S. Tsuyumine

On the Siegel modular function field of degree three


<http://www.numdam.org/item?id=CM_1987__63_1_83_0>
On the Siegel modular function field of degree three

S. TSUYUMINE
Sonderforschungsbereich 170, Mathematisches Institut, Bunsenstrasse 3–5, 3400 Göttingen, Federal Republic of Germany (Current address: Department of Mathematics, Mie University, Tsu, 514 Japan)

Received 19 September 1986; accepted 22 December 1986

Introduction

Let $H_n$ be the Siegel space of degree $n$, and let $\Gamma_n$ be the modular group. A (Siegel) modular function $f$ is defined to be a meromorphic function on $H_n$ which is invariant under $\Gamma_n$, where for $n = 1$, we need an additional condition that $f$ is meromorphic also at the cusp. Let $K_n$ denote the Siegel modular function field over $\mathbb{Q}$, namely the field generated over $\mathbb{Q}$ by modular functions with the rational Fourier coefficients. Then the modular function field is given by $K_n \cong \langle \mathbb{Q} \rangle$. When $n = 1$, namely the elliptic modular case, it is well-known that $K_1$ is generated by the absolute invariant, which has a nice arithmetic property, e.g. an elliptic curve $E$ has a model over the field generated over $\mathbb{Q}$ by its special value attached to $E$. In the higher dimensional case, several ways to get $K_n$ are known: for example, Siegel [16], [18] showed that $K_n$ is generated by $E_{kl}/E_k^l$ (even $k > n + 1$, $l = 1, 2, \ldots$) where $E_k$ denotes the Eisenstein series of weight $k$. Besides this, if we denote by $K(\Gamma_n(l))$ the modular function field for the principal congruence subgroup $\Gamma_n(l)$ of level $l$, then it is shown (Siegel [17]) that $K(\Gamma_n(l))$, $l \geq 3$, is generated by ratios of theta constants. Then $K_n$ is given as the invariant subfield $K(\Gamma_n(l))^{\Gamma_n/\pm \Gamma_n(l)}$. However, these methods seem not very effective to get a finite number of generators explicitly. In the case of $K_2$, Igusa determined three generators in his paper [3], [4], where they are written by Eisenstein series, or also by theta constants. In particular, $K_2$ is shown to be purely transcendental. In a previous paper [19], we gave 34 generators of the graded ring of Siegel modular forms of degree three. By this, we are able to find generators of $K_3$ systematically. However, a systematic calculation gives too many (actually thirty three) generators. The purpose of the presence paper is to give seven generators of $K_3$ explicitly, which are ratios of modular forms of weight at most 30.

The quotient space $H_3/\Gamma_3$ is naturally equipped with the structure of the moduli variety over $\mathbb{Q}$, of three-dimensional principally polarized Abelian
varieties. It is still an open problem if the number of generators of $K_3$ can be reduced one more, to six, which amounts to the rationality problem of $H_3/\Gamma_3$ since $K_3$ is the rational function field of the variety $H_3/\Gamma_3$. The moduli variety of curves of genus three is regarded as an open subvariety of $H_3/\Gamma_3$ by means of the Torelli map. Using the moduli theory of curves, Riemann [11], Weber [20], Frobenius [2] studied $K(\Gamma_3(2))$. They showed the rationality of the variety $H_3/\Gamma_3(2)$, and moreover gave six generators of $K(\Gamma_3(2))$ explicitly written in terms of derivatives of odd theta functions at the origin.

Prof. R. Sasaki has given a nice mimeograph [12] surveying this topic. So $H_3/\Gamma_3$ is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3: \Gamma_3(2)] = 1451520$, in other words, $K_3$ has a Galois extension of degree 1451520 which is purely transcendental. Also by the moduli theory of curves, $H_3/\Gamma_3$ is proved to be even stably rational (Kollár and Schreyer [6], see also Bogomolov and Katsylo [1]).

In some cases, generators of $K_n$ work as the absolute invariant of the elliptic modular case. More precisely by Shimura [13], [14] it is shown that if a principally polarized Abelian variety $A$ is with sufficiently many complex multiplication, under a certain condition, or generic of odd dimension (our case), then $A$ has a model over the field generated over $\mathbb{Q}$ by their special values attached to $A$ (see also [15], Theorem 9.5, Corollary 9.6). The author hopes that the result of the present paper will be of use for study of the rationality problem of $H_3/\Gamma_3$, or for that of arithmetic properties of three-dimensional Abelian varieties.

1. Notation and preliminary

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote as usual the ring of integers, the rational number field, the complex number field respectively. Let $A = \bigoplus A_k$, $B = \bigoplus B_k$ be graded $\mathbb{C}$-algebras. Then the tensor product $A \otimes B$ denotes a graded $\mathbb{C}$-algebra $\bigoplus \otimes A_k \otimes B_k$. For an integral graded algebra $A$, $F_0(A)$ denotes the field formed by elements of degree 0 in the field of fractions of $A$. We denote by $M_{k,l}(\ast)$, the set of $k \times l$ matrices with entries in $\ast$, and by $M_k(\ast)$, the set of square matrices of size $k$.

Let $H_n$ denote the Siegel space of degree $n \{Z \in M_n(\mathbb{C}) | \text{Im } Z > 0\}$, and let $\Gamma_n$ denote the modular group $Sp_{2n}(\mathbb{Z})$. $\Gamma_n$ acts on $H_n$ by the usual modular substitution

$$Z \rightarrow MZ = (AZ + B)(CZ + D)^{-1}, \quad M = \frac{A_B}{C_D} \in \Gamma_n.$$ 

$\Gamma_n(l)$ denotes the principal congruence subgroup of level $l \{M \in \Gamma_n | M \equiv 1_{2n} \mod l\}$, $1_{2n}$ being the identity matrix of size $2n$. For a congruence subgroup
A holomorphic function $f$ on $H_n$ is called a (Siegel) modular form for $\Gamma$ of weight $k$ if $f$ satisfies

$$f(MZ) = |CZ + D|^k f(Z) \quad \text{for} \quad M \in \Gamma$$

and if $f$ is holomorphic also at cusps which is automatic when $n > 1$. In the present paper, weight $k$ of a modular form is always supposed to be even. $A(\Gamma)_k$ denotes the vector space of modular forms of weight $k$, and $A(\Gamma) = \bigoplus A(\Gamma)_k$, the graded ring of modular forms. For $f \in A(\Gamma)_k$, and for $M \in \Gamma$, we define $(Mf)(Z)$ to be $|CZ + D|^{-k} f(MZ)$.

Let $m = (m''_m) \in M_{2,n}(\mathbb{Z})$. We define a theta function with a theta characteristic $m$ by setting

$$\theta[m](Z, x) = \sum_{g \in \mathbb{Z}^n} e\left(\frac{1}{2}(g + \frac{1}{2}m')Z'(g + \frac{1}{2}m') + (g + \frac{1}{2}m')'(x + \frac{1}{2}m'')\right)$$

where $x = (x_1, \ldots, x_n)$ is a variable on $\mathbb{C}^n$, and $e(\ ) = \exp (2\pi i (\ ) )$. $m$ is called even or odd according as $e(\frac{1}{2}m'm'')$ equals 1 or $-1$. We put $\theta[m](Z) = \theta[m](Z, 0)$, which is called a theta constant and which is not identically zero if and only if $m$ is even. $\theta[m](Z)$ has the integral Fourier coefficients. If $m$ is odd, then $(1/2\pi i) \partial/(\partial x_i) \theta[m](Z, 0)$ does not vanish identically and has the integral Fourier coefficients.

Let $\xi_0, \ldots, \xi_{r-1}$ be variables, and let $h$ be a homogeneous polynomial in $\xi_0, \ldots, \xi_{r-1}$, of degree $k$ in $\xi_0$, and of degree $s$ in each of $\xi_1, \ldots, \xi_{r-1}$ such that the identity

$$h\left(\ldots, \frac{a\xi_i + b}{c\xi_i + d}, \ldots\right) = (c\xi_0 + d)^{-k} \prod_{i=1}^{r-1} (c\xi_i + d)^{-s} h(\ldots, \xi_i, \ldots)$$

is satisfied for $(a_{cd}) \in SL_2(\mathbb{C})$. Let $S(r)$ denote the $\mathbb{C}$-algebra of such $h$ with $k = s$. $S(r)$ becomes a graded $\mathbb{C}$-algebra in terms of $s$. $S(2, r)$ is defined to be a subring of $S(r)$ composed of $h$ which is symmetric in $\xi_0, \ldots, \xi_{r-1}$, namely $S(2, r)$ is the invariant subring $S(r)^{\mathfrak{S}_r}$, where the symmetric group $\mathfrak{S}_r$ acts naturally on $\xi_0, \ldots, \xi_{r-1}$ as permutations. $S(2, r)$ is nothing else but the graded ring of invariants of a binary $r$-form (cf. Tsuyumine [19], Sect. 1), and its homogeneous element is called a (projective) invariant.

An element $h$ satisfying (1) is called a $(k, s)$-covariant if $h$ is symmetric in $\xi_1, \ldots, \xi_{r-1}$. The ring of $(s, s)$-covariants ($s \geq 0$) is equal to $S(r)^{\mathfrak{S}_{r-1}}$ where $\mathfrak{S}_{r-1}$ acts on $\xi_1, \ldots, \xi_{r-1}$ as permutations. We have inclusions of rings; $S(2, r) \subset S(r)^{\mathfrak{S}_{r-1}} \subset S(r)$. 


2. Modular forms of degree three

Let us recall some structures of the graded ring $A(\Gamma_3)$ of modular forms of degree three. The details are found in Tsuyumine [19]. For simplicity we write $A$ for $A(\Gamma_3)$ in what follows.

We decompose $Z \in H_3$ into

$$Z = \begin{pmatrix} Z_1 \\ \tau \\ z_3 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} z_1 \\ z_{12} \\ z_2 \end{pmatrix} \in H_2, \quad z_3 \in H_1, \quad \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathbb{C}^2.$$

$R$ denotes the subset of $H_3$ given by $\tau = 0$. A point of $H_3$ equivalent to some point in $R$ is called reducible, and the set of images of such points by the canonical projection of $H_3$ to $H_3/\Gamma_3$ is its algebraic subset, and called the reducible locus. Let $V \subset H_3$ denote the irreducible component of zeros of a theta constant $\theta_{[1110]}$ which contains $R$. A modular form $f \in A$, we define $v(f)$ to be the vanishing order of $f|_V$ at $R$ ($v(f) = \infty$ if $f|_V \equiv 0$). $v(f)$ is called the order of $f$. If $f|_V \not\equiv 0$, then $v(f)$ is a non-negative even integer since $f$ is of even weight, namely $f$ is invariant by changing $\tau$ for $-\tau$. For even $v \geq 0$, we define $A(v)$ to be a graded ideal generated by modular forms $f$ with $v(f) \geq v$. We have a sequence of inclusions $A = A(0) \supset A(2) \supset A(4) \supset \cdots$. Let

$$\chi_{18}(Z) = \prod_{m: \text{even}} \theta[m](Z).$$

Then $\chi_{18}$ is a modular form of weight 18, and it is a prime element of the ring $A$ (Igusa [5]). If $f \in A$ vanishes identically on $V$, then $f$ is divisible by $\chi_{18}$, i.e., $f/\chi_{18}$ is an element of $A$. $\chi_{18}$ is involved in every $A(v)$. Let us put

$$\bar{A}(v) = A(v)/A(v + 2).$$

$\bar{A}(0)$ is a graded $\mathbb{C}$-algebra and $\bar{A}(v)$'s can be regarded as $\bar{A}(0)$-modules. We have an isomorphism

$$A/(\chi_{18}) \simeq \bar{A}(0) \oplus \bar{A}(2) \oplus \cdots$$

(2)

of vector spaces, or more strongly, of (infinite) graded modules over some ring of Krull dimension five. If $f$ is a modular form of weight $k$ with $v(f) > \frac{3}{2}k$, then $f$ vanishes identically on $V$ ([19], Cor. 2 to Prop. 7) and hence $f$ is divisible by $\chi_{18}$. So the vector space $(A/(\chi_{18}))_k$ corresponding to modular forms of weight $k$ is isomorphic to the direct sum $\bar{A}(0)_k \oplus \bar{A}(2)_k \oplus \cdots \oplus A([\frac{3}{2}k]')_k$, $[\frac{3}{2}k]'$ denoting the maximal even integer not exceeding $\frac{3}{2}k$. To know the structure of $\bar{A}(v)$, we exhibit them as
subspaces of $A(\Gamma'_2) \otimes A(\Gamma_1)$ in the following way where $\Gamma'_2$ is the maximal congruence subgroup of $\Gamma_2$ which stabilizes an odd theta characteristic $(\frac{1}{2})$ mod 2.

Suppose that $g$ is a meromorphic modular form, but holomorphic on $V - \Gamma_3R$, $\Gamma_3R$ being the union $\cup M \cdot R$, $M \in \Gamma_3$, and that $g|_{V - \Gamma_3R}$ is locally bounded at $R$, hence at $\Gamma_3R \cap V$. For such $g$, and for $(Z_1, z_3) \in H_2 \times H_1$ we define

$$(\Psi g)(Z_1, z_3) = \lim_{Z \to Z_0 \atop Z \in V} g(Z), \quad Z_0 = \left( \begin{array}{l} Z_1 \\ 0 \\ z_3 \end{array} \right) \in R.$$  

By Riemann's removable singularity theorem $g|_{V - \Gamma_3R}$ extends to a holomorphic function on $V$, and hence $\Psi g$ is well-defined. $\Psi g$ is an element of the tensor product $A(\Gamma'_2) \otimes A(\Gamma_1)$ ([19], Sect. 14). Let $\chi_{28}$ be a modular form of weight 28 defined in Section 5 of the present paper (or [19], Sect. 22). It is a modular form of lowest weight having the property that $\chi_{28}|_V$ vanishes only at $\Gamma_3R \cap V$. Its order $v(\chi_{28})$ is eight. Now let us fix three modular forms $\beta', \gamma, \delta$ with $\beta' \in A(2) - A(4), \gamma \in A(4) - A(6), \delta \in A(6) - A(8)$. Then if $f \in A$ is of order $v = 0 \mod 8$ (resp. $2, 4, 6 \mod 8$), then

$$f/\chi_{28}^{(v)8} \quad (\text{resp. } f\delta/\chi_{28}^{(v+6)8}, f\gamma/\chi_{28}^{(v+4)8}, f\beta'/\chi_{28}^{(v+2)8})$$

is obviously holomorphic on $V - \Gamma_3R$ and moreover its restriction to $V - \Gamma_3R$ is locally bounded at $R$ ([19], Sect. 13). So its image by $\Psi$ is well-defined. We denote by $\Psi(v)$, the map $f \mapsto \Psi(f/\chi_{28}^{(v)8})$ (resp. $\Psi(f\delta/\chi_{28}^{(v+6)8})$, $\Psi(f\gamma/\chi_{28}^{(v+4)8})$, $\Psi(f\beta'/\chi_{28}^{(v+2)8})$), where we shall write simply $\Psi$ instead of $\Psi(0)$. (In [19], we have taken as $\beta', \gamma, \delta$, some particular modular forms.) $\Psi(v)$ is a map of $A(v)$ to $A(\Gamma'_2) \otimes A(\Gamma_1)$, and by definition the kernel of $\Psi(v)$ is just $A(v + 2)$. So $\Psi(v)$ is also considered to be an embedding of $\tilde{A}(v)$ to $A(\Gamma'_2) \otimes A(\Gamma_1)$. By definition $\Psi(v)(Z_1, z_3) = f(z_1, 0, z_3)$, hence $\tilde{A}(0)$ is contained in $A(\Gamma_2) \otimes A(\Gamma_1)$. If we identify $\tilde{A}(0)$ with $\Psi \tilde{A}(0)$, then the map $\Psi(v)$ of $\tilde{A}(v)$ to $A(\Gamma'_2) \otimes A(\Gamma_1)$ can be regarded as an $A(0)$-module homomorphism since $\Psi(v)(fg) = \Psi f \cdot \Psi(v)g$ for $f \in A, g \in A(\tilde{V})$. $\tilde{A}(0) \subset A(\Gamma_2) \otimes A(\Gamma_1)$ is equal to $\{ \Sigma \psi \otimes j \in A(\Gamma_2) \otimes A(\Gamma_1) | \Sigma \psi(z_1, 0, z_3) j(z_3) \}$ is symmetric in $z_1, z_2, z_3 \}$ ([19], Sect. 16), over which $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite as a module, hence $A(\Gamma'_2) \otimes A(\Gamma_1)$ is. Since $\chi_{28} A(v) \subset A(v + 8)$, we have sequences of inclusions of $A(0)$-submodules of $A(\Gamma'_2) \otimes A(\Gamma_1)$ by definition of $\Psi(v)$;

$$\Psi \tilde{A}(0) \subset \Psi(8) \tilde{A}(8) \subset \cdots$$

$$\Psi(2) \tilde{A}(2) \subset \Psi(10) \tilde{A}(10) \subset \cdots$$
\[ \Psi(4) \bar{A}(4) \subset \Psi(12) \bar{A}(12) \subset \cdots \]
\[ \Psi(6) \bar{A}(6) \subset \Psi(14) \bar{A}(14) \subset \cdots . \]

Since \( A(\Gamma') \otimes A(\Gamma_2) \) is a Noetherian \( \bar{A}(0) \)-module, there is a positive even integer \( v_* \) such that if \( v \geq v_* \), then \( \Psi(v) \bar{A}(v) = \Psi(v - 8) \bar{A}(v - 8) \), in other words
\[ \bar{A}(v) = \chi_{28} \bar{A}(v - 8) \quad \text{for} \quad v \geq v_* . \]

Then it is not difficult to see that any modular form \( f \in A(v) \), \( v \geq v_* \), is written as \( f = g \chi_{28} + h \chi_{18} \) for some \( g, h \in A \), combining (3) with the fact that \( f \) is divisible by \( \chi_{18} \) if \( v(f) > \frac{v}{2} \) weight \( f \). \( v_* \) is actually taken to be 14, and hence the isomorphism (2) becomes
\[ A/(\chi_{18}) \cong \bar{A}(0) \oplus \bar{A}(2) \oplus \bar{A}(4) \]
\[ \oplus \left( \bigoplus_{n=0}^{\infty} \left( \bar{A}(6) \oplus \bar{A}(8) \oplus \bar{A}(10) \oplus \bar{A}(12) \right) \chi_{28}^{n} \right) . \]

All the structures of \( \bar{A}(v) \), \( v \leq 12 \), have been determined in [19], and from this the structure of \( A/(\chi_{18}) \) is given, and that of \( A \) is too.

Finally in this section we give a comment on an alternate definition of \( \Psi(2) \). Restricting to \( V \), the Taylor expansion of \( \theta^{[111]}(Z) \) at \( Z_0 = (z_1, 0) \in R \) in terms of \( \tau \), we get
\[ 0 = \sum_{j=1}^{2} \left( \frac{\partial}{\partial x_j} \theta^{[111]}(Z_1, 0) (\theta^{[0]} \theta^{[0]} \theta^{[1]})(z_3) \right) \tau_i + \text{(higher degree terms of } \tau) . \]

At least one of \( \frac{\partial}{\partial x_j} \theta^{[111]}(Z_1, 0) \) is not zero since the theta divisor of degree two is nonsingular, and \( \theta^{[0]} \theta^{[0]} \theta^{[1]} \) vanishes nowhere on \( H_1 \). Hence one of the \( \tau_i \) is written as an analytic function of another on some neighborhood at \( Z_0 \). Let \( f \in A(2) \). Substituting it in the expansion of \( (f \delta/\chi_{28}) \) in terms of \( \tau \), and taking the limit as \( \tau_i \to 0 \), we get
\[ (\Psi(2)f)(Z_1, z_3) = (F_2F_6/F_8)(Z_1, z_3) . \]
where

\[
\begin{align*}
F_2(Z_1, z_3) &= \frac{1}{2! (2\pi)^4 (\sqrt{-1})^2} \sum_{l=0}^{2} (-1)^l \binom{2}{l} \frac{\partial}{\partial \tau_1^l} \frac{\partial}{\partial \tau_2^l} f(Z_0) \\
&\quad \times \left( \frac{\partial}{\partial x_1} \theta[10^l](Z_1, 0) \right)^{2-l} \left( \frac{\partial}{\partial x_2} \theta[10^l](Z_1, 0) \right)^l,
\end{align*}
\]

\[
\begin{align*}
F_6(Z_1, z_3) &= \frac{1}{6! (2\pi)^{12} (\sqrt{-1})^6} \sum_{l=0}^{6} (-1)^l \binom{6}{l} \frac{\partial}{\partial \tau_1^l} \frac{\partial}{\partial \tau_2^{6-l}} \delta(Z_0) \\
&\quad \times \left( \frac{\partial}{\partial x_1} \theta[10^l](Z_1, 0) \right)^{6-l} \left( \frac{\partial}{\partial x_2} \theta[10^l](Z_1, 0) \right)^l,
\end{align*}
\]

\[
\begin{align*}
F_8(Z_1, z_3) &= \frac{1}{8! (2\pi)^{16} (\sqrt{-1})^8} \sum_{l=0}^{8} (-1)^l \binom{8}{l} \frac{\partial}{\partial \tau_1^l} \frac{\partial}{\partial \tau_2^{8-l}} \chi_{28}(Z_0) \\
&\quad \times \left( \frac{\partial}{\partial x_1} \theta[10^l](Z_1, 0) \right)^{8-l} \left( \frac{\partial}{\partial x_2} \theta[10^l](Z_1, 0) \right)^l,
\end{align*}
\]

\(\binom{\cdot}{\cdot}\) denoting a binomial coefficient. \(\Psi(2)f\) is holomorphic and has a Fourier expansion on \(H_2 \times H_1\), and each of \(F_2, F_6, F_8\) has too. By definition \(\chi_{28}\) has integral Fourier coefficients. Now let us suppose that \(\delta\) has rational Fourier coefficients (with a bounded denominator). Then both of \(F_6, F_8\) have rational Fourier coefficients (with a bounded denominator). Hence there is a rational number \(N\) such that \(N\Psi(2)f\) has integral Fourier coefficients if and only if \(F_2\) does. In particular, for such \(N\), \(2N\Psi(2)f\) has the integral Fourier coefficients if \(f\) does.

Let us calculate a first term of \(F_2\) explicitly in terms of the Fourier coefficient of \(f \in A(2)\) for the identity matrix, i.e. for \(e(\text{tr}(Z))\). There are 23 positive symmetric semi-integral ternary matrices with merely one as their diagonal components, each of which is equivalent under the action \(S \to \sigma U \sigma^T, U \in GL_3(\mathbb{Z})\), to one of the following three matrices; the identity matrix; the matrix with 0 as its (1, 2), (1, 3)-components and with 1/2 as its (2, 3)-component; the matrix with 0 as its (1, 2)-component and with 1/2 as its (1, 3), (2, 3)-components. Let \(a_0, a_1, a_2\) be the Fourier coefficients of \(f\) corresponding to the first, second, third matrix respectively. From \(\Psi \beta = 0\), two relations among \(a_0, a_1, a_2\) are derived; \(a_0 + 4a_1 + 4a_2 = a_1 + 6a_2 = 0\), hence \(a_0 : a_1 : a_2 = 20 : -6 : 1\) if \(a_0 \neq 0\). Then a direct calculation shows

\[
F_2(Z_1, z_3) = -\frac{2}{5} a_0 e(\text{tr}( (\pm \frac{5}{4}, \pm \frac{3}{4}) Z_1 )) e(z_3) + \cdots.
\]
3. A subring of $A(\Gamma_3)$

$\Gamma_2/\Gamma_2(2)$ is isomorphic to the symmetric group $\mathfrak{S}_6$ of degree six, and it acts on the set of six odd theta characteristics (mod 2) of degree two as permutations. $\Gamma_2$ has been defined to be a stabilizer subgroup of $\Gamma_2$ at an odd theta characteristic $(\frac{11}{10})$, and hence $\Gamma_2/\Gamma_2(2)$ is isomorphic to $\mathfrak{S}_5$.

There is an injective homomorphism $\varphi_2$ of $A(\Gamma_2(2))$ to $S(6) \subset \mathbb{C}[\xi_0, \ldots, \xi_5]$ which is equivalent under $\mathfrak{S}_6$ (Igusa [5], Tsuyumine [19], Sect. 9, 11), where $\varphi_2$ induces an isomorphism between the field of fractions of $A(\Gamma_2(2))$ and that of $S(6)^{(2)}$, $S(6)^{(2)}$ denoting the subring of $S(6)$ consisting of homogeneous elements of even degree. We may assume that $\mathfrak{S}_5 \simeq \Gamma_2/\Gamma_2(2)$ acts on $\{\xi_1, \ldots, \xi_5\}$ as permutations. Hence we have a commutative diagram;

$$
\begin{array}{c}
A(\Gamma_2(2)) \xrightarrow{\varphi_2} S(6) \\
\cup \quad \cup \\
A(\Gamma_2') \longrightarrow S(6)^{\mathfrak{S}_5} \\
\cup \quad \cup \\
A(\Gamma_2) \longrightarrow S(2, 6) = S(6)^{\mathfrak{S}_6}.
\end{array}
$$

In particular, there is no proper intermediate field between $F_0(A(\Gamma_2'))$ and $F_0(A(\Gamma_2))$, and hence $F_0(A(\Gamma_2')) = F_0(A(\Gamma_2)[\psi])$ for any $\psi \in A(\Gamma_2') - A(\Gamma_2)$.

**Lemma 1.** Let $\beta$ be a modular form for $\Gamma_3$ of order $v$ with $v \equiv 2$ or 6 mod 8. Let us fix $z_3 \in H$, so that $\psi(Z_1) := (\Psi(4v)\beta^4)(Z_1, z_3)$ is not identically zero. Then $\psi \notin A(\Gamma_2)$. In particular, $F_0((A(\Gamma_2') \otimes A(\Gamma_1))[\Psi(4v)\beta^4]) = F_0(A(\Gamma_2') \otimes A(\Gamma_1))$.

**Proof.** We treat only the case $v \equiv 2$ mod 8, since a similar argument is applicable to the case $v \equiv 6$ mod 8. By the argument [19], Sect. 14, the proof of Lemma 12, $\varphi_2\phi$ is the form $H^4\mathcal{D}_0$ where $H$ is an $(s + 2, s)$-covariant and $\mathcal{D}_0$ denotes the $(0, 8)$-covariant $\Pi_{1 \leq i < j \leq 5}(\xi_i - \xi_j)^2$. It is enough to show that $H^4\mathcal{D}_0 \notin S(2, 6)$. Suppose otherwise. Dividing $H^4$ by a power of the discriminant $\Pi_{0 \leq i < j \leq 5}(\xi_i - \xi_j)^2 \in S(2, 6)$ if necessary, we may assume that $H$ is not divisible by $\Pi_{0 \leq i < j \leq 5}(\xi_i - \xi_j)^2$ (1 \leq i < j \leq 5) and since $H^4\mathcal{D}_0$ is symmetric in $\xi_0, \ldots, \xi_5$ by our assumption, it has a factor $\Pi_{i=1}^5(\xi_0 - \xi_i)^2$. Then $H$ is divisible by $\Pi_{i=1}^5(\xi_0 - \xi_i)$, and hence $H^4\mathcal{D}_0$, by $\Pi_{i=1}^5(\xi_0 - \xi_i)^4 \times \Pi_{1 \leq i < j \leq 5}(\xi_i - \xi_j)^2$. Again by symmetry $H^4\mathcal{D}_0/\Pi_{i=0}^5(\xi_0 - \xi_i)^4 \times \Pi_{1 \leq i < j \leq 5}(\xi_i - \xi_j)^2$ is still divisible by...
Let $\Lambda$ be a graded subring of $A$ such that $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite integral over $\bar{\Lambda} := \Psi \Lambda$, and that $\chi_{28}, \chi_{18} \in \Lambda$.

**Lemma 2.** $A$ is finite integral over $\Lambda$.

**Proof.** $\Psi(v)A(v)$ is a finite $\bar{\Lambda}$-module for every even $v \geq 0$. Let $\{f_{i,v}\}_i$ be a finite number of modular forms in $A(v)$ such that $\{\Psi(v)f_{i,v}\}_i$ generates $\Psi(v)A(v)$ over $\bar{\Lambda}$. We show that $A$ is generated as a $\Lambda$-module, by $f_{i,v}$'s with $v \leq v_0$, $v_0$ being as in (3).

We prove that any modular form $f$ of weight $k$ is written as a linear combination of $f_{i,v}$'s ($v \leq v_0$) over $\Lambda$, by induction on $k$. $\Psi f \in \Psi A(0)$ is written as $\Psi f = \sum_i \Psi(P_i f_{i,0})$ with $P_i \in \Lambda$. By taking $f = \sum P_i f_{i,0}$ instead of $f$, we may assume $\Psi f = 0$, namely $v(f) \geq 2$. Then $\Psi(2)f$ is written as $\sum_i \Psi(2)(P'_i f_{i,2})$ with $P'_i \in \Lambda$. By a similar argument as above, may assume $\Psi(2)f = 0$, and by a recursive argument, we may assume $v(f) > \frac{k}{2}$, where we make use of such elements as $\chi_{28}^m f_{i,v} (m > 0)$ instead of $f_{i,v}$, if the order $v(f)$ exceeds $v_0$. Then $f|_v$ vanishes identically and $f$ is written as $f = g \chi_{18}$ for some $g \in A$. By the induction hypothesis $g$ is a linear combination of $f_{i,v}$'s ($v \leq v_0$) over $\Lambda$, and hence $f$ is.

Q.E.D.

**Corollary.** $A(v)$ is a finite $\Lambda$-module for any even $v \geq 0$.

**Proposition 1.** Let $\Lambda$ be a graded subring of $A$ containing $\chi_{28}, \chi_{18}$ such that $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite integral over $\bar{\Lambda} := \Psi \Lambda$, and that $\text{g.c.d} \{k | \bar{\Lambda}_k \neq \{0\}\} = 2$ for $\bar{\Lambda} = \oplus \bar{\Lambda}_k$. If $\beta$ is a modular form of order two such that $F_0(\bar{\Lambda}[\Psi(8)\beta^4]) = F_0(\Lambda(\Gamma_2) \otimes A(\Gamma_1))$, then the modular function field of degree three is given by $F_0(\Lambda[\beta])$.

**Proof.** At first we show that there are a positive integer $v_1$ and a modular form $P \in \Lambda$ of order 0 such that

$$\Psi(v + v')(\beta^{v'/2}PA(v)) \subset \Psi(v + v')(\beta^{v'/2}(\Lambda[\beta] \cap A(v))$$  (5)

for any even $v \geq v_1$ where $v' \in \{0, 2, 4, 6\}$ is determined by $v + v' \equiv 0 \mod 8$. By our assumption, we can take $\bar{P} \in \bar{\Lambda}$, $\neq 0$ such that $\bar{P}(\Lambda(\Gamma_2) \otimes A(\Gamma_1))$ is contained in a $\bar{\Lambda}$-module generated by $\Psi(8)\beta^4, (\Psi(8)\beta^4)^2, \ldots, (\Psi(8)\beta^4)^m$ with $m = [F_0(\Lambda(\Gamma_2) \otimes A(\Gamma_1)): F_0(\Lambda)]$. Since $\Lambda[\beta] \cap A(v)$ has as a subset

$$\sum_{2n_1 + 8n_2 \geq v} \beta^{n_1} \chi_{28}^{n_2} \Lambda,$$
92 \quad S. Tsuyumine

\[ \Psi(v + v')(\beta^{v/2}(\Lambda[\beta] \cap A(v))) \text{ contains the } \bar{\Lambda}\text{-module generated by } \Psi(8)^{\beta}, \ldots, (\Psi(8)^{\beta})^m \text{ if } v \text{ is large enough. If } P \in \Lambda \text{ is such that } \bar{P} = \Psi P, \text{ then } \Psi(v + v')(\beta^{v/2} PA(v)) = \bar{P}\Psi(v + v')(\beta^{v/2} A(v)) = \bar{P}(A(\Gamma_2') \otimes A(\Gamma_1)). \]

Thus we have proved (5).

\[ A(2) \text{ is the prime ideal of } A \text{ defining the reducible locus of } H_3/\bar{\Lambda}, \text{ and hence } A(2) \cap \Lambda[\beta] \text{ is prime in } \Lambda[\beta]. \]

Let us take the ring \( \Lambda_0 := \Lambda[\beta, \chi_{18}/\chi_{28}^k (k = 0, 1, 2, \ldots)]. \) The ideal of \( \Lambda_0 \) generated by \( A(2) \cap \Lambda[\beta], \chi_{18}/\chi_{28}^k (k = 0, 1, 2, \ldots) \) is prime since \( \bar{\Lambda} = \Lambda_0/(A(2) \cap \Lambda[\beta], \chi_{18}/\chi_{28}^k (k = 0, 1, 2, \ldots)) \) is an integral domain. Let \( \Lambda_0 \) be the localization of \( \Lambda_0 \) at the prime ideal. Let \( v_2 \) be an even integer equal to or greater than each of \( v_0 \) and \( v_1, \) \( v_0 \) being as in (3). Since \( \Lambda \subset \Lambda_0, \) by Corollary to Lemma 2 there are a finite number of holomorphic modular forms \( f_1, \ldots, f_t \in A(v_2) \) such that \( A(v_2) \subset \Lambda_0 f_1 + \cdots + \Lambda_0 f_t. \) We may assume that \( \{f_1, \ldots, f_t\} \) is a minimal system with this property. Then we show \( t = 1. \) Suppose \( t \geq 2. \) Since \( v := v(f_t) \) is larger than \( v_1, \) we have \( \Psi(v + v')\beta^{v'/2} Pf_t = \Psi(v + v')\beta^{v'/2} q \) for some \( q \in A(v) \cap \Lambda[\beta]. \)

Since \( \Psi(v + v')\beta^{v'/2} (Pf_t - q) = 0, \) the order of \( Pf_t - q \) is at least \( v + 2. \) By repeating the similar argument four times, it is shown that there is \( Q \in \Lambda[\beta] \) satisfying the inequality \( v(P^4f_t - Q) \geq v + 8. \) Since \( v \geq v_0, \) by (3) there are \( g, h \) such that \( P^4f_t - Q = g\chi_{28} + h\chi_{18}, \) \( g(v) \) is obviously greater than or equal to \( v, \) and in particular \( g \in A(v_2) \) because \( v \geq v_2. \) \( h\chi_{28}^k \) is also involved in \( A(v_2) \) if \( k \) is sufficiently large. \( g, h\chi_{28}^k \in A(v_2) \subset \Lambda_0 f_1 + \cdots + \Lambda_0 f_t \) is written as \( g = \Sigma_{i=1}^t a_i f_i, \) \( h\chi_{28}^k = \Sigma_{i=1}^t b_i f_i \) with \( a_i, b_i \in \Lambda_0. \) Hence we have

\[
(P^4 - a_i\chi_{28} - b_i\chi_{18}/\chi_{28}^k)f_i = Q + \sum_{i=1}^{t-1} a_i\chi_{28}f_i + \sum_{i=1}^{t-1} b_i(\chi_{18}/\chi_{28}^k)f_i.
\]

Since \( P \) is of order 0, \( P^4 - a_i\chi_{28} - b_i\chi_{18}/\chi_{28}^k \) is a unit of the ring \( \Lambda_0. \) So \( f_i \) is written as a linear combination of other \( f_j. \) This contradicts to the minimality of a system of \( \{f_1, \ldots, f_t\}. \) Thus \( t = 1. \)

Now we have \( A(v_2) \subset \Lambda_0 f_1. \) \( A(v_2) \) and \( \Lambda_0 \) have a common non-trivial element (e.g., \( \chi_{18}). \) This implies that \( f_1 \) is contained in the field of fractions of \( \Lambda_0, \) and that \( A(v_2) \) is a subset of the field of fractions of \( \Lambda[\beta]. \) Since \( \chi_{28}^k A \subset A(v_2) \) for large \( k, \) the modular function field \( F_0(A) \) is equal to \( F_0(\Lambda[\beta]). \) Q.E.D.

Combining Proposition 1 with Lemma 1, we have the following corollary.

**Corollary.** Let \( \Lambda \) be a ring as in Proposition 1 satisfying the additional condition that \( F_0(\bar{\Lambda}) = F_0(A(\Gamma_2) \otimes A(\Gamma_1)). \) Let \( \beta \) be any modular form with \( v(\beta) = 2. \) Then the modular function field is given by \( F_0(\Lambda[\beta]). \)
4. Main theorem

$A(\Gamma_1)$ is generated by two algebraically independent modular forms $j_4, j_6$ of weight 4, 6 respectively where

$$j_4 = \frac{1}{2} \sum_{m: \text{even}} \theta[m]^8, \quad j_6 = \sum_{M: \Gamma_1/\Gamma_1(2)} M(\theta[^0_0]^8 \theta[^{10}_0]^6 \theta[^{11}_0]^4).$$

As Igusa [3], [4] showed, $A(\Gamma_2)$ is generated by four algebraically independent modular forms $\psi_4, \psi_6, \psi_{10}, \psi_{12}$ with their subscript as their weight where

$$\psi_4 = \frac{1}{4} \sum_{m: \text{even}} \theta[m]^8, \quad \psi_6 = \frac{1}{2} \sum_{M: \Gamma_2/\Gamma_2(2)} M(\theta[^{00}_0]^8 \theta[^{08}_0]^6 \theta[^{10}_0]^4 \theta[^{11}_0]^4),$$

$$\psi_{10} = \prod_{m: \text{even}} \theta[m]^2,$$

$$\psi_{12} = \frac{1}{288} \sum_{M: \Gamma_2/\Gamma_2(2)} M(\theta[^{00}_0]^4 \theta[^{06}_0]^2 \theta[^{10}_0]^4 \theta[^{10}_1]^4 \theta[^{11}_1]^4)^4$$

(note that we are considering only modular forms of even weight). Let $\alpha_4, \alpha_6, \alpha_{10}, \alpha_{12}, \alpha_{20}, \alpha_{30} \in A$ be as in Section 5, and let $\alpha_{20}' = (\alpha_{20} - 5\alpha_{10}^2)/7, \alpha_{30}' = (7\alpha_{30} - 313\alpha_{10}\alpha_{20} + 865\alpha_{10}^3)/7.$ By [19], Section 23, we have

$$\Psi \alpha_4 = \psi_4 \otimes j_4, \quad \Psi \alpha_6 = \psi_6 \otimes j_6, \quad \Psi \alpha_{12} = 3^{-3} \psi_{12} \otimes (-j_6^2 + 4j_4^3),$$

$$\Psi \alpha_{12}' = 2^4 3^{-3} \psi_4^3 \otimes (-j_6^2 + 4j_4^3)$$

$$- 3^{-3} \psi_6^3 \otimes (-j_6^2 + 4j_4^3) + 3^2 \psi_{12} \otimes (j_6^2 + 8j_4^3),$$

$$\Psi \alpha_{20}' = \psi_{10}^3 \otimes j_4^5, \quad \Psi \alpha_{30}' = \psi_{10}^3 \otimes j_6^5.$$

**Lemma 3.** Let $\overline{\Lambda}$ denote a graded $\mathbb{C}$-algebra generated by $\Psi$-images of i) $\alpha_4, \alpha_6, \alpha_{12}, \alpha_{20}, \alpha_{30}$, or ii) $\alpha_4, \alpha_6, \alpha_{12}, \alpha_{12}', \alpha_{20}', \alpha_{30}$, $k$ being any fixed positive integer. Then $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite integral over $\overline{\Lambda}$, and $F_0(\overline{\Lambda})$ equals $F_0(A(\Gamma_2) \otimes A(\Gamma_1)).$

**Proof.** The first assertion follows from the fact that $\Psi \alpha_4, \Psi \alpha_6, \Psi \alpha_{12}, \Psi \alpha_{12}', \Psi \alpha_{20}, \Psi \alpha_{30}$ do not vanish simultaneously at any point of the projective variety $(H_2/\Gamma_2)^* \times (H_1/\Gamma_1)^*, (H_n/\Gamma_n)^*$ denoting the Satake compactification,
which is not difficult to see. We treat only the case ii), because the similar argument is applicable to the case i). Put \( s = (j^3_5/j^3_3)(z_3) \). Then \( s \) is an element of degree five over \( \mathbb{C}[\omega_{30}/\omega_{230}] \). As easily seen, \( F_0(A(\Gamma_2) \otimes A(\Gamma_1)) \) is an extension over \( F_0(\mathbb{C}[\omega_4, \omega_6, \omega_{12}, \omega_{20}, \omega_{30}]) \) of degree five, since the former is obtained from the latter by adding \( s \). In particular, the extension is simple. Since an element \( \Psi(\omega_{12}/\omega_4) \) is not contained in the latter one, \( F_0(\Lambda) \) equals \( F_0(A(\Gamma_2) \otimes A(\Gamma_1)) \). Q.E.D.

**Theorem 1.** Let \( \lambda \) be any modular form of weight twenty with \( v(\lambda) = 2 \), and let \( c \) be a constant. Let us put \( \Lambda := \mathbb{C}[\omega_4, \omega_6, \omega_{12}, \omega_{20}, \omega_{30}, \omega_{28}, \omega_{18}] \). Then the modular function field of degree three is given by \( F_0(\Lambda) \), except at most one value of \( c \). (See Sect 5 for the definition of modular forms.)

**Remark.** Our argument will show that the assertion of Theorem 1 holds even if we replace \( \Lambda \) by other rings such as \( \mathbb{C}[\omega_4, \omega_6, \omega_{12}, \omega_{20}, \omega_{30}, \omega_{28}, \omega_{18}] \), \( \mathbb{C}[\omega_4, \omega_6, \omega_{12}, \omega_{20}, \omega_{30}, \omega_{28}, \omega_{18}] \), and so on, \( \lambda \) being a modular form of appropriate weight with \( v(\lambda) = 2 \).

**Proof:** Let us find an algebraic relation among \( \Psi(\omega_4, \omega_6, \omega_{12}, \omega_{20}, \omega_{30}) \). Let \( s \) be as in the proof of Lemma 3. If we put

\[
\begin{align*}
p_0 & = 16\omega_4^3, p_1 = -128\omega_4^3 - \omega_6^3 + 243\omega_{12} + 27\omega_{12}, \\
p_2 & = 256\omega_4^3 + 8\omega_6^3 + 1944\omega_{12} - 108\omega_{12}, p_3 = -16\omega_6^3,
\end{align*}
\]

then we have

\[
(\Psi p_0)s^3 + (\Psi p_1)s^2 + (\Psi p_2)s + \Psi p_3 = 0 \tag{6}
\]

by a direct computation. For an indeterminate \( X \), we put

\[
L(X) = p_0^5(\omega_{20} + X)^5 + (p_2^5 + 5p_0p_2^2p_3^2 + 5p_1^5p_3^2 - 5p_0p_1p_3^3 \omega_{30}(\omega_{20} + X)^6 + (p_1^5 + 5p_0^2p_2^2p_3 + 5p_0p_1p_3^2 \omega_{30}(\omega_{20} + X)^5 + p_0^3\omega_{30}^3.
\]

\[\text{Such a detail is not necessary to prove merely Theorem 1. However, it (or } L(X) \text{) will be used for other purposes later.}\]
Then $\Psi L(0)/(\Psi \alpha_{20})^9 = 0$ is a minimal algebraic relation among $\Psi \alpha_4$, $\Psi \alpha_6$, $\Psi \alpha_{12}$, $\Psi \alpha_{12}'$ and $(j_6/j_2)^5 = (s^5)$, given by eliminating $s$ from (6). Hence $\Psi L(0) = 0$, which is an algebraic relation among $\Psi \alpha_4$, \ldots, $\Psi \alpha_{10}$.

By Lemma 3 $\Lambda$ satisfies the condition in Corollary to Proposition 1. $\beta := L(c\lambda)$ is a modular form contain in $\Lambda$, which equals $L(0) + cL'(0)\lambda$ up to $A(4)$ where $L'$ is the derivative of $L$ in terms of $X$. Since $L(0)$, $\lambda \in A(2)$, $\beta$ is a modular form of order at least two. Since $L'(0) \in A(0) - A(2)$, we have, except for at most one value of $c$

$$\Psi(2)L(0) + c\Psi L'(0)\lambda \neq 0,$$

i.e., $\nu(\beta) = 2$. Then by the Corollary to Proposition 1, the modular function field is given by $F_0(\Lambda[\beta]) = F_0(\Lambda)$. Q.E.D.

Let us make $c\lambda$ explicit for which the assertion of Theorem 1 holds. By the above proof it is enough to find $c\lambda$ satisfying (7). From the definition, $\alpha_4$, $2^{-3}\alpha_6$, $2^3 3^2 \alpha_{12}$, $2^4 3^{-1} \alpha_{12}'$, $2^4 3^2 5 \cdot 7 \cdot 11 \alpha_{20}'$, $2^1 13^3 5^2 7^2 11^3 \alpha_{30}'$ are easily checked to have integral Fourier coefficients. By the way, $30\chi_{28}$, $\chi_{18}$ have too. Let $N$ be the rational number given in the last part of Section 2. Since $210^6 3^{24} 5^{16} 7^{16} 11^{24} L(0)$ has integral Fourier coefficients, also $2N$ times its $\Psi(2)$-image does. So (7) holds if $2N \cdot 210^6 3^{24} 5^{16} 7^{16} 11^{24} c\Psi L'(0)\Psi(2)\lambda$ has a non-integral Fourier coefficient.

We take as $\lambda$, $\alpha_6 \beta_{14}$ where $\beta_{14}$ is a cusp form of weight 14 and of order two which is defined in Section 5 (or, also in [19], Sect. 24). $\Psi(2)\alpha_6 \beta_{14}$ equals $\Psi \alpha_6 \Psi(2)\beta_{14}$. Now we must find a rational number $c$ such that $210^6 3^{24} 5^{16} 7^{16} 11^{24} c\Psi L'(0)\Psi \alpha_6 \cdot F_2$ has a non-integral Fourier coefficient, $F_2$ being the one given for $f = \beta_{14}$ in (4), which implies (7). $\alpha_6$ has the Fourier expansion starting from the constant term 8, and a direct calculation shows that $\Psi L'(0)$ has the Fourier expansion starting from

$$-2^{254} 3^7 5^2 \{2e(\text{tr}(Z_1)) - e(\text{tr}(\pm^1/2 \pm^3/12)(Z_1))\}^{16} e(2z_3).$$

Let $a$ be the Fourier coefficient of $\beta_{14}$ for $e(\text{tr}(Z))$. Combining the above calculation with that of the last part of Section 2, $210^6 3^{24} 5^{16} 7^{16} 11^{24} c\Psi L'(0)\Psi \alpha_6 \cdot F_2$ is shown to have $2^{264} 3^{31} 5^{17} 7^{16} 11^{24} ac$ as a Fourier coefficient. Here we give a rough estimate of $a$. $\beta_{14}$ is written as a sum of 2160 products with sign, of 28 theta constants, where each of products has the Fourier expansion starting from the terms corresponding to positive semi-integral ternary matrices with their diagonal components $\geq 1$. From this, $a \in \mathbb{Z}$, and a rough estimate shows $|a| < 2160 \times 2^3 \times 8 = 2^{28} 3^4 5$. On the other hand $a \neq 0$ is shown in the following way. So if $c$ is a rational number such that
$2^{364}3^{31}5^{17}7^{16}11^{24}ac \notin \mathbb{Z}$ for any positive integer $a$ less than $2^{28}3^45$, then (7) holds and hence $F_0(\Lambda)$ gives the modular function field of degree three, where $\Lambda = \mathbb{C}[\alpha_4, \alpha_6, \alpha_{12}, \alpha'_{12}, \alpha'_{20} + c\alpha_6\beta_{14}, \alpha_{30}, \chi_{28}, \chi_{18}]$. Since the generators have the rational Fourier coefficients, their ratios of weight zero generate the modular function field $K_3$ over $\mathbb{Q}$.

Let us prove $a \neq 0$. Let $E_{k,n}$ denote the normalized Eisenstein series of degree $n$ and of weight $k$, where 'normalized' implies that its constant term is one. By the structure theorem of $A(\Gamma_2)$ (Igusa [3], [4], [5]) and by the formulas for the Fourier coefficients of Eisenstein series of degree two in Maass [7], Satz 1, the identity $3^7 \cdot 11 \cdot 659E_{4,2}E_{6,2} - 2^2269 \cdot 43867E_{4,2}E_{10,2} + 53 \cdot 657931E_{14,2} = 0$ follows. Hence

$$3^7 \cdot 11 \cdot 659E_{4,3}E_{6,3} - 2^2269 \cdot 43867E_{4,3}E_{10,3} + 53 \cdot 657931E_{14,3} \quad (8)$$

is a cusp form of weight fourteen where $E_{4,3}$ is well-defined by Raghavan [10]. By virtue of Ozeki and Washio [8], [9], the Fourier coefficient of (8) for $e(\text{tr}(Z))$ can be calculated, namely $-2^73^85^27^211 \cdot 79973$. By [19], the vector space of cusp forms of weight 14 is one-dimensional, and hence (8) and $\beta_{14}$ are proportional. Thus $a \neq 0$. We have proved the following theorem.

**Theorem 2.** The Siegel modular function field $K_3$ of degree three over $\mathbb{Q}$ is generated by the following seven modular functions: $\alpha_6/\alpha_4^3$, $\alpha_{12}/\alpha_4^3$, $\alpha'_{12}/\alpha_4^3$, $\alpha_{20} - 5\alpha_{10} + 7c\alpha_6\beta_{14}/\alpha_4^3$, $\alpha_{30} - 313\alpha_{20}\alpha_{10} + 865\alpha_{10}^2)/\alpha_4^3\alpha_6$, $\chi_{28}/\alpha_4^2$, $\chi_{18}/\alpha_4^2\alpha_6$ where $c$ is any rational number exclusive of at most one value. If $c$ is such that $2^{364}3^{31}5^{17}7^{16}11^{24}ac \notin \mathbb{Q} - \mathbb{Z}$ for any positive integer $a$ less than $2^{28}3^45$, then our assertion holds. (see Sect. 5 for the definition of modular forms).

**Remark**

i) In Theorem 2, we may replace $\alpha_{12}/\alpha_4^3$ or $\alpha'_{12}/\alpha_4^3$ by its power for general $c \in \mathbb{Q}$. This implies for example, that $K_3$ is not a cyclic extension of $\mathbb{Q}(\alpha_6/\alpha_4^3, \alpha_{12}/\alpha_4^3, (\alpha_{20} - 5\alpha_{10} + 7c\alpha_6\beta_{14})/\alpha_4^3, (\alpha_{30} - 313\alpha_{20}\alpha_{10} + 865\alpha_{10}^2)/(\alpha_4^2\alpha_6, \chi_{28}/\alpha_4^2, \chi_{18}/\alpha_4^2\alpha_6)$ unless the extension is trivial.

ii) In Theorem 2 we can replace $\beta_{14}$ by the cusp form (8). Then $c$ is taken to be a rational number such that $2^{371}3^{39}5^{19}7^{18}11^{25}79973^c \notin \mathbb{Z}$, e.g., $c = 1/13$.

**5. Modular forms**

We give definition of modular forms $\alpha_4, \alpha_6, \alpha_{10}, \alpha_{12}, \alpha'_{12}, \alpha_{20}, \alpha_{30}, \beta_{14}, \chi_{28}$ with their subscripts as their weight. We denote by $E_k$, the Eisenstein series of degree three and of weight $k$. 
i) \( \alpha_4 = 2^{-3} \sum_m \theta[m]^8 \), \( m \) running over the set of all even theta characteristics (mod 2). \( \alpha_4 \) is equal to the Eisenstein series \( E_4 \).

ii) \( \alpha_6 = 2^{-6} 3^{-1} 7^{-1} \sum_{M: \Gamma_3/\Gamma_2} M(\theta[000]^4 \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000]) \), \( \theta[000] \), which is equal to \( 8E_6 \).

iii) \( \alpha_{10} = -2^{-4} 3^{-2} -11^{-1} \sum_{M: \Gamma_3/\Gamma_2} M(\theta[110000] \theta[110001] \theta[010000] \theta[010001] \theta[010100] \theta[010101] \theta[110001] \theta[110111] \theta[111110]) \), which is proportional to \( E_4 E_6 - E_{10} \).

iv) \( \alpha_{12} = 2^{-3} 3^{-2} \sum (\theta[m_1] \cdots \theta[m_6])^4 \) where \( \{m_1, \ldots, m_6\} \) runs through all the maximal azymetic sequences of even theta characteristics. Such an azymetic sequence is characterized by the property that a sum of any distinct three elements is odd (cf. Igusa [5]). \( \alpha_{12} \) cannot be written as a polynomial of Eisenstein series. Indeed \( \alpha_{12} \) is a cusp form, however, no non-trivial elements of the vector space spanned by \( E_4, E_6, E_{12} \) are cusp forms.

v) \( \alpha_{12} = 2^{-8} 3^{-5} 5^{-1} \sum_{M: \Gamma_3/\Gamma_2} M(\theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000]) \), \( \theta[000] \). Let \( P \) denote the product \( \theta[110111] \theta[110110] \theta[010001] \theta[010000] \theta[100100] \theta[100101] \theta[110111] \theta[110110] \theta[100011] \theta[100010] \theta[100000])^2 \). Then \( \alpha_{20} = 2^{-9} 3 \cdot 5^{-1} \sum_{M: \Gamma_3/\Gamma_2} M(\chi_{18}/P^2) \), \( \chi_{18} \) denoting as before the product of all theta constant with even characteristics.

vi) \( \alpha_{30} = 2^{-8} 3^{-4} 5^{-1} \sum_{M: \Gamma_3/\Gamma_2} M(\theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000]) \), \( \theta[000] \).

vii) \( \beta_{14} = 2^{-5} 3^{-1} 7^{-1} \sum_{M: \Gamma_3/\Gamma_2} M(\theta[111011]^6 \theta[111010] \theta[111000] \theta[111000] \theta[101000] \theta[101000] \theta[101000] \theta[101000] \theta[101000] \theta[101000] \theta[101000]) \). In the summation, the same term appears \( 2^3 3 \cdot 7 \) times, so \( \beta_{14} \) is actually a sum of \( 2^{-5} 3^{-1} 7^{-1}[\Gamma_3: \Gamma_3(2)] = 2160 \) terms. \( \beta_{14} \) is proportional to the cusp form (8).

ix) \( \chi_{28} = 2^{-10} 3^{-2} 5^{-1} 7^{-1} \sum_{M: \Gamma_3/\Gamma_2} M(\chi_{18} \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000] \theta[000]) \). In the summation, the same term appears \( 2^3 3 \cdot 7 \) times.

Correction to [19]

- p. 802 line 1 should be read as \( \psi_{10} = \Pi_{k: \text{even}} \theta[k]^2 \).
- Sect. 23, (1) should be read as follows:

\[
\alpha_4 = \frac{1}{8} \sum_{k: \text{even}} \theta[k]^8 = \frac{1}{21504} \sum_{M: \Gamma_3/\Gamma_2} M((131) \cap (132)).
\]

\[
\Sigma(1234, 5678)^2 = 8 \Sigma D^{1/2}/(12)(34)(56)(78) = \frac{9}{8} \Sigma D^{1/2}/(12)(36)(45)(78)
\]

\[
+ \frac{4}{8} \Sigma (34)(56) D^{1/2}/(12)(78)(35)(46)(36)(45).
\]

- p. 847 line 7 should be read as \( + \frac{8}{8} \Sigma M: \theta[\Gamma_3(2)] M(((115))^2 ((135))^2 / 21^4 24^4) \).
Acknowledgements

The author wishes to express his gratitude to Sonderforschungsbereich 170, Göttingen for the hospitality and for financial support.

References