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On the Siegel modular function field of degree three

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Introduction

Let H_n be the Siegel space of degree n, and let Γ_n be the modular group. A *(Siegel) modular function* f is defined to be a meromorphic function on H_n which is invariant under Γ_n, where for $n = 1$, we need an additional condition that f is meromorphic also at the cusp. Let K_n denote the Siegel modular function field over $\mathbb{Q}$, namely the field generated over $\mathbb{Q}$ by modular functions with the rational Fourier coefficients. Then the modular function field is given by $K_n \otimes_{\mathbb{Q}} \mathbb{C}$. When $n = 1$, namely the elliptic modular case, it is well-known that K_1 is generated by the absolute invariant, which has a nice arithmetic property, e.g. an elliptic curve E has a model over the field generated over $\mathbb{Q}$ by its special value attached to E. In the higher dimensional case, several ways to get K_n are known: for example, Siegel [16], [18] showed that K_n is generated by E_{kl}/E_k^l (even $k > n + 1$, $l = 1, 2, \ldots$) where E_k denotes the Eisenstein series of weight k. Besides this, if we denote by $K(\Gamma_n(l))$ the modular function field for the principal congruence subgroup $\Gamma_n(l)$ of level l, then it is shown (Siegel [17]) that $K(\Gamma_n(l))$, $l \geqslant 3$, is generated by ratios of theta constants. Then K_n is given as the invariant subfield $K(\Gamma_n(l))^{\Gamma_n/\pm\Gamma_n(l)}$. However, these methods seem not very effective to get a finite number of generators *explicitly*. In the case of K_2, Igusa determined three generators in his paper [3], [4], where they are written by Eisenstein series, or also by theta constants. In particular, K_2 is shown to be purely transcendental. In a previous paper [19], we gave 34 generators of the graded ring of Siegel modular forms of degree three. By this, we are able to find generators of K_3 systematically. However, a systematic calculation gives too many (actually thrity three) generators. The purpose of the presence paper is *to give seven generators of K_3 explicitly*, which are ratios of modular forms of weight at most 30.

The quotient space H_3/Γ_3 is naturally equipped with the structure of the moduli variety over $\mathbb{Q}$, of three-dimensional principally polarized Abelian varieties. It is still an open problem if the number of generators of K_3 can be reduce one more, to six, which amounts to the rationality problem of H_3/Γ_3 since K_3 is the rational function field of the variety H_3/Γ_3. The moduli variety of curves of genus three is regarded as an open subvariety of H_3/Γ_3 by means of the Torelli map. Using the moduli theory of curves, Riemann [11], Weber [20], Frobenius [2] studied $K(\Gamma_3(2))$. They showed the rationality of the variety $H_3/\Gamma_3(2)$, and moreover gave six generators of $K(\Gamma_3(2))$ explicitly written in terms of derivatives of odd theta functions at the origin. Prof. R. Sasaki has given a nice mimeograph [12] surveying this topic. So H_3/Γ_3 is a unirational variety with a Galois covering of a rational variety of degree $[\Gamma_3 : \Gamma_3(2)] = 1\,451\,520$, in other words, K_3 has a Galois extension of degree 1 451 520 which is purely transcendental. Also by the moduli theory of curves, H_3/Γ_3 is proved to be even stably rational (Kollár and Schreyer [6], see also Bogomolov and Katsylo [1]).

In some cases, generators of K_n work as the absolute invariant of the elliptic modular case. More precisely by Shimura [13], [14] it is shown that if a principally polarized Abelian variety A is with sufficiently many complex multiplication, under a certain condition, or generic of *odd dimension (our case)*, then A has a model over the field generated over $\mathbb{Q}$ by their special values attached to A (see also [15], Theorem 9.5, Corollary 9.6). The author hopes that the result of the present paper will be of use for study of the rationality problem of H_3/Γ_3, or for that of arithmetic properties of three-dimensional Abelian varieties.

1. Notation and preliminary

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ denote as usual the ring of integers, the rational number field, the complex number field respectively. Let $A = \oplus A_k$, $B = \oplus B_k$ be graded $\mathbb{C}$-algebras. Then the tensor product $A \otimes B$ denotes a graded $\mathbb{C}$-algebra $\oplus_k A_k \otimes B_k$. For an integral graded algebra A, $F_0(A)$ denotes the field formed by elements of degree 0 in the field of fractions of A. We denote by $M_{k,l}(*)$, the set of $k \times l$ matrices with entries in $*$, and by $M_k(*)$, the set of square matrices of size k.

Let H_n denote the Siegel space of degree n $\{Z \in M_n(\mathbb{C}) | {}^tZ = Z, \operatorname{Im} Z > 0\}$, and let Γ_n denote the modular group $Sp_{2n}(\mathbb{Z})$. Γ_n acts on H_n by the usual modular substitution

$$Z \to MZ = (AZ + B)(CZ + D)^{-1}, \quad M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n.$$

$\Gamma_n(l)$ denotes the principal congruence subgroup of level l $\{M \in \Gamma_n | M \equiv 1_{2n} \bmod l\}$, 1_{2n} being the identity matrix of size $2n$. For a congruence subgroup

Γ of Γ_n, a holomorphic function f on H_n is called a *(Siegel) modular form* for Γ of weight k if f satisfies

$$f(MZ) = |CZ + D|^k f(Z) \quad \text{for} \quad M \in \Gamma$$

and if f is holomorphic also at cusps which is automatic when $n > 1$. In the present paper, weight k of a modular form is always supposed to be even. $A(\Gamma)_k$ denotes the vector space of modular forms of weight k, and $A(\Gamma) = \oplus A(\Gamma)_k$, the graded ring of modular forms. For $f \in A(\Gamma)_k$, and for $M \in \Gamma_n$, we define $(Mf)(Z)$ to be $|CZ + D|^{-k} f(MZ)$.

Let $m = \binom{m'}{m''} \in M_{2,n}(\mathbb{Z})$. We define a theta function with a theta characteristic m by setting

$$\theta[m](Z, x) = \sum_{g \in \mathbb{Z}^n} e(\tfrac{1}{2}(g + \tfrac{1}{2}m')Z^t(g + \tfrac{1}{2}m') + (g + \tfrac{1}{2}m')^t(x + \tfrac{1}{2}m''))$$

where $x = (x_1, \ldots, x_n)$ is a variable on $\mathbb{C}^n$, and $e(\ ) = \exp(2\pi\sqrt{-1}\ )$. m is called even or odd according as $e(\frac{1}{2}m'^t m'')$ equals 1 or -1. We put $\theta[m](Z) = \theta[m](Z, 0)$, which is called a theta constant and which is not identically zero if and only if m is even. $\theta[m]\,(Z)$ has the integral Fourier coefficients. If m is odd, then $(1/2\pi)\partial/(\partial x_i)\theta[m]\,(Z, 0)$ does not vanish identically and has the integral Fourier coefficients.

Let $\xi_0, \ldots, \xi_{r-1}$ be variables, and let h be a homogeneous polynomial in $\xi_0, \ldots, \xi_{r-1}$, of degree k in ξ_0, and of degree s in each of $\xi_1, \ldots, \xi_{r-1}$ such that the identity

$$h\left(\ldots, \frac{a\xi_i + b}{c\xi_i + d}, \ldots\right) = (c\xi_0 + d)^{-k} \prod_{i=1}^{r-1} (c\xi_i + d)^{-s} h(\ldots, \xi_i, \ldots) \tag{1}$$

is satisfied for $\binom{ab}{cd} \in SL_2(\mathbb{C})$. Let $S(r)$ denote the $\mathbb{C}$-algebra of such h with $k = s$. $S(r)$ becomes a graded $\mathbb{C}$-algebra in terms of s. $S(2, r)$ is defined to be a subring of $S(r)$ composed of h which is symmetric in $\xi_0, \ldots, \xi_{r-1}$, namely $S(2, r)$ is the invariant subring $S(r)^{\mathfrak{S}_r}$ where the symmetric group $\mathfrak{S}_r$ acts naturally on $\xi_0, \ldots, \xi_{r-1}$ as permutations. $S(2, r)$ is nothing else but the graded ring of invariants of a binary r-form (cf. Tsuyumine [19], Sect. 1), and its homogeneous element is called a *(projective) invariant.*

An element h satisfying (1) is called a (k, s)-*covariant* if h is symmetric in $\xi_1, \ldots, \xi_{r-1}$. The ring of (s, s)-covariants $(s \geqslant 0)$ is equal to $S(r)^{\mathfrak{S}_{r-1}}$ where $\mathfrak{S}_{r-1}$ acts on $\xi_1, \ldots, \xi_{r-1}$ as permutations. We have inclusions of rings; $S(2, r) \subset S(r)^{\mathfrak{S}_{r-1}} \subset S(r)$.

2. Modular forms of degree three

Let us recall some structures of the graded ring $A(\Gamma_3)$ of modular forms of degree three. The details are found in Tsuyumine [19]. For simplicity we write A for $A(\Gamma_3)$ in what follows.

We decompose $Z \in H_3$ into

$$Z = \begin{pmatrix} Z_1 & \tau \\ {}^t\tau & z_3 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \in H_2, \quad z_3 \in H_1, \quad \tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \in \mathbb{C}^2.$$

R denotes the subset of H_3 given by $\tau = 0$. A point of H_3 equivalent to some point in R is called *reducible*, and the set of images of such points by the canonical projection of H_3 to H_3/Γ_3 is its algebraic subset, and called the *reducible locus*. Let $V \subset H_3$ denote the irreducible component of zeros of a theta constant $\theta\left[\begin{smallmatrix}111\\101\end{smallmatrix}\right]$ which contains R. For a modular form $f \in A$, we define $v(f)$ to be the vanishing order of $f|_V$ at R ($v(f) = \infty$ if $f|_V \equiv 0$). $v(f)$ is called the order of f. If $f|_V \not\equiv 0$, then $v(f)$ is a non-negative even integer since f is of even weight, namely f is invariant by changing τ for $-\tau$. For even $v \geqq 0$, we define $A(v)$ to be a graded ideal generated by modular forms f with $v(f) \geqq v$. We have a sequence of inclusions $A = A(0) \supset A(2) \supset A(4) \supset \cdots$. Let

$$\chi_{18}(Z) = \prod_{m:\ \text{even}} \theta[m](Z).$$

Then χ_{18} is a modular form of weight 18, and it is a prime element of the ring A (Igusa [5]). If $f \in A$ vanishes identically on V, then f is divisible by χ_{18}, i.e., f/χ_{18} is an element of A. χ_{18} is involved in every $A(v)$. Let us put

$$\bar{A}(v) = A(v)/A(v+2).$$

$\bar{A}(0)$ is a graded $\mathbb{C}$-algebra and $\bar{A}(v)$'s can be regarded as $\bar{A}(0)$-modules. We have an isomorphism

$$A/(\chi_{18}) \simeq \bar{A}(0) \oplus \bar{A}(2) \oplus \cdots \tag{2}$$

of vector spaces, or more strongly, of (infinite) graded modules over some ring of Krull dimension five. If f is a modular form of weight k with $v(f) > \frac{2}{7}k$, then f vanishes identically on V ([19], Cor. 2 to Prop. 7) and hence f is divisible by χ_{18}. So the vector space $(A/(\chi_{18}))_k$ corresponding to modular forms of weight k is isomorphic to the direct sum $\bar{A}(0)_k \oplus \bar{A}(2)_k \oplus \cdots \oplus \bar{A}([\frac{2}{7}k]')_k$, $[\frac{2}{7}k]'$ denoting the maximal even integer not exceeding $\frac{2}{7}k$. To know the structure of $\bar{A}(v)$, we exhibit them as subspaces of $A(\Gamma_2') \otimes A(\Gamma_1)$ in the following way where Γ_2' is the maximal congruence subgroup of Γ_2 which stabilizes an odd theta characteristic $\binom{11}{10}$ mod 2.

Suppose that g is a meromorphic modular form, but holomorphic on $V - \Gamma_3 R$, $\Gamma_3 R$ being the union $\cup M \cdot R$, $M \in \Gamma_3$, and that $g|_{V-\Gamma_3 R}$ is locally bounded at R, hence at $\Gamma_3 R \cap V$. For such g, and for $(Z_1, z_3) \in H_2 \times H_1$ we define

$$(\Psi g)(Z_1, z_3) = \lim_{\substack{Z \to Z_0 \\ Z \in V}} g(Z), \quad Z_0 = \begin{pmatrix} Z_1 & 0 \\ 0 & z_3 \end{pmatrix} \in R.$$

By Riemann's removable singularity theorem $g|_{V-\Gamma_3 R}$ extends to a holomorphic function on V, and hence Ψg is well-defined. Ψg is an element of the tensor product $A(\Gamma_2') \otimes A(\Gamma_1)$ ([19], Sect. 14). Let χ_{28} be a modular form of weight 28 defined in Section 5 of the present paper (or [19], Sect. 22). It is a modular form of lowest weight having the property that $\chi_{28}|_V$ vanishes only at $\Gamma_3 R \cap V$. Its order $v(\chi_{28})$ is eight. Now let us fix three modular forms β', γ, δ with $\beta' \in A(2) - A(4)$, $\gamma \in A(4) - A(6)$, $\delta \in A(6) - A(8)$. Then if $f \in A$ is of order $v \equiv 0 \bmod 8$ (resp. 2, 4, 6 mod 8), then

$$f/\chi_{28}^{(v/8)} \quad (\text{resp.}\ f\delta/\chi_{28}^{(v+6)/8}, f\gamma/\chi_{28}^{(v+4)/8}, f\beta'/\chi_{28}^{(v+2)/8})$$

is obviously holomorphic on $V - \Gamma_3 R$ and moreover its restriction to $V - \Gamma_3 R$ is locally bounded at R ([19], Sect. 13). So its image by Ψ is well-defined. We denote by $\Psi(v)$, the map $f \mapsto \Psi(f/\chi_{28}^{(v/8)})$ (resp. $\Psi(f\delta/\chi_{28}^{(v+6)/8})$, $\Psi(f\gamma/\chi_{28}^{(v+4)/8})$, $\Psi(f\beta'/\chi_{28}^{(v+2)/8})$), where we shall write simply Ψ instead of $\Psi(0)$. (In [19], we have taken as β', γ, δ, some particular modular forms.) $\Psi(v)$ is a map of $A(v)$ to $A(\Gamma_2') \otimes A(\Gamma_1)$, and by definition the kernel of $\Psi(v)$ is just $A(v + 2)$. So $\Psi(v)$ is also considered to be an embedding of $\bar{A}(v)$ to $A(\Gamma_2') \otimes A(\Gamma_1)$. By definition $(\Psi f)(Z_1, z_3) = f\left(\begin{smallmatrix} Z_1 & 0 \\ 0 & z_3 \end{smallmatrix}\right)$, hence $\Psi\bar{A}(0)$ is contained in $A(\Gamma_2) \otimes A(\Gamma_1)$. If we identify $\bar{A}(0)$ with $\Psi\bar{A}(0)$, then the map $\Psi(v)$ of $\bar{A}(v)$ to $A(\Gamma_2') \otimes A(\Gamma_1)$ can be regarded as an $\bar{A}(0)$-module homomorphism since $\Psi(v)(fg) = \Psi f \cdot \Psi(v)g$ for $f \in A$, $g \in A(v)$. $\bar{A}(0) \subset A(\Gamma_2) \otimes A(\Gamma_1)$ is equal to $\{\Sigma\psi \otimes j \in A(\Gamma_2) \otimes A(\Gamma_1) | \Sigma\psi\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right) j(z_3)$ is symmetric in $z_1, z_2, z_3\}$ ([19], Sect. 16), over which $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite as a module, hence $A(\Gamma_2') \otimes A(\Gamma_1)$ is. Since $\chi_{28} A(v) \subset A(v + 8)$, we have sequences of inclusions of $\bar{A}(0)$-submodules of $A(\Gamma_2') \otimes A(\Gamma_1)$ by definition of $\Psi(v)$;

$$\Psi\bar{A}(0) \subset \Psi(8)\bar{A}(8) \subset \cdots$$

$$\Psi(2)\bar{A}(2) \subset \Psi(10)\bar{A}(10) \subset \cdots$$

$$\Psi(4)\bar{A}(4) \subset \Psi(12)\bar{A}(12) \subset \cdots$$

$$\Psi(6)\bar{A}(6) \subset \Psi(14)\bar{A}(14) \subset \cdots .$$

Since $A(\Gamma_2') \otimes A(\Gamma_1)$ is a Noetherian $\bar{A}(0)$-module, there is a positive even interger v_0 such that if $v \geqq v_0$, then $\Psi(v)\bar{A}(v) = \Psi(v-8)\bar{A}(v-8)$, in other words

$$\bar{A}(v) = \chi_{28}\bar{A}(v-8) \quad \text{for} \quad v \geqq v_0. \tag{3}$$

Then it is not difficult to see that any modular form $f \in A(v)$, $v \geqq v_0$, is written as $f = g\chi_{28} + h\chi_{18}$ for some $g, h \in A$, combining (3) with the fact that f is divisible by χ_{18} if $v(f) > \frac{2}{7}$ weight (f). v_0 is actually taken to be 14, and hence the isomorphism (2) becomes

$$A/(\chi_{18}) \simeq \bar{A}(0) \oplus \bar{A}(2) \oplus \bar{A}(4)$$

$$\oplus \left(\bigoplus_{\mu=0}^{\infty} (\bar{A}(6) \oplus \bar{A}(8) \oplus \bar{A}(10) \oplus \bar{A}(12))\chi_{28}^{\mu} \right).$$

All the structures of $\bar{A}(v)$, $v \leqq 12$, have been determined in [19], and from this the structure of $A/(\chi_{18})$ is given, and that of A is too.

Finally in this section we give a comment on an alternate definition of $\Psi(2)$. Restricting to V, the Taylor expansion of $\theta\left[\begin{smallmatrix}111\\101\end{smallmatrix}\right](Z)$ at $Z_0 = \left(\begin{smallmatrix}Z_1 & 0\\ 0 & z_3\end{smallmatrix}\right) \in R$ in terms of τ, we get

$$0 = \sum_{i=1}^{2} \left(\frac{\partial}{\partial x_i} \theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)(\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])(z_3) \right)\tau_i$$

$$+ \text{(higher degree terms of } \tau).$$

At least one of $\partial/(\partial x_i)\theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)$ is not zero since the theta divisor of degree two is nonsingular, and $\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ vanishes nowhere on H_1. Hence one of the τ_i is written as an analytic function of another on some neighborhood at Z_0. Let $f \in A(2)$. Substituting it in the expansion of $(f\delta/\chi_{28})|_V$ in terms of τ, and taking the limit as $\tau_i \to 0$, we get

$$(\Psi(2)f)(Z_1, z_3) = (F_2F_6/F_8)(Z_1, z_3)$$

where

$$F_2(Z_1, z_3) = \frac{1}{2!(2\pi)^4(\sqrt{-1})^2} \sum_{l=0}^{2} (-1)^l \binom{2}{l} \frac{\partial}{\partial \tau_1^l} \frac{\partial}{\partial \tau_2^{2-l}} f(Z_0)$$

$$\times \left(\frac{\partial}{\partial x_1} \theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)\right)^{2-l} \left(\frac{\partial}{\partial x_2} \theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)\right)^{l},$$

$$F_6(Z_1, z_3) = \frac{1}{6!(2\pi)^{12}(\sqrt{-1})^6} \sum_{l=0}^{6} (-1)^l \binom{6}{l} \frac{\partial}{\partial \tau_1^l} \frac{\partial}{\partial \tau_2^{6-l}} \delta(Z_0)$$

$$\times \left(\frac{\partial}{\partial x_1} \theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)\right)^{6-l} \left(\frac{\partial}{\partial x_2} \theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)\right)^{l}, \tag{4}$$

$$F_8(Z_1, z_3) = \frac{1}{8!(2\pi)^{16}(\sqrt{-1})^8} \sum_{l=0}^{8} (-1)^l \binom{8}{l} \frac{\partial}{\partial \tau_1^l} \frac{\partial}{\partial \tau_2^{8-l}} \chi_{28}(Z_0)$$

$$\times \left(\frac{\partial}{\partial x_1} \theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)\right)^{8-l} \left(\frac{\partial}{\partial x_2} \theta\left[\begin{smallmatrix}11\\10\end{smallmatrix}\right](Z_1, 0)\right)^{l},$$

$\binom{k}{l}$ denoting a binomial coefficient. $\Psi(2)f$ is holomorphic and has a Fourier expansion on $H_2 \times H_1$, and each of F_2, F_6, F_8 has too. By definition $30\chi_{28}$ has integral Fourier coefficients. Now let us suppose that δ has rational Fourier coefficients (with a bounded denominator). Then both of F_6, F_8 have rational Fourier coefficients (with a bounded denominator). Hence there is a rational number N such that $N\Psi(2)f$ has integral Fourier coefficients if and only if F_2 does. In particular, for such N, $2N\Psi(2)f$ has the integral Fourier coefficients if f does.

Let us calculate a first term of F_2 explicitly in terms of the Fourier coefficient of $f \in A(2)$ for the identity matrix, i.e. for $e(\mathrm{tr}(Z))$. There are 23 positive symmetric semi-integral ternary matrices with merely one as their diagonal components, each of which is equivalent under the action $S \to {}^tUSU$, $U \in GL_3(\mathbb{Z})$, to one of the following three matrices; the identity matrix; the matrix with 0 as its (1, 2), (1, 3)-components and with 1/2 as its (2, 3)-component; the matrix with 0 as its (1, 2)-component and with 1/2 as its (1, 3), (2, 3)-components. Let a_0, a_1, a_2 be the Fourier coefficients of f corresponding to the first, second, third matrix respectively. From $\Psi\beta = 0$, two relations among a_0, a_1, a_2 are derived; $a_0 + 4a_1 + 4a_2 = a_1 + 6a_2 = 0$, hence $a_0 : a_1 : a_2 = 20 : -6 : 1$ if $a_0 \neq 0$. Then a direct calculation shows

$$F_2(Z_1, z_3) = -\tfrac{2}{5} a_0 e\left(\mathrm{tr}\left(\left(\begin{smallmatrix}5/4 & \pm 1/4\\ \pm 1/4 & 5/4\end{smallmatrix}\right) Z_1\right)\right) e(z_3) + \cdots .$$

3. A subring of $A(\Gamma_3)$

$\Gamma_2/\Gamma_2(2)$ is isomorphic to the symmetric group $\mathfrak{S}_6$ of degree six, and it acts on the set of six odd theta characteristics (mod 2) of degree two as permutations. Γ_2' has been defined to be a stabilizer subgroup of Γ_2 at an odd theta characteristic $\binom{11}{10}$, and hence $\Gamma_2'/\Gamma_2(2)$ is isomorphic to $\mathfrak{S}_5$.

There is an injective homomorphism ϱ_2 of $A(\Gamma_2(2))$ to $S(6) \subset \mathbb{C}[\xi_0, \ldots, \xi_5]$ which is equivalent under $\mathfrak{S}_6$ (Igusa [5], Tsuyumine [19], Sect. 9, 11), where ϱ_2 induces an isomorphism between the field of fractions of $A(\Gamma_2(2))$ and that of $S(6)^{(2)}$, $S(6)^{(2)}$ denoting the subring of $S(6)$ consisting of homogeneous elemens of even degree. We may assume that $\mathfrak{S}_5 \simeq \Gamma_2'/\Gamma_2(2)$ acts on $\{\xi_1, \ldots, \xi_5\}$ as permutations. Hence we have a commutative diagram;

$$\begin{array}{ccc}
A(\Gamma_2(2)) & \xrightarrow{\varrho_2} & S(6) \\
\cup & & \cup \\
A(\Gamma_2') & \longrightarrow & S(6)^{\mathfrak{S}_5} \\
\cup & & \cup \\
A(\Gamma_2) & \longrightarrow & S(2, 6) = S(6)^{\mathfrak{S}_6}.
\end{array}$$

In particular, there is no proper intermediate field between $F_0(A(\Gamma_2'))$ and $F_0(A(\Gamma_2))$, and hence $F_0(A(\Gamma_2')) = F_0(A(\Gamma_2)[\psi])$ for any $\psi \in A(\Gamma_2') - A(\Gamma_2)$.

LEMMA 1. *Let β be a modular form for Γ_3 of order v with $v \equiv 2$ or 6 mod 8. Let us fix $z_3 \in H_1$ so that $\psi(Z_1) := (\Psi(4v)\beta^4)(Z_1, z_3)$ is not identically zero. Then $\psi \notin A(\Gamma_2)$. In particular, $F_0((A(\Gamma_2) \otimes A(\Gamma_1))[\Psi(4v)\beta^4]) = F_0(A(\Gamma_2') \otimes A(\Gamma_1))$.*

Proof. We treat only the case $v \equiv 2 \bmod 8$, since a similar argument is applicable to the case $v \equiv 6 \bmod 8$. By the argument [19], Sect. 14, the proof of Lemma 12, $\varrho_2\phi$ is the form $H^4\mathscr{D}_0$ where H is an $(s+2, s)$-covariant and $\mathscr{D}_0$ denotes the $(0, 8)$-covariant $\Pi_{1 \leqq i < j \leqq 5}(\xi_i - \xi_j)^2$. It is enough to show that $H^4\mathscr{D}_0 \notin S(2, 6)$. Suppose otherwise. Dividing H^4 by a power of the discriminant $\Pi_{0 \leqq i < j \leqq 5}(\xi_i - \xi_j)^2 \in S(2, 6)$ if necessary, we may assume that H is not divisible by $\Pi_{0 \leqq i < j \leqq 5}(\xi_i - \xi_j)$. Since $H^4\mathscr{D}_0$ obviously has factors $(\xi_i - \xi_j)^2$ $(1 \leqslant i < j \leqq 5)$ and since $H^4\mathscr{D}_0$ is symmetric in $\xi_0, \ldots, \xi_5$ by our assumption, it has a factor $\Pi_{i=1}^5 (\xi_0 - \xi_i)^2$. Then H is divisible by $\Pi_{i=1}^5 (\xi_0 - \xi_i)$, and hence $H^4\mathscr{D}_0$, by $\Pi_{i=1}^5 (\xi_0 - \xi_i)^4 \times \Pi_{1 \leqq i < j \leqq 5} (\xi_i - \xi_j)^2$. Again by symmetry $H^4\mathscr{D}_0/\Pi_{i=0}^5(\xi_0 - \xi_i)^4 \times \Pi_{1 \leqq i < j \leqq 5} (\xi_i - \xi_j)^2$ is still divisible by

$(\xi_i - \xi_j)^2$ $(1 \leqq i < j \leqq 5)$, hence H, by $(\xi_i - \xi_j)$ $(1 \leqq i < j \leqq 5)$, a contradiction. Q.E.D.

Let Λ be a graded subring of A such that $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite integral over $\bar{\Lambda} := \Psi\Lambda$, and that $\chi_{28}, \chi_{18} \in \Lambda$.

LEMMA 2. *A is finite integral over Λ.*

Proof. $\Psi(v)A(v)$ is a finite $\bar{\Lambda}$-module for every even $v \geqq 0$. Let $\{f_{i,v}\}_i$ be a finite number of modular forms in $A(v)$ such that $\{\Psi(v)f_{i,v}\}_i$ generates $\Psi(v)A(v)$ over $\bar{\Lambda}$. We show that A is generated as a Λ-module, by $f_{i,v}$'s with $v \leqq v_0$, v_0 being as in (3).

We prove that any modular form f of weight k is written as a linear combination of $f_{i,v}$'s $(v \leqq v_0)$ over Λ, by induction on k. $\Psi f \in \Psi A(0)$ is written as $\Psi f = \Sigma_i \Psi(P_i f_{i,0})$ with $P_i \in \Lambda$. By taking $f - \Sigma P_i f_{i,0}$ instead of f, we may assume $\Psi f = 0$, namely $v(f) \geqq 2$. Then $\Psi(2)f$ is written as $\Sigma_i \Psi(2)(P'_i f_{i,2})$ with $P'_i \in \Lambda$. By a similar argument as above, may assume $\Psi(2)f = 0$, and by a recursive argument, we may assume $v(f) > \frac{2}{7}k$, where we make use of such elements as $\chi_{28}^m f_{i,v}$ $(m > 0)$ instead of $f_{i,v}$ if the order $v(f)$ exceeds v_0. Then $f|_V$ vanishes identically and f is written as $f = g\chi_{18}$ for some $g \in A$. By the induction hypothesis g is a linear combination of $f_{i,v}$'s $(v \leqq v_0)$ over Λ, and hence f is. Q.E.D.

COROLLARY. *$A(v)$ is a finite Λ-module for any even $v \geqq 0$.*

PROPOSITION 1. *Let Λ be a graded subring of A containing χ_{28}, χ_{18} such that $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite integral over $\bar{\Lambda} := \Psi\Lambda$, and that* g.c.d $\{k | \bar{\Lambda}_k \neq \{0\}\} = 2$ *for $\bar{\Lambda} = \oplus \bar{\Lambda}_k$. If β is a modular form of order two such that $F_0(\bar{\Lambda}[\Psi(8)\beta^4]) = F_0(A(\Gamma'_2) \otimes A(\Gamma_1))$, then the modular function field of degree three is given by $F_0(\Lambda[\beta])$.*

Proof. At first we show that there are a positive integer v_1 and a modular form $P \in \Lambda$ of order 0 such that

$$\Psi(v + v')(\beta^{v'/2}PA(v)) \subset \Psi(v + v')(\beta^{v'/2}(\Lambda[\beta] \cap A(v)) \tag{5}$$

for any even $v \geqq v_1$ where $v' \in \{0, 2, 4, 6\}$ is determined by $v + v' \equiv 0 \bmod 8$. By our assumption, we can take $\bar{P} \in \bar{\Lambda}$, $\neq 0$ such that $\bar{P}(A(\Gamma'_2) \otimes A(\Gamma_1))$ is contained in a $\bar{\Lambda}$-module generated by $\Psi(8)\beta^4$, $(\Psi(8)\beta^4)^2, \ldots, (\Psi(8)\beta^4)^m$ with $m = [F_0(A(\Gamma'_2) \otimes A(\Gamma_1)): F_0(\Lambda)]$. Since $\Lambda[\beta] \cap A(v)$ has as a subset

$$\sum_{2n_1 + 8n_2 \geqq v} \beta^{n_1} \chi_{28}^{n_2} \Lambda,$$

$\Psi(v+v')(\beta^{v'/2}(\Lambda[\beta] \cap A(v))$ contains the $\bar{\Lambda}$-module generated by $\Psi(8)\beta^4, \ldots, (\Psi(8)\beta^4)^m$ if v is large enough. If $P \in \Lambda$ is such that $\bar{P} = \Psi P$, then $\Psi(v+v')(\beta^{v'/2}PA(v)) = \bar{P}\Psi(v+v')(\beta^{v'/2}A(v)) \subset \bar{P}(A(\Gamma_2') \otimes A(\Gamma_1))$. Thus we have proved (5).

$A(2)$ is the prime ideal of A defining the reducible locus of H_3/Γ_3, and hence $A(2) \cap \Lambda[\beta]$ is prime in $\Lambda[\beta]$. Let us take the ring $\Lambda_0 := \Lambda[\beta, \chi_{18}/\chi_{28}^k$ $(k = 0, 1, 2, \ldots)]$. The ideal of Λ_0 generated by $A(2) \cap \Lambda[\beta]$, χ_{18}/χ_{28}^k $(k = 0, 1, 2, \ldots)$ is prime since $\bar{\Lambda} = \Lambda_0/(A(2) \cap \Lambda[\beta], \chi_{18}/\chi_{28}^k$ $(k = 0, 1, 2, \ldots))$ is an integral domain. Let Λ_0' be the localization of Λ_0 at the prime ideal. Let v_2 be an even integer equal to or greater than each of v_0 and v_1, v_0 being as in (3). Since $\Lambda \subset \Lambda_0'$, by Corollary to Lemma 2 there are a finite number of holomorphic modular forms $f_1, \ldots, f_t \in A(v_2)$ such that $A(v_2) \subset \Lambda_0' f_1 + \cdots + \Lambda_0' f_t$. We may assume that $\{f_1, \ldots, f_t\}$ is a minimal system with this property. Then we show $t = 1$. Suppose $t \geqq 2$. Since $v := v(f_t)$ is larger than v_1, we have $\Psi(v+v')\beta^{v'/2}Pf_t = \Psi(v+v')\beta^{v'/2}q$ for some $q \in A(v) \cap \Lambda[\beta]$. Since $\Psi(v+v')\beta^{v'/2}(Pf_t - q) = 0$, the order of $Pf_t - q$ is at least $v + 2$. By repeating the similar argument four times, it is shown that there is $Q \in \Lambda[\beta]$ satisfying the inequality $v(P^4 f_t - Q) \geqq v + 8$. Since $v \geqq v_0$, by (3) there are g, h such that $P^4 f_t - Q = g\chi_{28} + h\chi_{18}$. $v(g)$ is obviously greater than or equal to v, and in particular $g \in A(v_2)$ because $v \geqq v_2$. $h\chi_{28}^k$ is also involved in $A(v_2)$ if k is sufficiently large. g, $h\chi_{28}^k \in A(v_2) \subset \Lambda_0' f_1 + \cdots + \Lambda_0' f_t$ is written as $g = \Sigma_{i=1}^t a_i f_i$, $h\chi_{28}^k = \Sigma_{i=1}^t b_i f_i$ with a_i, $b_i \in \Lambda_0'$. Hence we have

$$(P^4 - a_t\chi_{28} - b_t\chi_{18}/\chi_{28}^k)f_t = Q + \sum_{i=1}^{t-1} a_i\chi_{28}f_i + \sum_{i=1}^{t-1} b_i(\chi_{18}/\chi_{28}^k)f_i.$$

Since P is of order 0, $P^4 - a_t\chi_{28} - b_t\chi_{18}/\chi_{28}^k$ is a unit of the ring Λ_0'. So f_t is written as a linear combination of other f_i. This contradicts to the minimality of a system of $\{f_1, \ldots, f_t\}$. Thus $t = 1$.

Now we have $A(v_2) \subset \Lambda_0' f_1$. $A(v_2)$ and Λ_0' have a common non-trivial element (e.g., χ_{18}). This implies that f_1 is contained in the field of fractions of Λ_0', and that $A(v_2)$ is a subset of the field of fractions of $\Lambda[\beta]$. Since $\chi_{28}^k A \subset A(v_2)$ for large k, the modular function field $F_0(A)$ is equal to $F_0(\Lambda[\beta])$. Q.E.D.

Combining Proposition 1 with Lemma 1, we have the following corollary.

COROLLARY. *Let Λ be a ring as in Proposition 1 satisfying the additional condition that $F_0(\bar{\Lambda}) = F_0(A(\Gamma_2) \otimes A(\Gamma_1))$. Let β be any modular form with $v(\beta) = 2$. Then the modular function field is given by $F_0(\Lambda[\beta])$.*

4. Main theorem

$A(\Gamma_1)$ is generated by two algebraically independent modular forms j_4, j_6 of weight 4, 6 respectively where

$$j_4 = \frac{1}{2} \sum_{m:\text{even}} \theta[m]^8, \quad j_6 = \sum_{M:\Gamma_1/\Gamma_1(2)} M(\theta[{}^0_0]^8 \theta[{}^0_1]^4).$$

As Igusa [3], [4] showed, $A(\Gamma_2)$ is generated by four algebraically independent modular forms ψ_4, ψ_6, ψ_{10}, ψ_{12} with their subscript as their weight where

$$\psi_4 = \frac{1}{4} \sum_{m:\text{even}} \theta[m]^8, \quad \psi_6 = \frac{1}{2} \sum_{M:\Gamma_2/\Gamma_2(2)} M(\theta[{}^{00}_{00}]^6 \theta[{}^{00}_{10}]^2 \theta[{}^{00}_{01}]^2 \theta[{}^{00}_{11}]^2),$$

$$\psi_{10} = \prod_{m:\text{even}} \theta[m]^2,$$

$$\psi_{12} = \frac{1}{288} \sum_{M:\Gamma_2/\Gamma_2(2)} M(\theta[{}^{00}_{10}]\theta[{}^{01}_{00}]\theta[{}^{10}_{00}]\theta[{}^{00}_{01}]\theta[{}^{10}_{01}]\theta[{}^{01}_{10}])^4$$

(note that we are considering only modular forms of even weight). Let α_4, α_6, α_{10}, α_{12}, α'_{12}, α_{20}, $\alpha_{30} \in A$ be as in Section 5, and let $\alpha'_{20} = (\alpha_{20} - 5\alpha_{10}^2)/7$, $\alpha'_{30} = (7\alpha_{30} - 313\alpha_{10}\alpha_{20} + 865\alpha_{10}^3)/7$. By [19], Section 23, we have

$$\Psi\alpha_4 = \psi_4 \otimes j_4, \Psi\alpha_6 = \psi_6 \otimes j_6, \Psi\alpha_{12} = 3^{-3}\psi_{12} \otimes (-j_6^2 + 4j_4^3),$$

$$\Psi\alpha'_{12} = 2^4 3^{-3}\psi_4^3 \otimes (-j_6^2 + 4j_4^3)$$
$$- 3^{-3}\psi_6^2 \otimes (-j_6^2 + 4j_4^3) + 3^2\psi_{12} \otimes (j_6^2 + 8j_4^3),$$

$$\Psi\alpha'_{20} = \psi_{10}^2 \otimes j_4^5, \Psi\alpha'_{30} = \psi_{10}^3 \otimes j_6^5.$$

LEMMA 3. *Let $\bar{\Lambda}$ denote a graded $\mathbb{C}$-algebra generated by Ψ-images of* i) α_4, α_6, α_{12}^k, α'_{12}, α'_{20}, α'_{30}, *or* ii) α_4, α_6, α_{12}, α'^k_{12}, α'_{20}, α'_{30}, *k being any fixed positive integer. Then $A(\Gamma_2) \otimes A(\Gamma_1)$ is finite integral over $\bar{\Lambda}$, and $F_0(\bar{\Lambda})$ equals $F_0(A(\Gamma_2) \otimes A(\Gamma_1))$.*

Proof. The first assertion follows from the fact that $\Psi\alpha_4$, $\Psi\alpha_6$, $\Psi\alpha_{12}$, $\Psi\alpha'_{12}$, $\Psi\alpha'_{20}$, $\Psi\alpha'_{30}$ do not vanish simultaneously at any point of the projective variety $(H_2/\Gamma_2)^* \times (H_1/\Gamma_1)^*$, $(H_n/\Gamma_n)^*$ denoting the Satake compactification, which is not difficult to see. We treat only the case ii), because the similar argument is applicable to the case i). Put $s = (j_6^2/j_4^3)(z_3)$. Then s is an element of degree five over $\mathbb{C}[\Psi\alpha_{30}'^2/\Psi\alpha_{20}'^3]$. As easily seen, $F_0(A(\Gamma_2) \otimes A(\Gamma_1))$ is an extension over $F_0(\mathbb{C}[\Psi\alpha_4, \Psi\alpha_6, \Psi\alpha_{12}, \Psi\alpha_{20}', \Psi\alpha_{30}'])$ of degree five, since the former is obtained from the latter by adding s. In particular, the extension is simple. Since an element $\Psi(\alpha_{12}'^k/\alpha_4^{3k})$ is not contained in the latter one, $F_0(\bar{\Lambda})$ equals $F_0(A(\Gamma_2) \otimes A(\Gamma_1))$. Q.E.D.

THEOREM 1. *Let λ be any modular form of weight twenty with $v(\lambda) = 2$, and let c be a constant. Let us put* $\Lambda := \mathbb{C}[\alpha_4, \alpha_6, \alpha_{12}, \alpha_{12}', (\alpha_{20} - 5\alpha_{10}^2)/7 + c\lambda, (7\alpha_{30} - 313\alpha_{10}\alpha_{20} + 865\alpha_{10}^3)/7, \chi_{28}, \chi_{18}]$. *Then the modular function field of degree three is given by $F_0(\Lambda)$, except at most one value of c. (See Sect 5 for the definition of modular forms.)*

REMARK. Our argument will show that the assertion of Theorem 1 holds even if we replace Λ by other rings such as $\mathbb{C}[\alpha_4, \alpha_6, \alpha_{12}, \alpha_{12}'^k + c\lambda, \alpha_{20}', \alpha_{30}', \chi_{28}, \chi_{18}]$, $\mathbb{C}[\alpha_4, \alpha_6, \alpha_{12}, \alpha_{12}', \alpha_{20}', \alpha_{30}' + c\lambda, \chi_{28}, \chi_{18}]$ and so on, λ being a modular form of appropriate weight with $v(\lambda) = 2$.

Proof: Let us find an algebraic relation[1] among $\Psi\alpha_4$, $\Psi\alpha_6$, $\Psi\alpha_{12}$, $\Psi\alpha_{12}'$, $\Psi\alpha_{20}' = \Psi(\alpha_{20}' + c\lambda)$, $\Psi\alpha_{30}'$. Let s be as in the proof of Lemma 3. If we put

$$p_0 = 16\alpha_4^3,\ p_1 = -128\alpha_4^3 - \alpha_6^2 + 243\alpha_{12} + 27\alpha_{12}',$$

$$p_2 = 256\alpha_4^3 + 8\alpha_6^2 + 1944\alpha_{12} - 108\alpha_{12}',\ p_3 = -16\alpha_6^2,$$

then we have

$$(\Psi p_0)s^3 + (\Psi p_1)s^2 + (\Psi p_2)s + \Psi p_3 = 0 \tag{6}$$

by a direct computation. For an indeterminate X, we put

$$\begin{aligned} L(X) = {} & p_3^5(\alpha_{20}' + X)^9 + (p_2^5 + 5p_0p_2^2p_3^2 + 5p_1^2p_2p_3^2 - 5p_0p_1p_3^3 \\ & - 5p_1p_2^3p_3)\alpha_{30}'^2(\alpha_{20}' + X)^6 + (p_1^5 + 5p_0^2p_1^2p_3 + 5p_0^2p_1p_2^2 \\ & - 5p_0^3p_2p_3 - 5p_0p_1^3p_2)\alpha_{30}'^4(\alpha_{20}' + X)^3 + p_0^5\alpha_{30}'^6. \end{aligned}$$

[1] Such a detail is not necessary to prove merely Theorem 1. However, it (or $L(X)$) will be used for other purposes later.

Then $\Psi L(0)/(\Psi\alpha'_{20})^9 = 0$ is a minimal algebraic relation among $\Psi\alpha_4$, $\Psi\alpha_6$, $\Psi\alpha_{12}$, $\Psi\alpha'_{12}$ and $(j_6^2/j_4^3)^5$ $(=s^5)$, given by eliminating s from (6). Hence $\Psi L(0) = 0$, which is an algebraic relation among $\Psi\alpha_4, \ldots, \Psi\alpha'_{30}$.

By Lemma 3 Λ satisfies the condition in Corollary to Proposition 1. $\beta := L(c\lambda)$ is a modular form contain in Λ, which equals $L(0) + cL'(0)\lambda$ up to $A(4)$ where L' is the derivative of L in terms of X. Since $L(0)$, $\lambda \in A(2)$, β is a modular form of order at least two. Since $L'(0) \in A(0) - A(2)$, we have, except for at most one value of c

$$\Psi(2)L(0) + c\Psi L'(0)\lambda \neq 0, \tag{7}$$

i.e., $v(\beta) = 2$. Then by the Corollary to Proposition 1, the modular function field is given by $F_0(\Lambda[\beta]) = F_0(\Lambda)$. Q.E.D.

Let us make $c\lambda$ explicit for which the assertion of Theorem 1 holds. By the above proof it is enough to find $c\lambda$ satisfying (7). From the definition, α_4, $2^{-3}\alpha_6$, $2^3 3^2\alpha_{12}$, $2^4 3^{-1}\alpha'_{12}$, $2^9 3^2 5\cdot 7\cdot 11\alpha'_{20}$, $2^{12}3^3 5^2 7^2 11^3\alpha'_{30}$ are easily checked to have integral Fourier coefficients. By the way, $30\chi_{28}$, χ_{18} have too. Let N be the rational number given in the last part of Section 2. Since $2^{106}3^{24}5^{16}7^{16}11^{24}L(0)$ has integral Fourier coefficients, also $2N$ times its $\Psi(2)$-image does. So (7) holds if $2N\cdot 2^{106}3^{24}5^{16}7^{16}11^{24}c\Psi L'(0)\,\Psi(2)\lambda$ has a non-integral Fourier coefficient.

We take as λ, $\alpha_6\beta_{14}$ where β_{14} is a cusp form of weight 14 and of order two which is defined in Section 5 (or, also in [19], Sect. 24). $\Psi(2)\alpha_6\beta_{14}$ equals $\Psi\alpha_6\Psi(2)\beta_{14}$. Now we must find a rational number c such that $2^{106}3^{24}5^{16}7^{16}11^{24}c\Psi L'(0)\Psi\alpha_6\cdot F_2$ has a non-integral Fourier coefficient, F_2 being the one given for $f = \beta_{14}$ in (4), which implies (7). α_6 has the Fourier expansion starting from the constant term 8, and a direct calculation shows that $\Psi L'(0)$ has the Fourier expansion starting from

$$-2^{254}3^7 5^2\{2e(\mathrm{tr}(Z_1)) - e(\mathrm{tr}(\begin{smallmatrix}1 & \pm 1/2\\ \pm 1/2 & 1\end{smallmatrix})(Z_1))\}^{16}e(2z_3).$$

Let a be the Fourier coefficient of β_{14} for $e(\mathrm{tr}(Z))$. Combining the above calculation with that of the last part of Section 2, $2^{106}3^{24}5^{16}7^{16}11^{24}c\Psi L'(0)\Psi\alpha_6\cdot F_2$ is shown to have $2^{364}3^{31}5^{17}7^{16}11^{24}ac$ as a Fourier coefficient. Here we give a rough estimate of a. β_{14} is written as a sum of 2160 products with sign, of 28 theta constants, where each of products has the Fourier expansion starting from the terms corresponding to positive semi-integral ternary matrices with their diagonal components $\geqq 1$. From this, $a \in \mathbb{Z}$, and a rough estimate shows $|a| < 2160 \times 2^{3\times 8} = 2^{28}3^4 5$. On the other hand $a \neq 0$ is shown in the following way. So if c is a rational number such that

$2^{364}3^{31}5^{17}7^{16}11^{24}ac \notin \mathbb{Z}$ for any positive integer a less than $2^{28}3^4 5$, then (7) holds and hence $F_0(\Lambda)$ gives the modular function field of degree three, where $\Lambda = \mathbb{C}[\alpha_4, \alpha_6, \alpha_{12}, \alpha'_{12}, \alpha'_{20} + c\alpha_6\beta_{14}, \alpha'_{30}, \chi_{28}, \chi_{18}]$. Since the generators have the rational Fourier coefficients, their ratios of weight zero generate the modular function field K_3 over $\mathbb{Q}$.

Let us prove $a \neq 0$. Let $E_{k,n}$ denote the normalized Eisenstein series of degree n and of weight k, where 'normalized' implies that its constant term is one. By the structure theorem of $A(\Gamma_2)$ (Igusa [3], [4], [5]) and by the formulas for the Fourier coefficients of Eisenstein series of degree two in Maass [7], Satz 1, the identity $3^5 7 \cdot 11 \cdot 659 E_{4,2}^2 E_{6,2} - 2^2 269 \cdot 43867 E_{4,2} E_{10,2} + 53 \cdot 657931 E_{14,2} = 0$ follows. Hence

$$3^5 7 \cdot 11 \cdot 659 E_{4,3}^2 E_{6,3} - 2^2 269 \cdot 43867 E_{4,3} E_{10,3} + 53 \cdot 657931 E_{14,3} \tag{8}$$

is a cusp form of weight fourteen where $E_{4,3}$ is well-defined by Raghavan [10]. By virtue of Ozeki and Washio [8], [9], the Fourier coefficient of (8) for $e(\mathrm{tr}(Z))$ can be calculated, namely $-2^7 3^8 5^2 7^2 11 \cdot 79973$. By [19], the vector space of cusp forms of weight 14 is one-dimensional, and hence (8) and β_{14} are proportional. Thus $a \neq 0$. We have proved the following theorem.

THEOREM 2. *The Siegel modular function field K_3 of degree three over $\mathbb{Q}$ is generated by the following seven modular functions; α_6^2/α_4^3, α_{12}/α_4^3, α'_{12}/α_4^3, $(\alpha_{20} - 5\alpha_{10}^2 + 7c\alpha_6\beta_{14})/\alpha_4^5$, $(7\alpha_{30} - 313\alpha_{20}\alpha_{10} + 865\alpha_{10}^3)/\alpha_4^6\alpha_6$, χ_{28}/α_4^7, $\chi_{18}/\alpha_4^3\alpha_6$ where c is any rational number exclusive of at most one value. If c is such that $2^{364}3^{31}5^{17}7^{16}11^{24}ac \in \mathbb{Q} - \mathbb{Z}$ for any positive integer a less than $2^{28}3^4 5$, then our assertion holds. (see Sect. 5 for the definition of modular forms).*

REMARK

i) In Theorem 2, we may replace α_{12}/α_4^3 or α'_{12}/α_4^3 by its power for general $c \in \mathbb{Q}$. This implies for example, that K_3 is not a cyclic extension of $\mathbb{Q}(\alpha_6^2/\alpha_4^3, \alpha_{12}/\alpha_4^3, (\alpha_{20} - 5\alpha_{10}^2 + 7c\alpha_6\beta_{14})/\alpha_4^5, (7\alpha_{30} - 313\alpha_{20}\alpha_{10} + 865\alpha_{10}^3)/(\alpha_4^6\alpha_6, \chi_{28}/\alpha_4^7, \chi_{18}/\alpha_4^3\alpha_6)$ unless the extension is trivial.

ii) In Theorem 2 we can replace β_{14} by the cusp form (8). Then c is taken to be a rational number such that $2^{371}3^{39}5^{19}7^{18}11^{25}79973c \notin \mathbb{Z}$, e.g., $c = 1/13$.

5. Modular forms

We give definition of modular forms $\alpha_4, \alpha_6, \alpha_{10}, \alpha_{12}, \alpha'_{12}, \alpha_{20}, \alpha_{30}, \beta_{14}, \chi_{28}$ with their subscripts as their weight. We denote by E_k, the Eisenstein series of degree three and of weight k.

i) $\alpha_4 = 2^{-3}\Sigma_m \theta[m]^8$, m running over the set of all even theta characteristics (mod 2). α_4 is equal to the Eisenstein series E_4.

ii) $\alpha_6 = 2^{-6}3^{-1}7^{-1}\,\Sigma_{M:\Gamma_3/\Gamma_3(2)}\, M(\theta{000 \brack 000}^5\theta{000 \brack 100}\theta{000 \brack 101}\theta{000 \brack 001}\theta{000 \brack 011}\theta{000 \brack 010}\theta{000 \brack 111}\theta{000 \brack 011})$, which is equal to $8E_6$.

iii) $\alpha_{10} = -2^{-4}3^{-2}5^{-1}11^{-1}\,\Sigma_{M:\Gamma_3/\Gamma_3(2)}\, M\{(\theta{110 \brack 000}\theta{010 \brack 000}\theta{010 \brack 100}\theta{110 \brack 110}\theta{100 \brack 010}\theta{100 \brack 000})^2\theta{000 \brack 100}\theta{000 \brack 101}\theta{000 \brack 000}\theta{000 \brack 001}\theta{000 \brack 011}\theta{000 \brack 010}\theta{000 \brack 111}\theta{000 \brack 110})\}$, which is proportional to $E_4E_6 - E_{10}$.

iv) $\alpha_{12} = 2^{-3}3^{-2}\,\Sigma\,(\theta[m_1]\cdots\theta[m_6])^4$ where $\{m_1, \ldots, m_6\}$ runs through all the maximal azygetic sequences of even theta characteristics. Such an azygetic sequence is characterized by the property that a sum of any distinct three elements is odd (cf. Igusa [5]). α_{12} cannot be written as a polynomial of Eisenstein series. Indeed α_{12} is a cusp form, however, no non-trivial elements of the vector space spanned by E_4^3, E_6^2, E_{12} are cusp forms.

v) $\alpha'_{12} = 2^{-8}3^{-5}5^{-1}\,\Sigma_{M:\Gamma_3/\Gamma_{3(2)}}\, M(\theta{110 \brack 000}\theta{110 \brack 001}\theta{010 \brack 000}\theta{010 \brack 001}\theta{010 \brack 100}\theta{010 \brack 101}\theta{110 \brack 111}\theta{110 \brack 110}\theta{100 \brack 011}\theta{100 \brack 010}\theta{100 \brack 000}\theta{100 \brack 001})^2$.

vi) Let P denote the product $\theta{011 \brack 111}\theta{011 \brack 011}\theta{111 \brack 000}\theta{111 \brack 011}\theta{011 \brack 000}\theta{011 \brack 100}\theta{111 \brack 110}\theta{101 \brack 111}\theta{001 \brack 100}\theta{001 \brack 000}\theta{101 \brack 010}\theta{101 \brack 000}\theta{001 \brack 010}\theta{001 \brack 110}\theta{101 \brack 101}\theta{111 \brack 101}$. Then $\alpha_{20} = 2^{-9}3\cdot 5^{-1}\,\Sigma_{M:\Gamma_3/\Gamma_3(2)}\, M(\chi_{18}^2/P^2)$, χ_{18} denoting as before the product of all theta constant with even characteristics.

vii) $\alpha_{30} = 2^{-8}3^4 5^{-1}\,\Sigma_{M:\Gamma_3/\Gamma_3(2)}\, M(\theta{000 \brack 001}^2\chi_{18}^3/\theta{000 \brack 000}^2P^3)$.

viii) $\beta_{14} = 2^{-5}3^{-1}7^{-1}\,\Sigma_{M:\Gamma_3/\Gamma_3(2)}\, M(\theta{011 \brack 111}^6\chi_{18}/\theta{110 \brack 001}\theta{010 \brack 000}\theta{010 \brack 101}\theta{110 \brack 111}\theta{111 \brack 110}\theta{001 \brack 100}\theta{001 \brack 000}\theta{100 \brack 010}\theta{101 \brack 010}\theta{100 \brack 000}\theta{000 \brack 011}\theta{000 \brack 010}\theta{101 \brack 101}\theta{111 \brack 101})$. In the summation, the same term appears $2^5 3\cdot 7$ times, so β_{14} is actually a sum of $2^{-5}3^{-1}7^{-1}[\Gamma_3:\Gamma_3(2)]$ $(=2160)$ terms. β_{14} is proportional to the cusp form (8).

ix) $\chi_{28} = 2^{-10}3^{-2}5^{-1}7^{-1}\Sigma_{M:\Gamma_3/\Gamma_3(2)}\, M(\chi_{18}/\theta{000 \brack 100}\theta{000 \brack 101}\theta{000 \brack 000}\theta{000 \brack 001}\theta{000 \brack 011}\theta{000 \brack 010}\theta{000 \brack 111}\theta{000 \brack 110})^2$. In the summation, the same term appears $2^9 3\cdot 7$ times.

Correction to [19]

- p. 802 line 1 should be read as $\psi_{10} = \Pi_{k:\text{even}}\,\theta[k]^2$.
- Sect. 23, (1) should be read as follows:

$$\alpha_4 = \frac{1}{8}\sum_{k:\text{even}}\theta[k]^8 = \sum_{i=1}^{135}((i)) = \frac{1}{21504}\sum_{M:\Gamma_3/\Gamma_3(2)} M((131)\cap(132)).$$

$$\Sigma(1234,5678)^2 = 8\Sigma D^{1/2}/(12)(34)(56)(78) = \tfrac{8}{7}\Sigma D^{1/2}/(12)(36)(45)(78)$$

$$+\tfrac{4}{7}\Sigma\ (34)(56)D^{1/2}/(12)(78)(35)(46)(36)(45).$$

- p. 847 line 7 should be read as $+8\,\Sigma_{M:\theta/\Gamma_3(2)}\, M(((115))^2((135))^2/(21)^4(24)^4)$.

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