(K_P) \quad (4.1.5)$$

$$\Omega_{F_P}^{q-2} \rightarrow V^i K_q^M(K_P)/U^i K_q^M(K_P) \quad (4.1.6)$$

where  $F_P$  is the residue field of  $K_P$  (hence, the function field of  $Y$ ). The above homomorphisms induce two global surjective homomorphisms.

$$H^d(Y, \Omega_Y^{q-1}) \rightarrow U^i I_q^{\text{top}}/V^{i+1} I_q^{\text{top}} \quad (4.1.7)$$

$$H^d(Y, \Omega_Y^{q-2}) \rightarrow V^i I_q^{\text{top}}/U^i I_q^{\text{top}} \quad (4.1.8)$$

This can be verified by using the explicit presentation of both groups as in [14] Lemma (1.6.3). (Cf. a similar homomorphism is constructed in [5] Th. (2.9).)

**PROPOSITION (4.1.9).** *For  $i$  sufficiently large (for example,  $i > 2(m+1)$  where  $m = \text{length}_R(\hat{\Omega}_R^1)$ ), (4.1.8) is the 0-map. Further, if the absolute ramification index  $e_k$  is smaller than  $p-1$ , (4.1.8) is the 0-map for any  $i \geq 1$ .*

*Proof.* By definition of (4.1.8), it suffices to show that (4.1.6) is the 0-map for  $i > 2(m + 1)$ . Since  $\mathfrak{X}$  is smooth over  $R$ , the claim follows from Corollary (1.5) and Theorem (2.2) (cf. the proof of Th. (1.4)). For the second assertion, if  $e_k < p - 1$ , we have  $p \cdot U^i = U^{i+e_k}$  and  $p \cdot V^i = V^{i+e_k}$  for  $i \geq 1$ . Since we can take  $r$  and  $j$  such that  $i = re_k + j$ ,  $r \geq 0$ , and  $0 < j < p$ , the claim reduces to the case  $0 < i < p$ . In this case, it is trivial that  $V^i = U^i$  (cf. Remark (2.3)). Q.E.D.

Next, we consider  $gr^0 I_q^{\text{top}} = I_q^{\text{top}}(X)/U^1 I_q^{\text{top}}(X)$ . By (4.1.2), we have  $gr^0 I_q^{\text{top}} = H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi))$ . From the exact sequence:

$$0 \rightarrow K_q^M(\mathcal{O}_{\mathfrak{X}}, \pi) \rightarrow K_q^M(\mathcal{O}_{\mathfrak{X}}) \rightarrow K_q^M(\mathcal{O}_Y) \rightarrow 0,$$

taking cohomology groups, we obtain an exact sequence

$$H^d(Y, K_q^M(\mathcal{O}_Y)) \rightarrow gr^0 I_q^{\text{top}} \rightarrow H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) \rightarrow 0. \quad (4.1.10)$$

We shall compute the group  $H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}}))$ .

We have the following isomorphisms ([13] Th. 2)

$$H_x^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) = K_{q-d-1}^M(\kappa(x)) \quad \text{for } x \in Y_0$$

$$H_y^d(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) = K_{q-d}^M(\kappa(y)) \quad \text{for } y \in Y_1$$

where  $\kappa(x)$  is the residue field at  $x$  and we define  $K_m^M(\kappa(x)) = 0$  if  $m < 0$ . By (4.1.3), we have an exact sequence

$$\bigoplus_{y \in Y_1} K_{q-d}^M(\kappa(y)) \rightarrow \bigoplus_{x \in Y_0} K_{q-d-1}^M(\kappa(x)) \rightarrow H_Y^{d+1}(\mathfrak{X}, K_q^M(\mathcal{O}_{\mathfrak{X}})) \rightarrow 0. \quad (4.1.11)$$

In particular, we obtain

$$H_Y^{d+1}(\mathfrak{X}, K_{d+1}^M(\mathcal{O}_{\mathfrak{X}})) = CH_0(Y) \quad (4.1.12)$$

(the group of zero cycles modulo rational equivalence).

(4.2) We define  $SK_1^{\text{top}}(X) = I_{d+1}^{\text{top}}(X)$ . This group is more useful than the group  $SK_1^M(X)$  which is defined to be  $H^d(X, K_{d+1}^M(\mathcal{O}_X))$ , when we consider the arithmetic of  $X$ . Let  $k$  be a local field (which means a finite extension of  $\mathbb{Q}_p$ ). By using this group  $SK_1^{\text{top}}(X)$ , we can construct the class field theory of  $X$ . In fact, we can construct the reciprocity map:  $SK_1^{\text{top}}(X) \rightarrow \pi_1^{ab}(X)$  by using the “explicit” presentation of  $SK_1^{\text{top}}(X)$  (cf. (4.1.4)) and patching the

reciprocity maps of higher dimensional local fields (cf. [10]) as in the case of arithmetical schemes (cf. [14]). This reciprocity map has dense image. Indeed, the quotient of  $\pi_1^{ab}(X)$  by the closure of the image of this homomorphism corresponds to the abelian covering in which any closed point of  $\mathfrak{X}$  splits completely. Such a covering is necessarily trivial because  $\pi_1^{ab}(\mathfrak{X}) = \pi_1^{ab}(Y)$  and we already know that over a smooth scheme over a finite field, such a covering is trivial (cf. [15], [14]).

Again by using the “explicit” presentation of  $SK_1^{\text{top}}(X)$ , we can construct the norm map  $N: SK_1^{\text{top}}(X) \rightarrow k^*$ , which is obtained by patching the residue homomorphisms (cf. [12]). This time, the residue theorem affirms the well-definedness of  $N$ . Then, we have the following commutative diagram

$$\begin{array}{ccc} SK_1^{\text{top}}(X) & \xrightarrow{N} & k^* \\ \downarrow & & \downarrow \\ \pi_1^{ab}(X) & \longrightarrow & \text{Gal}(k^{ab}/k) \end{array} \quad (4.2.1)$$

where the vertical arrows are the reciprocity maps.

**THEOREM (4.2.2).** *Let  $k$  be a finite extension of the  $p$ -adic number field  $\mathbb{Q}_p$ , and  $\mathfrak{X}, X, Y$  be as above. Then, the kernel of the norm map  $N: SK_1^{\text{top}}(K) \rightarrow k^*$  is finite.*

*Proof.* The filtration  $U^i K_1^M(k)$  on  $k^*$  is denoted by  $U_k^i$  as usual. Since the residue preserves the filtration (cf. [12]),  $N$  also. To be precise,  $N(V^i SK_1^{\text{top}}) \subset U_k^i$  for  $i \geq 1$ . Hence, we can consider  $gr^i(N): V^i SK_1^{\text{top}}/V^{i+1} SK_1^{\text{top}} \rightarrow U_k^i/U_k^{i+1}$ .

(4.2.3) For  $gr^0 SK_1^{\text{top}} = SK_1^{\text{top}}(X)/U^1 SK_1^{\text{top}}$ , we have the following diagram which is commutative by definition of the homomorphisms.

$$\begin{array}{ccccccc} H^d(Y, K_{d+1}^M(\mathcal{O}_Y)) & \longrightarrow & gr^0 SK_1^{\text{top}} & \longrightarrow & CH_0(Y) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \bar{F}^* & \longrightarrow & gr^0(k^*) & = & k^*/U_k^1 & \longrightarrow \mathbb{Z} \longrightarrow 0 \end{array}$$

In this diagram, the top row is exact by (4.1.10) and (4.1.12), and the bottom row is a well-known exact sequence. The left vertical arrow is bijective by the generalized Moore’s exact sequence (cf. [14], in which the henselian topology is used but we can replace it by the Zariski topology in the proof of bijectivity). The right vertical arrow is the degree map of Chow group, and the class field theory of  $Y$  tells us that its kernel is finite (cf. [14] Th. (6.1)). Thus,  $gr^0(N)$  has finite kernel.

(4.2.4) Next, we consider  $gr^i SK_1^{\text{top}} = V^i SK_1^{\text{top}} / V^{i+1} SK_1^{\text{top}}$  for  $i \geq 1$ . We already know its finiteness by (4.1.7) and (4.1.8) because  $H^q(Y, \Omega_Y)$  is finite. Moreover, we have the following commutative diagram.

$$\begin{array}{ccc} H^d(Y, \Omega_Y^d) & \xrightarrow{\varrho} & gr^i SK_1^{\text{top}} \\ & \searrow \text{trace} \simeq & \downarrow gr^i(N) \\ & & gr^i(k^*) = F \end{array}$$

In this diagram,  $\varrho$  is the map in (4.1.7) and  $H^d(Y, \Omega_Y^d) \rightarrow F$  is the trace map of Serre duality, which is an isomorphism. On the other hand, by Proposition (4.1.9), if  $i$  is sufficiently large,  $\varrho$  is surjective. Hence,  $\varrho$  and  $gr^i(N)$  are bijective for  $i \geq 0$ .

Since the filtration  $V^i SK_1^{\text{top}}$  is separated, the theorem follows from (4.2.3) and (4.2.4). Q.E.D.

**COROLLARY (4.2.5).** *If  $e_k < p - 1$ , we have*

$$\text{Ker}(SK_1^{\text{top}}(X) \rightarrow k^*) = \text{Ker}(CH_0(Y) \rightarrow \mathbb{Z}).$$

*Proof.* This follows from Proposition (4.1.9) and the proof of Theorem (4.2.2).

The fact that the reciprocity maps in (4.2.1) have dense images implies the following corollaries.

**COROLLARY (4.2.6).** *The kernel of the natural homomorphism:  $\pi_1^{ab}(X) \rightarrow \text{Gal}(k^{ab}/k)$  is finite.*

**COROLLARY (4.2.7).** *Every abelian etale covering of  $X$  comes from an abelian extension of  $k$  and an abelian etale covering of  $Y$  if  $e_k < p - 1$ .*

**REMARK (4.2.8).** This result (4.2.7) was essentially proved by Coombes [5] in the case of  $\dim X = 1$  (cf. [5] the proof of (3.6), and (2.11), (2.12)). Our method of proof of (4.2.7) is essentially the same as his.

(4.3) Finally, we give a remark concerning the case  $q = d$ . We define  $CH_0^{\text{top}}(X) = I_d^{\text{top}}(X)$ . On the other hand, let  $\text{Alb}_X$  be the Albanese variety of

$X$ . By analogy with  $N: SK_1^{\text{top}}(X) \rightarrow k^*$ , I expect that there exists a canonical homomorphism having finite kernel

$$CH_0^{\text{top}}(X)^{\deg 0} \rightarrow \text{Alb}_X(k)$$

which induces on  $CH_0(X)^{\deg 0}$  the classical Albanese mapping.

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### References

1. S. Bloch: Algebraic K-theory and crystalline cohomology. *Publ. Math. I.H.E.S.* 47 (1977) 187–268.
2. S. Bloch: Algebraic K-theory and classfield theory for arithmetic surfaces. *Ann. Math.* 114 (1981) 229–265.
3. S. Bloch and K. Kato: p-adic étale cohomology. *Publ. Math. I.H.E.S.* 63 (1986) 107–152.
4. N. Bourbaki: *Algèbre commutative*, Chap. 8 et 9. Masson, Paris (1983).
5. K.R. Coombes: Local class field theory for curves. *Contemporary Math.* 55, Part I (1986) 117–134.
6. M. Demazure: Lectures on  $p$ -divisible groups. *Lecture Notes in Math.*, 302. Springer, Berlin (1972).
7. A. Grothendieck: *Éléments de Géométrie algébrique*, IV (Première Partie). *Publ. Math. I.H.E.S.* 20 (1964).
8. O. Hyodo: Wild ramification in the imperfect residue field case, preprint.
9. L. Illusie: Complexe de De Rham-Witt et cohomologie cristalline. *Ann. Sc. Ec. Norm. Sup.* 4ème série, 12 (1979) 501–661.
10. K. Kato: A generalization of local class field theory by using K-groups. II. *J. Fac. Sci. Univ. Tokyo* 27 (1980) 603–683.
11. K. Kato: Galois cohomology of complete discrete valuation fields. *Lecture Notes in Math.*, 967. Springer (1982).
12. K. Kato: Residue homomorphisms in Milnor K-theory. *Adv. St. Pure Math.* 2 (1983) 153–172.
13. K. Kato: Milnor K-theory and the Chow group of zero cycles. *Contemporary Math.* 55, Part I (1986) 241–253.
14. K. Kato and S. Saito: Global class field theory of arithmetic schemes. *Contemporary Math.* 55, Part I (1986) 255–331.
15. S. Lang: Unramified class field theory over function fields in several variables. *Ann. Math.* 64 (1956) 285–325.

16. H. Miki: *On cyclic extensions of  $p$ -power degree over complete  $p$ -adic fields* (in Japanese), Master's thesis. Univ. Tokyo (1973).
17. J.P. Serre: Sur les corps locaux à corps résiduel algébriquement clos. *Bull. Soc. Math. France* 89 (1961) 105–154.
18. J.P. Serre: *Corps locaux* (3<sup>e</sup> édition). Hermann, Paris (1968).