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On the birational automorphism groups of algebraic varieties

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Introduction

Let X be a projective variety over an algebraically closed field of characteristic zero. The purpose of this paper is to define the scheme Bir (X) of birational automorphisms of X and study its structure.

It was Weil [14] who introduced the notion of birational action of an algebraic group on a variety. Many authors up to today have worked on such algebraic groups (see Rosenlicht [12], for example).

On the other hand, since the construction of Hilbert schemes due to Grothendieck [2], a fruitful general philosophy has been established in the study of certain algebraic objects which appear in algebraic geometry – subschemes of a given scheme, vector bundles on a given variety, all curves of fixed genus, polarized varieties, etc. This suggests first to construct the universal space (scheme) which parametrizes all algebraic objects we are interested in. The second step is the investigation of this universal parameter space (and the universal family over it). This philosophy is often quite essential and useful, and has much advantage over just looking at each algebraic object separately. Basically we shall follow this principle.

In §1 of this paper, we naively define the scheme Bir(X) which parametrizes all the birational automorphisms of X, and treat some formal consequences in §2. It turns out, however, that the scheme Bir(X) has some nasty properties; it is not a group scheme in general; even when X and X' are birationally equivalent, Bir(X) and Bir(X') may not be isomorphic (see (2.9)).

One way to remedy this situation is to take suitable birational models. In §3, which is the heart of the present paper, we assume X to be a terminal minimal model (see §3 for the definition). Then Bir(X) is shown to behave well by the following theorems.

(3.3) THEOREM. *Let X be a terminal minimal model,* $\mathrm{Bir}(X)$ *and* $\mathrm{Aut}(X)$ *the schemes defined as in* §1. *Then*

1. $\mathrm{Bir}(X)$ *is a group scheme in such a way that the multiplication rule is described as follows: for any pair of k-rational points* $([f], [g])$ *of* $\mathrm{Bir}(X)$, *the product* $[f] \cdot [g]$ *coincides with* $[f \circ g]$.
2. $\mathrm{Aut}(X)$ *is an open and closed group subscheme of* $\mathrm{Bir}(X)$.
3. $\mathrm{Aut}^0(X) = \mathrm{Bir}^0(X)$ *and it is an Abelian variety.*

(3.7) THEOREM. *Let X be a terminal minimal model.*

1. (*universality of* $\mathrm{Bir}(X)$) *The group scheme* $\mathrm{Bir}(X)$ *acts birationally on X. Suppose we are given a group scheme which is locally of finite type and a birational action* $\sigma\colon G \times X \dashrightarrow X$ *of G on X. Then there exists a homomorphism of group schemes* $\varrho\colon G \to \mathrm{Bir}(X)$ *such that the action of G is induced from that of* $\mathrm{Bir}(X)$ *by* ϱ.
2. (*birational invariance of* $\mathrm{Bir}(X)$) *Let X' be another terminal minimal model which is birationally equivalent to X. If we fix a birational map* φ: $X \dashrightarrow X'$, *there exists a natural isomorphism of group schemes* $\varphi^{\#}$: $\mathrm{Bir}(X) \to \mathrm{Bir}(X')$ *such that* $\varphi^{\#}([f]) = [\varphi \circ f \circ \varphi^{-1}]$ *for* $[f] \in \mathrm{Bir}(X)$.

Under the additional assumption of goodness on X (see (3.10) for the definition), we have an explicit expression for $\dim \mathrm{Bir}(X)$:

(3.10) THEOREM (cf. COROLLARY (4.8) where the goodness is not assumed). *let X be a good terminal minimal model and* $\varphi = \Phi_{|mK_X|}\colon X \to Y$ *be the canonical fibering of X, where m is a suitable positive integer. Then*

$$\begin{aligned}\dim \mathrm{Bir}(X) &= \dim H^0(Y, R^1\varphi_* \mathcal{O}_X) \\ &= q(X) - q(Y)\end{aligned}$$

where $q(X) := \dim H^1(X, \mathcal{O}_X)$ *and* $q(Y) := \dim H^1(Y, \mathcal{O}_Y)$.

The terminologies shall be explained in §3.

It is conjectured that a variety has a terminal minimal model unless it is uniruled. Thus the results in §3 are hopefully applicable to all varieties which are not uniruled.

In §4, where we assume X to be a projective variety which is not ruled, the structure of the canonical homomorphism $\lambda\colon \mathrm{Bir}^0(X) \to \mathrm{Aut}(A)$ is studied. Here A denotes the Albanese variety of X (see §4 for the definition of λ). Thence we derive some results concerning $\dim \mathrm{Bir}^0(X)$ (see THEOREM (4.6) and its corollaries).

§1. Scheme structure on Bir(X)

Throughout this paper we consider schemes over an algebraically closed field k of characteristic zero.

(1.1) Let X be an algebraic variety. A birational automorphism of X is defined to be a birational map from X to X itself. If a birational automorphism of X is moreover biregular, it is called an automorphism of X.

Let Y be another algebraic variety. To give a rational map $f: X \dashrightarrow Y$ is equivalent to giving the graph $\Gamma_f \subset X \times Y$ of f. Γ_f is defined to be the closure in $X \times Y$ of the subset $\{(x, f(x)) \in X \times Y | x \in \operatorname{dom}(f)\}$, where $\operatorname{dom}(f)$ denotes the domain of the rational map f.

Given a rational map $f: X \dashrightarrow X$, f is a birational automorphism if and only if the graph $\Gamma_f \subset X \times X$ is a birational correspondence, i.e. letting $p_i: X \times X \to X (i = 1, 2)$ be the projection to the i-th factor, $p_i|\Gamma_f: \Gamma_f \to X$ are birational morphisms for $i = 1, 2$.

(1.2) Let X be a projective variety. Define the abstract groups:
- Aut(X) := the group of automorphisms of X;
- Bir(X) := the group of birational automorphisms of X.

Bir(X) contains Aut(X) as a subgroup.

Aut(X) has a natural structure of a group scheme (see (2.3)). In particular it is finite dimensional.

On the other hand, Bir(X) can be a bit strange; for example Bir($\mathbb{P}^n$) is "infinite dimensional" for $n \geqq 2$, thus far from being a group scheme. More precisely, for any positive integer m, the additive algebraic group G_a^m acts birationally and effectively on $\mathbb{P}^n$.

Nevertheless, the notion of "algebraic group contained in Bir(X)" has traditionally been defined and studied; an algebraic group G is said to be contained in Bir(X) if G acts on X birationally and effectively (see (3.6) for the definition of birational action).

(1.3) Assume that X is a curve or surface which is non-singular and projective.

If X is a curve, Bir(X) = Aut(X).

In case X is a surface which is not ruled, X has the uniquely determined minimal model $X_{\min}$. Then we have

$$\operatorname{Bir}(X) = \operatorname{Bir}(X_{\min}) = \operatorname{Aut}(A_{\min}),$$

which can be different from Aut(X). Since Aut($X_{\min}$) is a group scheme, Bir(X) can also be considered to be a group scheme by the above identification.

(1.4) Let X be a projective variety. With (1.2) and (1.3) in mind, we may well suppose that the abstract group Bir(X) inherits some algebraic structure, which is something like a group scheme (and in particular finite-dimensional) in case X is not uniruled.

Here remember the newly developing theory of minimal models for higher dimensional varieties (see Mori [10], Reid [11] and Kawamata [5]). The working hypothesis of the theory is the minimal model conjecture:

– Any variety X which is not uniruled has a terminal minimal model.

The meaning of the terminology will be explained in (3.2).

Even if we assume X to be a terminal minimal model, $\mathrm{Bir}(X) \supsetneqq \mathrm{Aut}(X)$ in general because of the existence of "elementary transformations". (Such examples can be found in Beauville [18], for instance.) However, we shall see that $\mathrm{Bir}(X)$ has a natural structure of a group scheme in case X is a terminal minimal model.

(1.5) We shall quickly review the theory of Hilbert schemes (see Grothendieck [3] for details).

Let X be a projective variety. For a locally Noetherian scheme S, let

$$\mathrm{Hilb}_X(S) := \{\text{closed subschemes } Y \subset X \times S, \text{ which are flat over } S.\}.$$

This defines a contravariant functor from the category of locally Noetherian schemes to the category of sets. The fundamental theorem asserts that this functor is representable by the Hilbert scheme $\mathrm{Hilb}(X)$, which is a disjoint union of at most countably many projective schemes. In other words, there exists a closed subscheme (called the universal family) $Y \hookrightarrow X \times \mathrm{Hilb}(X)$ which is flat over $\mathrm{Hilb}(X)$ and satisfies the following universal property: given a closed subscheme $Z \hookrightarrow X \times S$, flat over S, there is a uniquely determined morphism $u\colon S \to \mathrm{Hilb}(X)$ such that Z is induced by u, i.e., $Y \times_{\mathrm{Hilb}(X)} S = Z$ as closed subschemes of $X \times S$.

(1.6) Definition. Let X be a projective variety, $\mathrm{Hilb}(X \times X)$ be the Hilbert scheme of $X \times X$ and $p_i\colon X \times X \to X$ be the projection to the i-th factor for $i = 1, 2$. There exists the universal family $Z \hookrightarrow X \times X \times \mathrm{Hilb}(X \times X)$, flat over $\mathrm{Hilb}(X \times X)$:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "\pi"'] & X \times X \times \mathrm{Hilb}(X \times X) \arrow[d, "p_3"] \\ & \mathrm{Hilb}(X \times X) \end{tikzcd}$$

where p_3 denotes the projection to the third factor and π the restriction of p_3 to Z. For a point $t \in \mathrm{Hilb}(X \times X)$, define $k(t)$ to be the residue field of the local ring $\mathcal{O}_{\mathrm{Hilb}(X\times X),t}$ at t, Z_t to be the fiber of π at t and $X_{k(t)} := X \otimes_k k(t)$. Let

$$\mathrm{Bir}(X) := \{t \in \mathrm{Hilb}(X \times X) | Z_t \hookrightarrow X_{k(t)} \times_{k(t)} X_{k(t)} \text{ is a birational correspondence.}\};$$

$$\mathrm{Aut}(X) := \{t \in \mathrm{Hilb}(X \times X) | \, Z_t \hookrightarrow X_{k(t)} \times_{k(t)} X_{k(t)} \text{ is a biregular correspondence.}\}.$$

Here we define a closed subscheme $Z_t \subset X_{k(t)} \times_{k(t)} X_{k(t)}$ to be a birational correspondence (resp. biregular correspondence) if and only if Z_t is a geometrically integral subscheme and both projections $p_i|Z_t$: $Z_t \to X_{k(t)}$ ($i = 1, 2$) are birational morphisms (resp. isomorphisms).

(1.7) PROPOSITION. *Under the above definitions,* Bir(X) *and* Aut(X) *are both open subsets of* Hilb($X \times X$).

Proof. Since π is flat the set

$$S := \{t \in \mathrm{Hilb}(X \times X) | Z_t \hookrightarrow X_{k(t)} \times_{k(t)} X_{k(t)} \text{ is a subscheme which is geometrically integral and } \dim Z_t = \dim X_{k(t)}.\}$$

is open in Hilb($X \times X$) by Grothendieck [2], IV.

Take an ample line bundle L on X. For a point $t \in S$, the first projection $p_{1,t} := p_1|_{Z_t}$: $Z_t \to X_{k(t)}$ is surjective if and only if the intersection number $(p_{1,t}^* L)^n > 0$ where $n = \dim X$. Under this condition,

$$\deg\,(p_{1,t}\colon Z_t \to X_{k(t)}) \;=\; (p_{1,t}^* L)^n/(L)^n.$$

Thus $p_{1,t}$: $Z_t \to X_{k(t)}$ is a birational morphism if and only if $(p_{1,t}^* L)^n = (L)^n$. The intersection number $(p_{1,t}^* L)^n$ is constant on each connected component of S. Therefore the set

$$S_1 := \{t \in S | p_{1,t}\colon Z_t \to X_{k(t)} \text{ is a birational morphism.}\}$$

is an open subset of S. Arguing similarly for p_2, we see that the set

$$S_2 := \{t \in S | p_{2,t}\colon Z_t \to X_{k(t)} \text{ is a birational morphism.}\}$$

is also open in S. Since $\mathrm{Bir}(X) = S_1 \cap S_2$, we conclude that Bir($X$) is an open subset of Hilb($X \times X$).

The claim for Aut(X) follows from Grothendieck [2], III.

(1.8) Definition. Let X be a projective variety. The scheme structures of $\mathrm{Bir}(X)$ and $\mathrm{Aut}(X)$ are defined as open subschemes of $\mathrm{Hilb}(X \times X)$. They are called the scheme of birational automorphisms and the scheme of automorphisms, respectively. The restriction to $\mathrm{Bir}(X)$ of the universal family over $\mathrm{Hilb}(X \times X)$ will be denoted by Γ:

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[dr, "\pi"'] & X \times X \times \mathrm{Bir}(X) \arrow[d, "p_3"] \\ & \mathrm{Bir}(X) \end{tikzcd}$$

There is a 1–1 correspondence between the set of k-rational points of $\mathrm{Bir}(X)$ (resp. $\mathrm{Aut}(X)$) and the set of birational automorphisms (resp. automorphisms) of X.

§2. First properties of Bir(X)

In this section, we let X be a projective variety unless otherwise stated.

(2.1) Definition. Let S be a locally Noetherian scheme. A *flat family of birational automorphisms* (*resp. automorphisms*) of X over S is a closed subscheme $Z \hookrightarrow X \times X \times S$, flat over S and such that for all points $t \in S$, the fibres Z_t over t are birational correspondences (resp. biregular correspondences).

(2.2) Proposition (universality of $\mathrm{Bir}(X)$, resp. $\mathrm{Aut}(X)$). *Given a flat family of birational automorphisms* (*resp. automorphisms*) τ: $Z \hookrightarrow X \times X \times S \to S$ *over a locally Noetherian scheme S, there exists a uniquely determined morphism* u: $S \to \mathrm{Bir}(X)$ (*resp.* u: $S \to \mathrm{Aut}(X)$) *such that* τ: $Z \to S$ *is induced from the universal family by* u.

Proof. Clear from the definition of $\mathrm{Bir}(X)$ (resp. $\mathrm{Aut}(X)$) and the universality of $\mathrm{Hilb}(X \times X)$.

(2.3) Proposition. $\mathrm{Aut}(X)$ *has a natural structure of a group scheme.*

Proof. Standard formal arguments using the universality of $\mathrm{Aut}(X)$ prove the assertion (cf. (2.5)).

REMARK. In case char $k = 0$, $\mathrm{Aut}(X)$ is smooth.

(2.4) DEFINITION. For a birational automorphism $f\colon X \dashrightarrow X$, denote by $[f]$ the corresponding k-rational point of $\mathrm{Bir}(X)$.

The irreducible component of $\mathrm{Bir}(X)$ (resp. $\mathrm{Aut}(X)$) containing the point $[id]$, with the reduced subscheme structure, shall be denoted by $\mathrm{Bir}^0(X)$ (resp. $\mathrm{Aut}^0(X)$). We obtain four schemes:

$$\begin{array}{ccc} \mathrm{Bir}(X) & \hookleftarrow & \mathrm{Aut}(X) \\ \uparrow\cup & & \uparrow\cup \\ \mathrm{Bir}^0(X) & \hookleftarrow & \mathrm{Aut}^0(X) \end{array}$$

$\mathrm{Aut}^0(X)$ is a connected algebraic group and $\mathrm{Bir}^0(X)$ contains $\mathrm{Aut}^0(X)$ as an open subscheme.

(2.5) PROPOSITION. *Let X be a projective variety. The group scheme* $\mathrm{Aut}(X)$ *acts naturally on* $\mathrm{Bir}(X)$ *from the left. More precisely, there exists a morphism of schemes*

$$\sigma\colon \mathrm{Aut}(X) \times \mathrm{Bir}(X) \longrightarrow \mathrm{Bir}(X)$$

such that

$$\sigma([\alpha], [f]) = [\alpha \circ f]$$

for k-rational points $[\alpha] \in \mathrm{Aut}(X)$ *and* $[f] \in \mathrm{Bir}(X)$.

REMARK. We have a similar action from the right.

Proof. We use the notation in (1.8). The universal family of automorphisms of X over $\mathrm{Aut}(X)$ gives rise to an isomorphism

$$\varphi\colon X \times \mathrm{Aut}(X) \xrightarrow{\sim} X \times \mathrm{Aut}(X)$$

over $\mathrm{Aut}(X)$.

Consider the following diagram:

$$\begin{tikzcd}
& \tilde{\Gamma} \arrow[r] & X \times X \times \mathrm{Aut}(X) \times \mathrm{Bir}(X) \\
\mathrm{Aut}(X) \times \Gamma \arrow[ur, "\bar{\varphi}'", "\cong"'] \arrow[r] \arrow[drr, "\pi'"'] & & X \times X \times \mathrm{Aut}(X) \times \mathrm{Bir}(X) \arrow[u, "\bar{\varphi}"', "\cong"] \arrow[d, "p_{34}"] \\
& & \mathrm{Aut}(X) \times \mathrm{Bir}(X)
\end{tikzcd}$$

where

p_{34} := the natural projection to the factor $\mathrm{Aut}(X) \times \mathrm{Bir}(X)$,
π' := $id_{\mathrm{Aut}(X)} \times \pi$,
i := $s \circ j$ where j: $\mathrm{Aut}(X) \times \Gamma \hookrightarrow \mathrm{Aut}(X) \times X \times X \times \mathrm{Bir}(X)$ is the product of $id_{\mathrm{Aut}(X)}$ and the closed immersion $\Gamma \hookrightarrow X \times X \times \mathrm{Bir}(X)$, and s is the isomorphism $\mathrm{Aut}(X) \times X \times X \times \mathrm{Bir}(X) \xrightarrow{\sim} X \times X \times \mathrm{Aut}(X) \times \mathrm{Bir}(X)$ which sends the closed point $([\alpha], x, y, [f])$ to $(x, y, [\alpha], [f])$,
$\bar{\varphi}$:= $\mathrm{id}_X \times \varphi \times id_{\mathrm{Bir}(X)}$,
$\tilde{\Gamma}$:= the closed subscheme of $X \times X \times \mathrm{Aut}(X) \times \mathrm{Bir}(X)$ which is isomorphic to $\mathrm{Aut}(X) \times \Gamma$ via $\bar{\varphi}$,
$\bar{\varphi}'$:= the induced isomprohism $\mathrm{Aut}(X) \times \Gamma \to \tilde{\Gamma}$.

Then

$$\pi' \circ \bar{\varphi}'^{-1} \colon \tilde{\Gamma} \hookrightarrow X \times X \times \mathrm{Aut}(X) \times \mathrm{Bir}(X) \to \mathrm{Aut}(X) \times \mathrm{Bir}(X)$$

defines a flat family of birational automorphisms.

By (2.2), we obtain a morphism σ: $\mathrm{Aut}(X) \times \mathrm{Bir}(X) \to \mathrm{Bir}(X)$ such that the family $\pi' \cdot \bar{\varphi}'^{-1}$ is induced by σ from the universal family π. We see immediately that

$$\sigma([\sigma], [f]) = [\alpha \circ f]$$

for $[\alpha] \in \mathrm{Aut}(X)$ and $[f] \in \mathrm{Bir}(X)$ from the construction.

(2.6) Definition. Let Z and Z' be two reduced schemes locally of finite type over k. A rational map from Z to Z' is defined to be a morphism f: $U \to Z'$ where U is an open subset of Z containing the generic point of each irreducible component of Z. Note that if f: $Z \dashrightarrow Z'$ and g: $Z' \dashrightarrow Z''$ are rational maps and f is dominating, the composition $g \circ f$: $Z \dashrightarrow Z''$ can be defined.

(2.7) PROPOSITION. *Let X and X' be two projective varieties which are birational. Fixing a birational map φ: $X \dashrightarrow X'$, we can construct a rational map*

$$\varphi^{\#}: \mathrm{Bir}(X)_{\mathrm{red}} \dashrightarrow \mathrm{Bir}(X')_{\mathrm{red}}$$

satisfying

$$\varphi^{\#}([f]) = [\varphi \circ f \circ \varphi^{-1}]$$

for any k-rational point $[f] \in U$, U being an open dense subset of $\mathrm{Bir}(X)_{\mathrm{red}}$. *Here the symbol red denotes the reduced part.*

Proof. Denote by π: $\Gamma \hookrightarrow X \times X \times \mathrm{Bir}(X) \to \mathrm{Bir}(X)$ the universal family. Take an irreducible component C or $\mathrm{Bir}(X)$ and consider it as a reduced subscheme. Restricting the base of π to C, we have a flat family π_C: $\Gamma_C \hookrightarrow X \times X \times C \to C$ of birational automorphisms of X.

Let π'_C: $\Gamma'_C \hookrightarrow X' \times X' \times C \to C$ be the strict transform of Γ_C. Over some open subset U of C, π'_C is flat family of birational automorphisms of X'. By the universal property of $\mathrm{Bir}(X')$, a morphism u: $U \to \mathrm{Bir}(X)$ is induced. If C varies over all the irreducible components of $\mathrm{Bir}(X)_{\mathrm{red}}$, we obtain the desired rational map $\varphi^{\#}: \mathrm{Bir}(X)_{\mathrm{red}} \dashrightarrow \mathrm{Bir}(X')_{\mathrm{red}}$.

(2.8) COROLLARY. *If X and X' are birationally equivalent projective varieties,* $\dim \mathrm{Bir}(X) = \dim \mathrm{Bir}(X')$ (*possibly infinite*).

(2.9) REMARK. It may actually occur that X and X' are birational while $\mathrm{Bir}(X)$ and $\mathrm{Bir}(X')$ are not isomorphic.

For example we let A be an abelian variety of dimension $n \geqq 2$, and μ: $\tilde{A} \to A$ the blow-up of a point $p \in A$. Then $\dim \mathrm{Bir}^0(A) = n$ but $\dim \mathrm{Bir}^0(\tilde{A}) = 0$, thus $\mathrm{Bir}(A)$ and $\mathrm{Bir}(\tilde{A})$ are not isomorphic.

$\mathrm{Bir}(\tilde{A})$ is not a group scheme. In fact, $\dim \mathrm{Bir}^0(\tilde{A}) < \dim \mathrm{Bir}(\tilde{A}) = n$, hence $\mathrm{Bir}(\tilde{A})$ is not even equi-dimensional.

(2.10) REMARK. Let X be a projective variety over k. Demazure [15] defined a functor $\mathbb{P}\mathit{saut}(X)$ as follows:

for any locally Noetherian scheme S over k,

$\mathbb{P}\mathit{saut}(X)(S) := \{$rational maps $f\colon X \times S \to X \times S$ with inverse rational maps such that the domain of f intersects with the fiber X_s of any point $s \in S.\}$.

Contrary to this we are restricting our attention to a subfunctor:

$\mathscr{B}\mathit{ir}(X)(S) := \{$flat families of birational automorphisms of X over $S\}$.

The merit of our approach is that the subfunctor is representable by the scheme Bir(X), giving us a clear picture of what this functor looks like. The author would like to express his gratitude to Professor Y. Namikawa for the information about Demazure's works.

§3. Bir(X) of a terminal minimal model X

(3.1). Let X be a normal projective variety of dimension d. We denote by $Z_{d-1}(X)$ the group of Weil divisors of X, and by Div(X) the group of Cartier divisors of X.

We have a natural injection $\mathrm{Div}(X) \to Z_{d-1}(X)$. An element $D \in Z_{d-1}(X) \otimes \mathbb{Q}$ is called a $\mathbb{Q}$-divisor. A $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-Cartier if D is in the image of the map $\mathrm{Div}(X) \otimes \mathbb{Q} \to Z_{d-1}(X) \otimes \mathbb{Q}$.

There is a 1–1 correspondence between the isomorphism classes of reflexive sheaves of rank one on X and the linear equivalence classes of Weil divisors on X. For a Weil divisor D, the corresponding reflexive sheaf is denoted by $\mathcal{O}_X(D)$. By K_X we mean the canonical divisor of X, that is, the Weil divisor satisfying $\mathcal{O}_X(K_X) = (\Omega_X^d)^{**}$, the right hand side denoting the double dual of Ω_X^d. We shall also write $\omega_X^{[s]}$ instead of $\mathcal{O}_X(sK_X)$.

(3.2) Definition (Reid [11]). Let X be a normal projective variety, and $f\colon Y \to X$ be a resolution of singularities of X.

X is said to be a *terminal minimal model* if the following three conditions are satisfied:

1. The Weil divisor K_X is $\mathbb{Q}$-Cartier, i.e. for some positive integer r, $\omega_X^{[r]} = \mathcal{O}_X(rK_X)$ is an invertible sheaf.
2. For an integer r satisfying (1), writing

$$rK_Y = f^*(rK_X) + \sum_{i=1}^{N} a_i E_i$$

where E_i ($i = 1, \ldots, N$) vary all the prime divisors on Y exceptional with respect to f, we have $a_i > 0$ for all i.

3. The Cartier division rK_X is nef, in other words, the intersection number $(\omega_X^{[r]} \cdot C)$ is non-negative for any irreducible curve C in X.

This definition is independent of the choice of a resolution $f\colon Y \to X$.

We note that a terminal minimal model X is not uniruled i.e., there does not exist a dominating rational map $\varphi\colon Y \to X$ where Y is a ruled variety such that $\dim X = \dim Y$.

(3.3) THEOREM. *Let X be a terminal minimal model,* $\mathrm{Bir}(X)$ *and* $\mathrm{Aut}(X)$ *the schemes defined as in* §1. *Then*

1. $\mathrm{Bir}(X)$ *is a group scheme in such a way that the multiplication rule is described as follows: for any pair of k-rational points* $([f], [g])$ *of* $\mathrm{Bir}(X)$, *the product* $[f] \cdot [g]$ *coincides with* $[f \circ g]$.
2. $\mathrm{Aut}(X)$ *is an open and closed group subscheme of* $\mathrm{Bir}(X)$.
3. $\mathrm{Aut}^0(X) = \mathrm{Bir}^0(X)$ *and it is an Abelian variety.*

We first prove a lemma.

(3.4) LEMMA. *A birational automorphism $f\colon X \dashrightarrow X$ of a terminal minimal model X is an isomorphism in codimension* 1.

Proof. Resolve the indeterminacy of f and the singularities of X by a proper birational morphism $\alpha\colon X' \to X$. We put $\beta := f \circ \alpha$ and obtain the following diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] & & X \\ & X' \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

Since X has only terminal singularities, we have

$$K_{X'} = \alpha^* K_X + E_\alpha; \tag{3.4.1}$$

$$K_{X'} = \beta^* K_X + E_\beta \tag{3.4.2}$$

where E_α (resp. E_β) is an effective $\mathbb{Q}$-divisor such that $\mathrm{Supp}\,(E_\alpha)$ (resp. $\mathrm{Supp}\,(E_\beta)$) coincides with the union of all the exceptional divisors of α (resp. β).

By Fujita's theory of Zariski decompositions (see Fujita [1]), (3.4.1) and (3.4.2) both give the Zariski decomposition of $K_{X'}$. Thus $\alpha^* K_X = \beta^* K_X$ in $\mathrm{Div}\,(X) \otimes \mathbb{Q}$ and $E_\alpha = E_\beta$. The latter equality implies that f is an isomorphism in codimension 1.

(3.5) Proof of Theorem (3.3). Take an arbitrary birational automorphism f of X. We shall evaluate $\dim T_{[f]}$ where $[f]$ denotes the point of $\mathrm{Bir}(X)$ corresponding to f, and $T_{[f]}$ the Zariski tangent space of $\mathrm{Bir}(X)$ at $[f]$.

Since f is an isomorphism in codimension 1, there are Zariski open subsets X_0 and X_0' of X such that $\operatorname{codim}(X \backslash X_0) \geqq 2$, $\operatorname{codim}(X \backslash X_0') \geqq 2$ and f induces an isomorphism $f_0 : X_0 \xrightarrow{\sim} X_0'$.

Let $\Gamma_f \subset X \times X$ and $\Gamma_{f_0} \subset X_0 \times X_0'$ be the graphs of f and f_0 respectively and $\Delta_{X_0} \subset X_0 \times X_0$ be the diagonal. Then we get:

$$\begin{tikzcd}
X & \Gamma_f \arrow[l, "p_1"'] \arrow[r, hook] & X \times X \\
X_0 \arrow[u, hook] & \Gamma_{f_0} \arrow[l, "p_1"'] \arrow[r, hook] \arrow[u, hook] & X_0 \times X_0' \arrow[u, hook] \\
X_0 \arrow[u, equal] & \Delta_{X_0} \arrow[l, "p_1"'] \arrow[r, hook] \arrow[u, "\cong"] & X_0 \times X_0 . \arrow[u, "\cong"', "id_{X_0} \times f_0"]
\end{tikzcd}$$

Here p_1 denotes the first projection.

Let I be the defining ideal sheaf of Γ_f in $X \times X$. Denote by $N_{\Gamma_f/X \times X}$ the normal sheaf of Γ_f in $X \times X$, which is defined to be the sheaf $\mathrm{Hom}_{\mathcal{O}_{\Gamma_f}}(I/I^2, \mathcal{O}_{\Gamma_f})$. Then we have

$$N_{\Gamma_f/X \times X} | \Gamma_{f_0} = N_{\Gamma_{f_0}/X_0 \times X_0'} = p_1^* \Theta_{X_0},$$

where Θ_{X_0} indicates the tangent sheaf of X_0.

Consider the composition of the maps:

$$H^0(\Gamma_f, N_{\Gamma_f/X \times X}) \xrightarrow{r} H^0(\Gamma_{f_0}, N_{\Gamma_{f_0}/X_0 \times X_0'}) \cong H^0(X_0, \Theta_{X_0})$$
$$\cong H^0(X, \Theta_X),$$

where r is the restriction. The last isomorphism holds because Θ_X is a reflexive sheaf as the dual of the coherent sheaf Ω_X^1 (see Hartshorne [19], for example). Since $N_{\Gamma_f/X \times X}$ is a torsion-free sheaf, r is injective. Using $T_{[f]} \cong H^0(\Gamma_f, N_{\Gamma_f/X \times X})$, we get

$$\dim T_{[f]} \leqq \dim H^0(X, \Theta_X) = \dim \mathrm{Aut}(X).$$

On the other hand, $\dim_{[f]} \mathrm{Bir}(X) \geqq \dim \mathrm{Aut}(X)$ since $\mathrm{Aut}(X)$ acts on $\mathrm{Bir}(X)$ freely ((2.5)). Thus $\mathrm{Bir}(X)$ is non-singular at $[f]$ and $\dim_{[f]} \mathrm{Bir}(X) = \dim \mathrm{Aut}(X)$.

(3) follows from $\mathrm{Bir}^0(X) \supset \mathrm{Aut}^0(X)$ and the fact that $\mathrm{Aut}^0(X)$ is an Abelian variety since X is not ruled (see Rosenlicht [12]).

For any point $[f] \in \mathrm{Bir}(X)$, the dimensions of $[f] \cdot \mathrm{Aut}^0(X)$ and $\mathrm{Aut}^0(X) \cdot [f]$ are both equal to $\dim \mathrm{Aut}^0(X) = \dim_{[f]} \mathrm{Bir}(X)$. Thus the two subvarieties must coincide and it is the connected component of $[f]$. Hence we have three isomorphisms:

$$\begin{aligned} L_f&\colon \mathrm{Aut}^0(X) \to [f] \cdot \mathrm{Aut}^0(X), \quad L_f([\alpha]) = [f \circ \alpha];\\ R_f&\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(X) \cdot [f], \quad R_f([\alpha]) = [\alpha \circ f];\\ I_f&:= (L_f)^{-1} \circ R_f\colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(X), \quad I_f([\alpha]) = [f^{-1} \circ \alpha \circ f]. \end{aligned}$$

As schemes we can write

$$\mathrm{Bir}(X) = \coprod_{[\bar{f}] \in \mathrm{Bir}(X)/\mathrm{Aut}^0(X)} [f] \cdot \mathrm{Aut}^0(X);$$

$$\mathrm{Aut}(X) = \coprod_{[\bar{f}] \in \mathrm{Aut}(X)/\mathrm{Aut}^0(X)} [f] \cdot \mathrm{Aut}^0(X),$$

where $\coprod$ denotes disjoint union of schemes, and $[\bar{f}]$ denotes the left coset of $[f]$. Hence $\mathrm{Bir}(X)$ contains $\mathrm{Aut}(X)$ as an open and closed subscheme.

We now show (1). Taken any two components $[f] \cdot \mathrm{Aut}^0(X)$ and $[g] \cdot \mathrm{Aut}^0(X)$ of $\mathrm{Bir}(X)$.

Define μ: $([f] \cdot \mathrm{Aut}^0(X)) \times ([g] \cdot \mathrm{Aut}^0(X)) \to [f \circ g] \cdot \mathrm{Aut}^0(X)$ as the composition of

$$\begin{aligned} ([f] \cdot \mathrm{Aut}^0(X)) \times ([g] \cdot \mathrm{Aut}^0(X)) &\xrightarrow{(L_f)^{-1} \times (L_g)^{-1}} \mathrm{Aut}^0(X) \times \mathrm{Aut}^0(X)\\ &\xrightarrow{I_g \times id} \mathrm{Aut}^0(X) \times \mathrm{Aut}^0(X)\\ &\xrightarrow{\mu^0} \mathrm{Aut}^0(X)\\ &\xrightarrow{L_{f \cdot g}} [f \circ g] \cdot \mathrm{Aut}^0(X), \end{aligned}$$

where μ^0: $\mathrm{Aut}^0(X) \times \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(X)$ denotes the multiplication of the group scheme $\mathrm{Aut}^0(X)$. Glueing μ together for all components, we define a morphism μ: $\mathrm{Bir}(X) \times \mathrm{Bir}(X) \to \mathrm{Bir}(X)$ which maps a point $([f], [g])$ to $[f \circ g]$, the point corresponding to the composition of f and g as birational maps.

We also define the inversion ι: $\mathrm{Bir}(X) \to \mathrm{Bir}(X)$. This is obtained by glueing the following morphisms for all components $[f] \cdot \mathrm{Aut}^0(X)$:

$$[f] \cdot \mathrm{Aut}^0(X) \xrightarrow{(L_f)^{-1}} \mathrm{Aut}^0(X)$$

$$\xrightarrow{\iota^0} \mathrm{Aut}^0(X)$$

$$\xrightarrow{R_{f^{-1}}} \mathrm{Aut}^0(X) \cdot [f^{-1}].$$

Here ι^0: $\mathrm{Aut}^0(X) \to \mathrm{Aut}^0(X)$ indicates the inversion of $\mathrm{Aut}^0(X)$. We denote by e: Spec $k \to \mathrm{Bir}(X)$ the identity as usual. Then (μ, ι, e) naturally satisfies the axioms of group scheme. Thus (1) is proved.

(3.6) DEFINITION (Weil [14]). Let G be a group scheme which is locally of finite type, and X be a projective variety. We say that G acts on X birationally (from the left) if a dominating rational map σ: $G \times X \dashrightarrow X$ is given and satisfies the law of action (from the left) at the generic point of each irreducible component of G. To fix the notion, we only consider left actions.

We can show that, using THEOREM (3.3), for a terminal minimal model X, the group scheme $\mathrm{Bir}(X)$ acts birationally on X.

(3.7) THEOREM. *Let X be a terminal minimal model.*

1. (*universality of* $\mathrm{Bir}(X)$) *Suppose we are given a group scheme which is locally of finite type and a birational action σ: $G \times X \dashrightarrow X$ of G on X. Then there exists a homomorphism of group schemes ϱ: $G \to \mathrm{Bir}(X)$ such that the action of G is induced from that of* $\mathrm{Bir}(X)$ *by ϱ.*
2. (*birational invariance of* $\mathrm{Bir}(X)$) *Let X' be another terminal minimal model which is birationally equivalent to X. If we fix a birational map φ: $X \dashrightarrow X'$, there exists a natural isomorphism of group schemes $\varphi^{\#}$:* $\mathrm{Bir}(X) \to \mathrm{Bir}(X')$ *such that* $\varphi^{\#}([f]) = [\varphi \circ f \circ \varphi^{-1}]$ *for* $[f] \in \mathrm{Bir}(X)$.

Proof

1. Note that for all $g \in G$, the domain of σ intersects with $\{g\} \times X$. Thus we have a natural morphism of abstract groups ϱ: $G \to \mathrm{Bir}(X)$. We shall show that this is in fact a homomorphism of group schemes.

Let Γ_σ be the graph of σ. We have a family of birational automorphisms of X:

$$\begin{tikzcd} \Gamma_\sigma \arrow[r, hook] \arrow[dr, "\pi"'] & X \times X \times G \arrow[d, "p_3"] \\ & G \end{tikzcd}$$

where π denotes the induced projection.

Restricting π to some open subset U of the connected component G^0 of G containing $[id]$, we have a flat family $\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U$.

By the definition of $\mathrm{Bir}(X)$, we have an induced morphism $\varrho'\colon U \to \mathrm{Bir}(X)$. Taking any point $g \in G$, we consider the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, "\varrho'"] \arrow[d, "L_g", "\cong"'] & \mathrm{Bir}(X) \arrow[d, "L_{\varrho(g)}", "\cong"'] \\ {[g] \cdot U} \arrow[r, "\varrho'_g"] & \mathrm{Bir}(X) \end{tikzcd}$$

where ϱ'_g is the induced morphism from ϱ' via the isomorphisms L_g and $L_{\varrho(g)}$. This means that ϱ is a morphism on $[g] \cdot U$. Since $[g] \cdot U$ covers G as g varies in G, we see that ϱ is a morphism from G to $\mathrm{Bir}(X)$.

2. $\mathrm{Bir}(X)$ acts on X birationally. Via φ, this action gives rise to a birational action of $\mathrm{Bir}(X)$ on X'. Using (1), we obtain a naturally induced homomorphism of group schemes $\varphi^{\#}\colon \mathrm{Bir}(X) \to \mathrm{Bir}(X')$. On the other hand, we also get a homomorphism $(\varphi^{-1})^{\#}\colon \mathrm{Bir}(X') \to \mathrm{Bir}(X)$ which is induced similarly by $\varphi^{-1}\colon X \dashrightarrow X$. It is clear that $\varphi^{\#}$ and $(\varphi^{-1})^{\#}$ are inverses to each other.

(3.8) Corollary. *A birational action of a connected algebraic group G on a terminal minimal model X is in fact a biregular action.*

Proof. The image of the induced homomorphism $\varrho\colon G \to \mathrm{Bir}(X)$ is contained in $\mathrm{Bir}^0(X) = \mathrm{Aut}^0(X)$.

(3.9) Definition. A terminal minimal model X is called *good* if K_X is semi-ample, i.e., mK_X is a Cartier divisor which is generated by global sections for some positive integer m.

In this case we can talk of the canonical fibering of X which is uniquely determined as follows: take a positive integer m such that the linear system $|mK_X|$ is base-point free and the associated morphism $\Phi_{|mK_X|}\colon X \to \mathbb{P}^{\dim|mK_X|}$ subject to the following conditions:

1. The image Y of $\Phi_{|mK_X|}$ is a normal variety;
2. The induced morphism $\varphi: X \to Y$ is a fiber space, i.e., a general fiber of φ is irreducible.

The fiber space $\varphi: X \to Y$ thus obtained is independent of the choice of m so long as the two conditions above are satisfied. This is called the Iitaka fibration or the canonical fibering of the good terminal minimal model X. Note that dim $Y = \kappa(X)$ (— the Kodaira dimension of X) and Y has only rational singularities (see Kollar [8]). For the definition of Kodaira dimension, we refer to Iitaka [4].

REMARK. The notion of goodness of a (not necessarily terminal) minimal model was introduced by Kawamata [6]. He proved that the goodness is equivalent to the condition $\kappa(X) = \nu(X)$ (:= the numerical Kodaira dimension). He conjectured that a minimal model is in fact good. It can be proven that if X is a good minimal model and X' is another minimal model birationally equivalent to X, then X' is also good.

(3.10) THEOREM (cf. COROLLARY (4.8) where the goodness is not assumed). *Let X be a good terminal minimal model and $\varphi = \Phi_{|mK_X|}: X \to Y$ be the canonical fibering of X, where m is a suitable positive integer. Then*

$$\begin{aligned} \dim \mathrm{Bir}(X) &= \dim H^0(Y, R^1\varphi_*\mathcal{O}_X) \\ &= q(X) - q(Y) \end{aligned}$$

where $q(X) := \dim H^1(X, \mathcal{O}_X)$ *and* $q(Y) := \dim H^1(Y, \mathcal{O}_Y)$.

Proof. Every birational automorphism of X gives rise to an automorphism of $H^0(X, \mathcal{O}_X(mK_X))$. We thus have a morphism of group schemes:

$$\varrho: \mathrm{Bir}(X) \to GL(H^0(X, \mathcal{O}_X(mK_X)).$$

Since the component $\mathrm{Bir}^0(X)$ is an Abelian variety and $GL(H^0(X, \mathcal{O}_X(mK_X))$ is a linear algebraic group, ϱ is constant on$\mathrm{Bir}^0(X) = \mathrm{Aut}^0(X)$. Thus we have

$$\dim \mathrm{Bir}(X) = \dim \mathrm{Aut}^0_Y(X).$$

Here $\mathrm{Aut}^0_Y(X)$ denotes the identity component of the group scheme $\mathrm{Aut}_Y(X)$, which is the scheme parametrizing all the birational automorphism of X over Y. $\mathrm{Aut}_Y(X)$ can be defined similarly as $\mathrm{Aut}(X)$. The tangent space of $\mathrm{Aut}^0_Y(X)$ at the point $[id]$ id $H^0(X, \Theta_{X/Y})$. By Kawamata [7], there exists an isomorphism $\varphi_*\Theta_{X/Y} \xrightarrow{\sim} R^1\varphi_*\mathcal{O}_X$. Hence

$$\dim \mathrm{Bir}(X) \ = \ \dim H^0(Y, R^1\varphi_*\mathcal{O}_X).$$

On the other hand we can prove

$$\mathbb{R}\varphi_*\mathcal{O}_X \simeq \sum_i R^i\varphi_*\mathcal{O}_X[-i] \tag{3.10.1}$$

where $\simeq$ denotes isomorphism in the derived category.

To show this, we take a general member $D \in |mK_X|$ and construct the associated cyclic cover

$$\pi\colon X' := \mathrm{Spec}\ \oplus_{j=0}^{m-1}\mathcal{O}_X(-jK_X) \to X$$

where we make $\oplus_{j=0}^{m-1}\mathcal{O}(-jK_X)$ into an $\mathcal{O}_X$-algebra by the morphism $\mathcal{O}_X(-mK_X) \to \mathcal{O}_X$ induced by D. Kawamata [6] proved that X' has only rational Gorenstein singularities. Thus we have

$$\mathbb{R}(\varphi \cdot \pi)_*\omega_{X'} \ = \ \sum_i R^i(\varphi \cdot \pi)_*\omega_{X'}[-i] \tag{3.10.2}$$

by Kollár [8]. By Grothendieck duality we see

$$\pi_*\omega_{X'} \ = \ \mathrm{Hom}(\pi_*\mathcal{O}_{X'}, \omega_X) \ = \ \oplus_{j=0}^{m-1}\omega_X^{[j+1]}.$$

Hence $\mathbb{R}(\varphi \cdot \pi)_*\omega_{X'} = \oplus_{j=0}^{m-1}\mathbb{R}\varphi_*(\omega_X^{[j+1]})$. Taking the direct summand corresponding to $j = m - 1$ and noting that the invertible sheaf $\omega_X^{[m]}$ is the pull-back of a line bundle on Y, we deduce (3.10.1) from (3.10.2). Thus

$$H^1(X, \mathcal{O}_X) \cong H^1(Y, \mathcal{O}_Y) \oplus H^0(Y, R^1\varphi_*\mathcal{O}_X).$$

(3.11) Corollary (cf. Matsumura [9]). *Let X as above and G be a group scheme locally of finite type acting birationally and effectively on X. Then*

$$\dim G \leqq q(X) - q(Y).$$

Proof. There is an induced injective homomorphism of group schemes ϱ: $G \to \mathrm{Bir}(X)$ by (3.7).

§4 Relation with Albanese maps

We recall a theorem of Nishi and Matsumura:

(4.1) THEOREM (Matsumura [9]). *Let G be an Abelian variety and X be a projective variety. Assume G acts birationally and effectively on X by a rational map σ: $G \times X \dashrightarrow X$.*

Denote by α: $X \to A$ the Albanese map of X. Then there exists an isogeny (onto the image) λ: $G \to A$ such that

$$\alpha(g \cdot x) = \lambda(g) + \alpha(x) \quad (x \in X, \quad g \in G)$$

whenever both sides are defined.

(4.2) Let X be a projective variety and H be a line bundle on X. We define a homomorphism

$$\varrho_H\colon H^0(X, \Theta_X) \to H^1(X, \mathcal{O}_X)$$

as follows.

Let $\bar{X} = X \otimes_k k[\varepsilon]$ and $\bar{H} = H \otimes_k k[\varepsilon]$ where $k[\varepsilon]$ indicates the ring of dual numbers. Then $\theta \in H^0(X, \Theta_X)$ gives rise to an automorphism $\bar{\theta}$ of $\bar{X}$ over $k[\varepsilon]$ which is the identity on X. The invertible sheaf $\bar{\theta}^*\bar{H} \otimes \bar{H}^{-1}$ on $\bar{X}$ is trivial on X and thus defines an element $\varrho_H(\theta) \in H^1(X, \mathcal{O}_X)$.

We note that ϱ_H is the derivative at $[id]$ of the morphism φ_H: $\mathrm{Aut}^0(X) \to \mathrm{Pic}^0(X)$, which is defined by $\varphi_H([f]) = [f^*H \otimes H^{-1}]$.

(4.3). In the following throughout this section, X is a projective variety, which is not ruled. We denote by α: $X \to A$ the Albanese map, Θ_X and Θ_A the tangent sheaves of X and A, respectively. Fix, once and for all, an ample line bundle L on A.

The algebraic group $\mathrm{Aut}^0(X)$ is an Abelian variety (Rosenlicht [12]) and coincides with $\mathrm{Bir}^0(X)$; thus by Theorem (4.1), there exists an homomorphism of algebraic groups λ: $\mathrm{Aut}^0(X) \to A$ which is an isogeny to its image. We note that φ_H: $\mathrm{Aut}^0(X) \to \mathrm{Pic}^0(X)$ is a homomorphism of group schemes for any $H \in \mathrm{Pic}(X)$ (cf. [17] p. 117, Corollary 6.4).

(4.4) PROPOSITION *Consider the following diagram*:

$$\begin{tikzcd} \mathrm{Aut}^0(X) \arrow[r, "\lambda"] \arrow[d, "\varphi_{\alpha^*L}"'] & A \arrow[d, "\varphi_L"'] \\ \mathrm{Pic}^0(X) & \mathrm{Pic}^0(A), \arrow[l, "\alpha^*"'] \end{tikzcd}$$

where α^* is a morphism canonically induced from α.

This diagram is commutative, φ_L is an isogeny, and λ and α^* are isogenies onto the images. In particular, $\varphi_{\alpha^* L}$ is also an isogeny onto its image.

Proof. The commutativity is immediately seen. φ_L is an isogeny by [16]. Since α is the Albanese map, α^* is an isogeny of Abelian varieties.

REMARK. Taking the derivatives at $[id]$, we obtain the infinitesimal version: Consider the following diagram:

$$\begin{array}{ccc} H^0(X, \Theta_X) & \xrightarrow{\lambda_*} & H^0(A, \Theta_A) \\ \Big\downarrow{\scriptstyle \varrho_{\alpha^* L}} & & \Big\downarrow{\scriptstyle \varrho_L} \\ H^1(X, \mathcal{O}_X) & \xleftarrow{\alpha^*} & H^1(A, \mathcal{O}_A). \end{array}$$

Here λ_* is the derivative of λ at $[id]$, α^* is the canonical pull-back isomorphism, and ϱ_L and $\varrho_{\alpha^* L}$ are as in (4.2).

Then the diagram is commutative, ϱ_L is an isomorphism, and λ_* and $\varrho_{\alpha^* L}$ are injective. Thus under the identification $H^0(A, \Theta_A) = H^1(X, \mathcal{O}_X)$, λ_* coincides with $\varrho_{\alpha^* L}$.

(4.5) COROLLARY. *Assume that*
1. *X is good terminal minimal model, and*
2. *Letting φ: $X \to Y$ be the canonical fibering, $q(Y) = 0$.*
Then λ_* and $\varrho_{\alpha^* L}$ are isomorphisms.

Proof. In this case, $\dim H^0(X, \Theta_X) = q(X)$ by Theorem (3.10).

(4.6) THEOREM. *Assumptions and notations as in* (4.3). *The map φ_H: $\mathrm{Aut}^0(X) \to \mathrm{Pic}^0(X)$ is an isogeny onto the image in each of the following cases:*
1. *H is an ample invertible sheaf on X.*
2. *$A = \alpha^*(L)$, where L is an ample invertible sheaf on A.*

Proof. (2) was proved in Proposition (4.4). Since $\varphi_{kH} = (\varphi_H)^k$ for any invertible sheaf H and integer k, we may assume that H is very ample to show (1). Let $X \to \mathbb{P}^N$ be the embedding of X by $|H|$. The kernel $\varphi_H^{-1}([\mathcal{O}_X])$ of the map φ_H: $\mathrm{Aut}^0(X) \to \mathrm{Pic}^0(X)$ consists of the automorphisms f of X which come from automorphisms of $\mathbb{P}^N$:

$$\varphi_H^{-1}([\mathcal{O}_X]) = \mathrm{Aut}^0(X) \cap \mathrm{Aut}(\mathbb{P}^N, X).$$

Since $\mathrm{Aut}(\mathbb{P}^N, X)$ is a linear algebraic group as a subgroup of $\mathrm{Aut}^0(\mathbb{P}^N)$, it must be discrete (Rosenlicht [12]); thus $\varphi_H^{-1}([\mathcal{O}_X])$ is also discrete, hence θ_H is injective.

(4.7) Corollary. *For a projective variety X which is not ruled,* $\dim \mathrm{Bir}^0(X) \leqq \dim H^1(X, \mathcal{O}_X)$.

(4.8) Corollary. *For a terminal minimal model X (which is not assumed to be good)*, $\dim \mathrm{Bir}(X) \leqq q(X)$.

Proof. $\dim \mathrm{Bir}(X) = \dim \mathrm{Aut}^0(X)$ by Theorem (3.3).

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