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A mapping theorem for topological sigma-compact manifolds

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1. Introduction

It is the purpose of this paper to prove a generalization to σ-compact manifolds of a well known result due to M. Brown (see [4]), which asserts the existence of a special kind of continuous, "non-pathological" surjections from the unit n-dimensional cube onto a given compact connected manifold M^n.

In the more general setting when M^n is σ-compact, the space $\mathscr{E}(M)$ of ends of M^n plays an important role: Since $\mathscr{E}(M)$ is a totally disconnected, compact, metrizable space, a set E contained in the boundary of the unit cube I^n can be constructed in such a way that E is homeomorphic to $\mathscr{E}(M)$. Now $I^n\backslash E$ and M^n are two manifolds with the same set of ends. Broadly speaking, our result states that M^n is the identification space obtained from $I^n\backslash E$ by identifying points within the boundary of $I^n\backslash E$ alone.

The set $\mathscr{E}(M)$ is empty exactly when M is compact. In this case, the arguments are reduced to those given by M. Brown for compact manifolds. Some applications are mentioned afterwards.

§1. The set of ends

The concept of the set of ends of a space is due to Freudenthal. Here we recall some basic notions.

Let X be a locally compact, Hausdorff space. Denote by $\mathscr{K}(X)$ the set of all compact subsets of X partially ordered by inclusion. If $K \in \mathscr{K}(X)$, denote by $\mathscr{C}(X\backslash K)$ the set of connected components of $X\backslash K$ considered as a discrete topological space.

If K, $L \in \mathscr{K}(X)$ with $K \subset L$, then there is a well defined continuous function

$$\varrho_K^L : \mathscr{C}(X\backslash L) \to \mathscr{C}(X\backslash K)$$

such that for each $V \in \mathscr{C}(X\backslash L)$, $\varrho_K^L(V)$ is the unique component of $X\backslash K$ containing V. In this manner, the collection

$$\{\mathscr{C}(X\backslash K), \varrho_K^L \mid K, L \in \mathscr{K}(X) \quad \text{and} \quad K \subset L\}$$

constitutes an inverse system of topological spaces indexed over the directed set $\mathscr{K}(X)$.

An *end* of X is, by definition, a point in the inverse limit space of this system. In other words, an end of X is a function e which assigns to each compact set K of X a non-empty connected component $e(K)$ of $X\backslash K$, in such a way that $K_1 \subset K_2$ implies $e(K_2) \subset e(K_1)$. Let $\mathscr{E}(X)$ be the set of all ends. There is a topology on $X \cup \mathscr{E}(X)$ having as a basis of neighbourhoods of $e_0 \in \mathscr{E}(X)$ the $\mathscr{N}_K(e_0) = e_0(K) \cup \{\text{ends } e \mid e(K) = e_0(K)\}$, $K \in \mathscr{K}(X)$. With this topology $X \cup \mathscr{E}(X)$ is a Hausdorff space containing $\mathscr{E}(X)$, with its inverse limit topology, as a closed (nowhere dense) subspace.

If $f: X \to Y$ is a continuous *proper* function (i.e., $F \subset Y$ compact implies $f^{-1}(F)$ compact), then f is extended uniquely and continuously to a function

$$f \cup f_\varepsilon : X \cup \mathscr{E}(X) \to Y \cup \mathscr{E}(Y)$$

such that for $e \in \mathscr{E}(X)$ and $F \subset Y$ compact $f_\varepsilon(e)F$ is the (unique) component of $Y\backslash F$ containing $f(e(f^{-1}(F)))$.

Let X be a space, and let $K \in \mathscr{K}(X)$. A connected component V of $X\backslash K$ is said to be *bounded* if its closure is compact, and otherwise we say that V is *unbounded*. Define

$$\hat{K} = X\backslash \cup \{V \in \mathscr{C}(X\backslash K) \mid V \text{ is unbounded}\}.$$

The proof of the following lemma may be found in Berlanga and Epstein [2].

1.1 Lemma. *Let X be a connected, locally connected, locally compact, Hausdorff space and let $K \in \mathscr{K}(X)$. Then $X\backslash K$ has only finitely many unbounded components and $\hat{K}$ is compact.*

1.2 Remark. It follows that $\mathscr{E}(X)$ is compact since $\hat{\mathscr{K}}(X) = \{\hat{K} \mid K \in \mathscr{K}(X)\}$ is cofinal in $\mathscr{K}(X)$ and each $\mathscr{C}(X\backslash\hat{K})$ is finite. It is also known that $X \cup \mathscr{E}(X)$ is compact and that $\mathscr{E}(X)$ is totally disconnected. Also if X is metric $X \cup \mathscr{E}(X)$ is metrizable.

§2. Definitions

Let X be a subset of a topological space Y. We define $\mathring{X}$ and Cl X to be, respectively, the topological interior and the topological closure of X in Y. Call X a (closed) *n-cell* if X is homeomorphic to the unit n-cube $I^n = [0, 1]^n$. For a subset X of a manifold M we define Int X to be $(M \backslash \partial M) \cap \mathring{X}$, where ∂M denotes the boundary of M.

§3. The main theorem

Let M^n be a connected, second countable manifold of dimension n. Then there exists a compact set $E \subset \partial I^n$ and a continuous proper surjection $\psi : I^n \backslash E \to M$ such that

(1) $\psi|_{\text{Int}\, I^n} : \text{Int}\, I^n \to \psi\,(\text{Int}\, I^n)$ *is a homeomorphism*;

(2) $\psi\,(\text{Int}\, I^n) \cap \psi(\partial I^n \backslash E) = \varnothing$;

(3) ψ *extends naturally to* $\tilde{\psi} : I^n \to M \cup \mathscr{E}(M)$ *in such a way that* $\tilde{\psi}|_E$ *is a homeomorphism from E onto* $\mathscr{E}(M)$.

Furthermore, if $n \geqslant 2$ then E can be chosen to be contained in $[1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\}$.

§4. Definitions, lemmas and proof of the main theorem

4.1. Definitions. An $(n - 1)$-dimensional submanifold B of an n-manifold M is *bicollared in M* if there is a homeomorphism P of $B \times \langle -1, 1\rangle$ onto a neighbourhood of B in M such that $P(b, 0) = b$, for all $b \in B$. If B is closed in M we require also that P can be extended to a closed embedding of $B \times [1, 1]$ into M.

If B is the boundary of an n-dimensional submanifold C of M, then $B \times \langle -1, 0]$ and $B \times [0, 1\rangle$ denote the *inner* and *outer* collars of B. In general, we will not distinguish $(b, t) \in B \times \langle -1, 1\rangle$ from $P((b, t))$.

Define $\mathscr{H}(M)$ to be the group of homeomorphisms of M onto itself. If $h : M \to M$ is a homeomorphism, then supp h denotes the *support* of h, that is, the closure of the set of points of M which are actually moved by h.

The following result is a straightforward generalization of Lemma 2 in M. Brown [4] (or Lemma 6 in Berlanga and Epstein [2]).

4.2. Lemma. *Let M^n be a manifold with $n \geqslant 3$ and let d be a metric on M. Let C^n be a closed n-dimensional manifold with bicollared boundary ∂C in M.*

Let $\varepsilon > 0$ be given and suppose $\Lambda = \{D_j\}_{j \in J}$ is a locally finite family of sets in M such that each D_j is a closed n-cell of diameter less than $\varepsilon/2$ whose interior intersects C. *Let $X = \{x_i\}_{i \in L}$ be a locally finite set of points in $\bigcup_j \operatorname{Int} D_j$.*

Suppose that $0 < \gamma < 1$. Then there is an ε-homeomorphism f in $\mathscr{H}(M)$ such that $f(C) \supset f(\mathring{C}) \supset C \cup X$ and $\operatorname{supp} f \subset (\bigcup_j \operatorname{Int} D_j \backslash C) \cup \partial C \times \langle -\gamma, \gamma \rangle$. In particular, f fixes pointwise the inner n-manifold bounded by $\partial C \times \{-\gamma\}$.

4.3 Lemma. *Let M be a connected, second countable, n-dimensional manifold with $n \geqslant 3$ and let $X \subset \operatorname{Int} M$ be a locally finite set of points. Then there exists a compact set $E \subset [1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\} \subset \partial I^n$, and a proper embedding $\psi_* : I^n \backslash E \to M$ with bicollared boundary such that $\psi_*(\operatorname{Int} I^n) \supset X$ and ψ_* extends naturally to $\tilde{\psi}_* : I^n \to M \cup \mathscr{E}(M)$ in such a way that $\tilde{\psi}_*|_E$ is a homeomorphism from E onto $\mathscr{E}(M)$.*

Proof. Define a *clean (closed) n-cube in I^n* to be a cube C of the form $[0, \beta]^n + v$, for some $\beta > 0$ and $v \in \mathbb{R}^n$, such that $C \subset I^n$ and $C \cap \partial I^n = ([0, \beta]^{n-1} \times \{\beta\}) + v$.

Observe that if $C_1, \ldots, C_k$ is a disjoint collection of clean cubes then $\operatorname{Cl}(I^n \backslash \bigcup_i C_i)$ is homeomorphic to I^n. We divide the proof in three steps.

Step 1. Let $\{K_i\}_{i \in \mathbb{N}}$ be any collection of $K_i \in \mathscr{K}(M)$ such that $M = \bigcup_i K_i$ and $K_i \subset \mathring{K}_{i+1}$; further properties of the K_i will be specified in Step 2. It is not difficult now to define a sequence $\{L_i\}_{i \in \mathbb{N}}$ of n-cells in I^n with $L_i \subset \mathring{L}_{i+1}$ and such that

(a) The complement of $\mathring{L}_i$ is the finite disjoint union of clean cubes of diameter less or equal $1/2^i$, and such that, for each $A \in \mathscr{C}(I^n \backslash L_i)$, we have, $A \cap [1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\} \neq \varnothing$. Hence, $E = \bigcap_i I^n \backslash L_i$ is contained in $[1/3, 2/3] \times \{(1/2, 1/2, \ldots, 1/2, 1)\}$;

(b) For each $i \in \mathbb{N}$ there exists a bijection $\lambda_i : \mathscr{C}(I^n \backslash L_i) \to \mathscr{C}(M \backslash \hat{K}_i)$ such that the diagrams

$$\begin{array}{ccc} \mathscr{C}(I^n \backslash L_j) & \xrightarrow{\lambda_j} & \mathscr{C}(M \backslash \hat{K}_j) \\ \downarrow \varrho_i^j & & \downarrow \varrho_i^j \\ \mathscr{C}(I^n \backslash L_i) & \xrightarrow{\lambda_i} & \mathscr{C}(M \backslash \hat{K}_i) \end{array}$$

commute ($i < j$).

The reader can readily verify the following assertion:

Assertion. $E = \bigcap_i I^n \backslash L_i$, $\mathscr{E}(I^n \backslash E)$ and $\mathscr{E}(M)$ are homeomorphic. Furthermore, the identity map $I^n \backslash E \to I^n \backslash E$ extends naturally to a homeomorphism of $I^n = (I^n \backslash E) \cup E$ onto $(I^n \backslash E) \cup \mathscr{E}(I^n \backslash E)$.

We now proceed into Step 2 of this lemma.

Step 2. Let the K_i be constructed as to satisfy also the following properties:
– $\mathring{K}_i$ is connected;
– $M \backslash \hat{K}_i$ has exactly the same number of components as $\mathring{K}_{i+1} \backslash \hat{K}_i$.
If $\psi_0 : I^n \to \mathring{K}_0$ is an embedding with bicollared boundary, then there exists a homeomorphism h_1 of M with compact support such that
(1) supp $h_1 \cap \psi_0(L_0) = \varnothing$;
(2) supp $h_1 \subset \mathring{K}_1$;
(3) If $A \in \mathscr{C}(I^n \backslash L_1)$ then
  (a) $h_1(\psi_0(A)) \subset \lambda_0(\varrho_0^1(A))$;
  (b) $h_1(\psi_0(A))$ and $\lambda_1(A)$ are not separated in M by $h_1(\psi_0(I^n \backslash A)) \cup \hat{K}_0$, (that is, $h_1(\psi_0(A))$ and $\lambda_1(A)$ lie in the same connected component of $M \backslash (h_1(\psi_0(I^n \backslash A)) \cup \hat{K}_0)$.

Proof. Let $A_1, A_2, \ldots, A_k$ be the components of $I^n \backslash L_1$. It is not difficult to construct a family of disjoint arcs, say $\{\gamma_i : [0, 2] \to M \mid 1 \leqslant i \leqslant k\}$ and a family $\{U_i \mid 1 \leqslant i \leqslant k\}$ of disjoint connected open sets in $\mathring{K}_1$ such that, for each i,

$$U_i \cap \psi_0(I^n) \subset \psi_0(A_i);$$

$$\gamma_i([0, 1]) \subset U_i;$$

$$\gamma_i([1, 2]) \subset \mathring{K}_2 \backslash \hat{K}_0;$$

$$\gamma_i(0) \in \psi_0(A_i);$$

$$\gamma_i(1) \in \lambda_0(\varrho_0^1(A_i));$$

$$\gamma_i(2) \in \lambda_1(A_i).$$

This can be done because $M \backslash \psi_0(I^n)$ is connected and an n-dimensional manifold cannot be disconnected by a set of dimension $n - 2$ (see Hurewicz and Wallman [5]).

Since the group of compactly supported homeomorphisms of a connected manifold acts transitively on interior points, we can find, for each i, a homeomorphism $h_{1,i}$ compactly supported on U_i which sends $\gamma_i(0)$ to $\gamma_i(1)$.

For each $i = 1, 2, \ldots, k$, let $\tau_i \in [1, 2]$ be the last parameter such that its image under γ_i lies in $h_{1,i}(\psi_0(I^n))$.

Consequently, there is a unique x_i in $\partial I^n \cap A_i$ with $\gamma_i(\tau_i) = h_{1,\iota}(\psi_0(x_i))$. Now choose a clean closed cube B_i such that $x_i \in B_i \subset A_i$ and $h_{1,i}(\psi_\iota(B_i)) \subset \lambda_0(\varrho_0^1(A_i))$.

With a homeomorphism of M sending $\psi_0(I^n)$ onto itself and supported in a small neighbourhood of $\psi_0(\mathrm{Cl}\ A_i)$ we can shrink $\psi_0(\mathrm{Cl}\ A_i)$ onto $\psi_0(B_i)$ before applying $h_{1,i}$. Therefore, without loss of generality we can assume that $A_i = B_i$ and that supp $h_{1,i} \cap$ supp $h_{1,j} = \varnothing$ for $i \neq j$. Hence, $h_{1,i}(\psi_0(A_i)) \subset \lambda_0(\varrho_0^1(A_i))$ and $h_{1,i}(\psi_0(A_i))$, $\lambda_1(A_i)$ are not separated in M by $h_{1,i}(\psi_0(I^n \backslash A_i)) \cup \hat{K}_0$.

Finally, the homeomorphism $h_1 = h_{1,1} \circ h_{1,2} \circ \ldots \circ h_{1,k}$ has the required properties.

Step 3. By induction, we can construct a sequence $\{h_i\}_{i \in \mathbb{N}}$ of homeomorphisms with compact support such that, for each i,

(1) supp $h_{i+1} \cap (h_i \circ h_{i-1} \circ \ldots \circ h_1 \circ \psi_0(L_i)) = \varnothing$;
(2) supp $h_{i+1} \subset \mathring{K}_{i+1}$;
(3) supp $h_{i+1} \cap \hat{K}_{i-1} = \varnothing$;
(4) If $A \in \mathscr{C}(I^n \backslash L_{i+1})$ then
  (a) $h_{i+1} \circ h_i \circ \ldots \circ h_1 \circ \psi_0(A) \subset \lambda_i(\varrho_i^{i+1}(A))$;
  (b) $h_{i+1} \circ \ldots \circ h_1 \circ \psi_0(A)$ and $\lambda_{i+1}(A)$ are not separated in M by $h_{i+1} \circ h_i \circ \ldots \circ h_1 \circ \psi_0(I^n \backslash A) \cup \hat{K}_i$.

Define $\psi_i = h_i \circ \ldots \circ h_1 \circ \psi_0$, $i \in \mathbb{N}$. Therefore, the following properties hold:

(5) $\psi_i|_{L_i} = \psi_{i+k}|_{L_i}$ for all $i, k \in \mathbb{N}$;
(6) $\psi_{i+k}(A) \subset \lambda_i(\varrho_i^{i+1}(A))$ for all $i \in \mathbb{N}$, $k \in \mathbb{N} \backslash \{0\}$, and all $A \in \mathscr{C}(I^n \backslash L_{i+1})$.
  It follows that $\lim_{i \to \infty} \psi_i = \psi_*$ exists in $\bigcup_i L_i$ and is such that
(7) $\psi_*|_{L_i} = \psi_i|_{L_i}$ for all $i \in \mathbb{N}$;
(8) $\psi_*(A) \subset \lambda_i(\varrho_i^{i+1}(A))$ for all $i \in \mathbb{N}$ and all $A \in \mathscr{C}((I^n \backslash E) \backslash L_{i+1})$;
(9) $\psi_*^{-1}(\hat{K}_i) \subset L_{i+1}$.

Property (7) says that ψ_* is continuous and injective. Property (9) (which follows from (8)) tells us that $\psi_* : I^n \backslash E \to M$ is proper, and therefore induces a map $\psi_* \cup \psi_\varepsilon : (I^n \backslash E) \cup \mathscr{E}(I^n \backslash E) = I^n \to M \cup \mathscr{E}(M)$ such that if e is an end of $I^n \backslash E$, $\psi_\varepsilon(e)\hat{K}_i$ is the component of $M \backslash \hat{K}_i$ containing $\psi_*(e(\psi_*^{-1}(\hat{K}_i)))$, hence, by (9), it is equal to the component of $M \backslash \hat{K}_i$ containing $\psi_*(e(L_{i+1}))$, but, by (8), this is just $\lambda_i(\varrho_i^{i+1}(e(L_{i+1}))) = \lambda_i(e(L_i))$. That is, we have proved that the following diagram commutes:

$$\begin{array}{ccc}
\mathscr{E}(I^n\backslash E) & \xrightarrow{\ \Pi_i\ } & \mathscr{C}(I^n\backslash L_i) \\
\Big\downarrow{\scriptstyle \psi_\varepsilon} & & \Big\downarrow{\scriptstyle \lambda_i} \\
\mathscr{E}(M) & \xrightarrow{\ \Pi_i\ } & \mathscr{C}(M\backslash \hat{K}_i)
\end{array}$$

Since each λ_i is bijective, ψ_ε must be a homeomorphism. Therefore, we have constructed a proper embedding $\psi_*: I^n\backslash E \to M$ inducing a homeomorphism on ends.

In order to complete the proof of Lemma 4.3 we need to produce a bicollar of $\psi_*(\partial I^n\backslash E)$ and we need to "expand" the image C of $I^n\backslash E$ in M as to contain X in its interior.

Let E' be the projection of E into I^{n-1}, so $E = E' \times \{1\}$. It is not difficult to see that the spaces $W = [-1, 2]^n\backslash(E' \times [1/2, 2])$ and $T = W \cap I^n$ are homeomorphic to $I^n\backslash E$ and that the inclusion map $T \to W$ is a proper map inducing a homeomorphism on ends.

Therefore without loss of generality, we can assume that the domain of the map ψ_* is W. But now $\psi_*|_T$ has the same properties of ψ_* with the advantage that ∂T has a natural bicollar contained in M.

It now only remains to "expand" the image of ψ_*. To this purpose we can construct a locally finite family $\Lambda_0 = \{D_j\}_{j\in J}$ of closed n-cells such that X is contained in $\bigcup_j \operatorname{Int} D_j$ and $\operatorname{Int} D_j \cap C \neq \varnothing$ for all $j \in J$.

Therefore, by an application of Lemma 4.2, say with $\gamma = 1/2$ and $\varepsilon = \infty$, we get the desired expansion.

4.4. Proof of the main theorem. When the dimension of the manifold M is less or equal two, the theorem follows from the classification of second countable manifolds of dimensions one and two (see Ahlfors and Sario [1]).

Assume now that the dimension of M is greater or equal to three: Let d be a complete metric on M. Let $\Lambda_1, \Lambda_2, \ldots$ be a sequence of locally finite covers of M such that each element of Λ_i is a closed n-cell of diameter less than $1/2^{i+1}$ and $\operatorname{Int} M = \cup\{\operatorname{Int} D \mid D \in \Lambda_i\}$. For each i, let X_i be a locally finite set of points such that $X_i \subset \operatorname{Int} M$ and $\operatorname{Int} D \cap X_i \neq \varnothing$ if $D \in \Lambda_i$.

Let C_1 be the image under ψ_* where ψ_* is the embedding given by the above lemma, and assume that $X_1 \subset \operatorname{Int} C_1$. Applying Lemma 4.2 with $X = X_2$, $\Lambda = \Lambda_1$ and γ small, we get a 1/2-homeomorphism f_1 of M onto itself such that

$$M \supset C_2 = f_1(C_1) \supset f_1(\mathring{C}_1) \supset C_1 \cup X_2 \quad \text{and} \quad f_1|_{(1-\gamma)C_1} = Id,$$

where

$$(1 - \gamma)C_1 = C_1 \backslash \partial C_1 \times \langle -\gamma, 0].$$

Repeated applications of 4.2 give a sequence $f_1, f_2, \ldots$ of homeomorphisms of M such that for each $m \in \mathbb{N}\backslash\{0\}$,
- f_m is a $(1/2)^m$-homeomorphism;
- $M \supset f_m \circ \ldots \circ f_1(C_1) \supset f_m \circ \ldots \circ f_1(\mathring{C}_1)$
$\supset C_1 \cup \bigcup_i \{X_i \mid 1 \leqslant i \leqslant m + 1\}$;
- f_{m+1} restricted to $f_m \circ \ldots \circ f_1((1 - \gamma/2^m)C_1)$ is the identity.

Clearly $f_m \circ \ldots \circ f_1$ converges to a map ψ such that
- $\psi(C_1) = \lim_{m \to \infty} f_m \circ \ldots \circ f_1(C_1) = M$;
- ψ is a homeomorphism on $\mathring{C}_1$;
- $\psi^{-1}(\psi(\partial C_1)) = M \backslash \mathring{C}_1$;

So that when ψ is restricted to C_1 we get the required map.

4.5 Remark. Let $\psi : I^n \backslash E \to M$ be a mapping given by the main theorem above. Then, measures (having the boundary of the unit n-cube as a null set) and homeomorphisms of the unit n-cube fixing ∂I^n point-wise can be thrown, respectively, into measures and homeomorphisms of M via ψ. This provides us with a tool for the topological and algebraic study of various groups of (measure preserving) homeomorphisms of M (see [3]).

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