COMPOSITIO MATHEMATICA

MITSUHIRO ITOH

Yang-Mills connections over a complex surface and harmonic curvature

Compositio Mathematica, tome 62, nº 2 (1987), p. 95-106

<http://www.numdam.org/item?id=CM_1987__62_2_95_0>

Compositio Mathematica 62: 95–106 (1987)

Yang–Mills connections over a complex surface and harmonic curvature

MITSUHIRO ITOH
Institute of Mathematics, University of Tsukuba, Ibaraki 305, Japan

Received 27 November 1985; accepted in revised form 22 September 1986

1. Introduction

A connection on a principal bundle is called Yang–Mills when it gives a critical point of the Yang–Mills functional, that is, it satisfies the Yang–Mills equation $d_A^* F_A = 0$ with respect to the curvature F_A. From the Bianchi identity $d_A F_A = 0$ a Yang–Mills connection is nothing but a connection whose curvature is harmonic with respect to the covariant exterior derivative d_A.

Over an oriented Riemannian 4-manifold M a connection A being Yang–Mills is equivalent to either $d_A F^+ = 0$ or $d_A F^- = 0$ where F^+ (or F^-) denotes the self-dual (or anti-self-dual) part of F_A. An (anti-) self-dual connection, namely a connection satisfying $F^- = 0$ (or $F^+ = 0$) yields a Yang–Mills connection minimizing the Yang–Mills functional from the Chern–Weil theorem.

There are many arguments focused on the gauge orbit space (moduli space) of (anti-) self-dual connections [4, 6, 8]. But we have a small knowledge of general solutions of the Yang–Mills equations. In fact only theorems with respect to the weakly stability and an estimate of the index of the Hessian of the functional are obtained over manifolds of special type [3, 10, 14] and we have isolation theorems relative to L^∞- (or L^2-) norm of the curvature [3, 12, 13].

Now we assume that the base manifold is a complex surface with a Kähler metric. Then the curvature splits into $F_A = F^{2,0} + F^{0,2} + F^{1,1}$, where $F^{p,q}$ is the (p, q)-component. We have from the Bianchi identity $\partial_A F^{2,0} = 0$ and $\bar{\partial}_A F^{0,2} = 0$ with respect to the partial covariant derivatives.

We notice that the complex surface carries the natural orientation and the self-dual part F^+ relative to this orientation is given as $F^+ = F^{2,0} + F^{0,2} + F^0 \otimes \omega$ (F^0 is a 0-form and ω is the Kähler form) and the anti-self-dual part F^- is a form of type (1, 1) which is orthogonal to ω [7].

An anti-self-dual connection relates to a semi-stable holomorphic structure together with an Einstein–Hermitian structure on the associated complex vector bundle [5, 9].

By the way we observe in general that a connection is Yang–Mills over a complex Kähler surface if and only if $\partial_A^* F^{2,0} = -i\partial_A F^0$ (Proposition 3.1). Then that F^0 is parallel is equivalent to $F^{2,0}$ being harmonic by the Bianchi identity.

Definition. A connection on a complex Kähler surface is said to be with harmonic curvature if $F^{2,0}$ is harmonic (i.e., $\partial_A F^{2,0} = 0$ and $\partial_A^* F^{2,0} = 0$).

We investigate in this article relations between the harmonicity and the anti-self-duality of the Yang–Mills connections. The following is one of the main results.

THEOREM 1. *Let M be either a compact complex surface with a Kähler metric of positive scalar curvature, a complex flat torus or a K^3 surface with a Ricci flat metric. Suppose that an irreducible connection A is Yang–Mills and is with harmonic curvature. Then A is anti-self-dual.*

A connection is irreducible when it admits no nontrivial covariantly constant Lie algebra-valued 0-form.

We have the following observation from Weitzenböck formulae that if the components of F^+ pointwise commute each other, that is, either $[F^0 \wedge F^{2,0}] = 0$ or $[F^{2,0} \wedge F^{0,2}] = 0$, then $F^{2,0}$ is harmonic (Proposition 3.3).

Denote by $\hat{\mathscr{A}}_{\mathrm{YM}}$ the space of irreducible Yang–Mills connections and $\hat{\mathscr{A}}_0$ the space of irreducible connections whose curvature satisfies $F^0 = 0$. Then the theorem asserts that $\hat{\mathscr{A}}_{\mathrm{YM}} \cap \hat{\mathscr{A}}_0 = \hat{\mathscr{A}}_-$, here $\hat{\mathscr{A}}_-$ is the space of irreducible anti-self-dual connections. These spaces are gauge invariant with respect to the group $\mathscr{G}$ of gauge transformations. So the moduli space of irreducible anti-self-dual connections $\hat{\mathscr{A}}_-/\mathscr{G}$ is described as $\hat{\mathscr{A}}_-/\mathscr{G} = \hat{\mathscr{A}}_{\mathrm{YM}}/\mathscr{G} \cap \hat{\mathscr{A}}_0/\mathscr{G}$.

The following gives an isolation phenomenon relative to L^2-norm of F^0.

THEOREM 2. *Let M be a compact complex surface with a Kähler metric of positive scalar curvature and $P = P(M, G)$ a principal bundle. Then there is a constant $c > 0$ which depends only on the invariant inner product on the Lie algebra $\mathfrak{g}$ of G and the Riemannian structure of M such that if A is an irreducible Yang–Mills connection on P satisfying $\|F_A^0\|_{L^2} < c$, then A is anti-self-dual.*

We have then from this theorem an open subset $W = \{[A];\ \|F_A^0\|_{L^2} < c\}$ in the orbit space $\hat{\mathscr{A}}/\mathscr{G}$ of irreducible connections with property $\hat{\mathscr{A}}_-/\mathscr{G} = W \cap \hat{\mathscr{A}}_{\mathrm{YM}}/\mathscr{G}$.

We discuss next the weakly stability of a Yang–Mills connection with harmonic curvature over a general complex Kähler surface.

We obtain the following by an averaging trick with respect to the complex structure of M. Similar arguments are used in [3] and [14].

THEOREM 3. *Let M be a compact Kähler surface and P an $SU(2)$-principal bundle with* $\mathrm{Pont}_1(\mathfrak{g}_P) < 0$, $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$, *satisfying $4k \geqq 3p_a(M)$. Let A be an irreducible Yang–Mills connection on P. If A is weakly stable (that is, the Hessian of the Yang–Mills functional is positive semi definite) and satisfies* $[F^{2,0} \wedge F^{0,2}] = 0$, *then*

(i) A is anti-self-dual, or

(ii) A is not anti-self-dual and the space $\mathrm{Ker}\,\square_A$, *of harmonic* (1,0)*-forms with values in $\mathfrak{g}_P^{\mathbb{C}}$ has dimension at least* $1 + 4k - 3p_a(M)$, $\square_A = \partial_A \partial_A^* + \partial_A^* \partial_A$, *and moreover for any $\alpha' \in$* $\mathrm{Ker}\,\square_A$, *$A + \alpha$, $\alpha = \alpha' + \bar{\alpha}'$ gives a Yang–Mills connection with harmonic curvature and $F_{A+\alpha}^+ = F_A^+$. Here $k = c_2(E)$, $E = P \times_{SU(2)} \mathbb{C}^2$ and $p_a(M)$ is the arithmetic genus of M.*

Thus the nullity n_A of the Hessian at the non anti-self-dual connection in the theorem is given by $n_A \geqq 1 + 4k - 3p_a(M)$. In fact conclusion (ii) implies that there is a subset N in $\hat{\mathscr{A}}_{\mathrm{YM}}$, $A \in N$, with effective l parameters, $l \geqq 1 + 4k - 3p_a(M)$ in the way that (1) each connection in N is Yang–Mills and with harmonic curvature, (2) if A_1 and A_2 in N are gauge equivalent, then $A_1 = A_2$ and (3) the self-dual part F^+ coincides for any connection in N.

We remark finally on relations among special subspaces of the orbit space $\hat{\mathscr{A}}/\mathscr{G}$ over a complex Kähler surface. For simplicity we set $G = SU(n)$. We have then the group $\mathscr{G}^{\mathbb{C}}$ of complex gauge transformations of the bundle so that $(\mathscr{G}, \mathscr{H}_+)$ gives an Iwasawa decomposition of $\mathscr{G}^{\mathbb{C}}$, $\mathscr{H}_+ = \Gamma(M\colon P \times_{SU(n)} H_+(n))$. Here $H_+(n)$ is the space of positive definite Hermitian matrices with determinant 1. The group $\mathscr{G}^{\mathbb{C}}$ acts on $\hat{\mathscr{A}}$ in a twisted manner to induce the quotient space $\hat{\mathscr{A}}/\mathscr{G}^{\mathbb{C}}$ and we get a fibration $\hat{\mathscr{A}}/\mathscr{G} \to \hat{\mathscr{A}}/\mathscr{G}^{\mathbb{C}}$ with fibre $\mathscr{H}_+$ which is diffeomorphic to the space of infinitesimal gauge transformations $\Omega^0(\mathfrak{g}_P)$. Hence the homotopy groups of these spaces are isomorphic.

The space $\hat{\mathscr{A}}/\mathscr{G}^{\mathbb{C}}$ inherits a canonical complex structure, since $\mathrm{Ker}\,\bar{\partial}_A^* \subset \Omega^{0,1}(\mathfrak{g}_P^{\mathbb{C}})$ is identified with its tangent space. The subspace $\hat{\mathscr{A}}_0/\mathscr{G}$ of $\hat{\mathscr{A}}/\mathscr{G}$ admits also a complex structure because its tangent space is isomorphic to

Ker $\bar{\partial}_A^*$. Thus $\hat{\mathscr{A}}_0/\mathscr{G}$ is considered as a representative of the complex space $\hat{\mathscr{A}}/\mathscr{G}^{\mathbb{C}}$ in the ambient space $\hat{\mathscr{A}}/\mathscr{G}$ by a unitary group description.

We denote by $\hat{\mathscr{A}}_{1,1}$ the space $\{A \in \hat{\mathscr{A}};\, F_A \text{ is type } (1, 1)\}$. Then we have the canonical projection $\hat{\mathscr{A}}_{1,1}/\mathscr{G} \to \hat{\mathscr{A}}_{1,1}/\mathscr{G}^{\mathbb{C}}$. The latter space is the moduli of holomorphic structures on the associated bundle. The restriction of this canonical projection to the moduli space $\hat{\mathscr{A}}_-/\mathscr{G}$ is shown to be injective and open [8]. Thus the moduli space represents exactly the moduli $\hat{\mathscr{A}}_{1,1}/\mathscr{G}^{\mathbb{C}}$ in a unitary group way. Moreover $\hat{\mathscr{A}}_-/\mathscr{G}$ is a deformation retract of $\hat{\mathscr{A}}_{1,1}/\mathscr{G}$, in fact it is diffeomorphic to the product $\hat{\mathscr{A}}_-/\mathscr{G} \times \mathscr{H}_+$ by the aid of the moment map [5]. We have also $\hat{\mathscr{A}}_{\mathrm{YM}}/\mathscr{G} \cap \hat{\mathscr{A}}_{1,1}/\mathscr{G} = \hat{\mathscr{A}}_-/\mathscr{G}$ which is an easy application of our argument.

2. Weitzenböck formulae

Let M be a compact complex surface with a Kähler metric g and $P = P(M, G)$ be a smooth principal bundle over M with a compact semi-simple Lie group G. Over the bundle $\Lambda^k \otimes \mathfrak{g}_P$ of Lie algebra-valued k-forms the inner product $\langle\ ,\ \rangle$ is canonically defined. For any connection A on P we have the covariant exterior derivatives $\mathrm{d}_A;\ \Omega^k(\mathfrak{g}_P) \to \Omega^{k+1}(\mathfrak{g}_P)$ where $\Omega^k(\mathfrak{g}_P)$ denotes the space of Lie algebra-valued k-forms. Like the canonical splitting of the exterior derivatives $\mathrm{d} = \partial + \bar{\partial}$, d_A decomposes over M into $\mathrm{d}_A = \partial_A + \bar{\partial}_A$. Then we get from the Bianchi identity

$$\begin{aligned} &\partial_A F^{2,0} = 0, && \bar{\partial}_A F^{0,2} = 0, \\ &\bar{\partial}_A F^{2,0} + \partial_A F^{1,1} = 0 && \partial_A F^{0,2} + \bar{\partial}_A F^{1,1} = 0. \end{aligned} \tag{2.1}$$

We now give the Weitzenböck formulae for Lie algebra-valued $(k, 0)$-forms which just correspond to the formulae in [3].

We define a Hermitian inner product $\langle\langle\ ,\ \rangle\rangle$ on $\Omega^{p,q}(\mathfrak{g}_P^{\mathbb{C}})$, the space of $\mathfrak{g}_P^{\mathbb{C}}$-valued (p, q)-forms by

$$\begin{aligned} \langle\langle \phi, \psi \rangle\rangle_M &= \int_M \langle\langle \phi, \psi \rangle\rangle(x)\, \mathrm{d}v, \\ \langle\langle \phi, \psi \rangle\rangle(x)\, \mathrm{d}v &= \langle \phi(x) \wedge *\overline{\psi(x)} \rangle. \end{aligned} \tag{2.2}$$

where $*$ is the $\mathbb{C}$-linearly extended Hodge operator over complex forms and $\bar{\ }$ is the conjugation on the bundle of $\mathfrak{g}_P^{\mathbb{C}}$-forms which is defined naturally. Remark that $\langle \xi \wedge *\bar{\eta} \rangle = 0$ for k-forms ξ and η which are of different type.

We denote by d_A^* and d_A^{+*} the formal adjoint of d_A and the self-dual part d_A^+ of d_A with respect to $\langle\ ,\ \rangle_M$ and by ∂_A^* and $\bar\partial_A^*$ the formal adjoint of the partial derivatives ∂_A and $\bar\partial_A$ relative to the Hermitian inner product.

PROPOSITION 2.1. *Let A be a connection on P. For $\alpha \in \Omega^1(\mathfrak{g}_P)$, $\alpha = \alpha' + \alpha''$, $\alpha' = \bar\alpha'' \in \Omega^{1,0}(\mathfrak{g}_P^{\mathbb{C}})$ we have*

$$(\tfrac{1}{2}\mathrm{d}_A\mathrm{d}_A^* + \mathrm{d}_A^{+*}\mathrm{d}_A^+)\alpha = \Box_A\alpha' + \overline{\Box}_A\alpha'' \tag{2.3}$$

where $\Box_A = \partial_A\partial_A^* + \partial_A^*\partial_A$ *and* $\overline{\Box}_A = \bar\partial_A\bar\partial_A^* + \bar\partial_A^*\bar\partial_A$.

Remark. This formula corresponds to the decomposition of the real Laplacians operating on the scalar field forms over a Kähler manifold.

Before proving this proposition we introduce an operator which is convenient for the expression of d_A^{+*} by $\mathrm{d}_A^{\mathbb{C}} = J\,\mathrm{d}_A; \Omega^0(\mathfrak{g}_P) \to \Omega^1(\mathfrak{g}_P)$, where J is the complex structure on $\Omega^1(\mathfrak{g}_P)$ induced from the base manifold, $J\alpha = i(\alpha' - \alpha'')$ for $\alpha = \alpha' + \alpha''$.

LEMMA 2.2. (i) *For $\alpha \in \Omega^1(\mathscr{G}_P)$ we have*

$$\mathrm{d}_A^*\alpha = \partial_A^*\alpha' + \bar\partial_A^*\alpha'', \tag{2.4}$$

$$\mathrm{d}_A^{\mathbb{C}*}\alpha = -i(\partial_A^*\alpha' - \bar\partial_A^*\alpha''), \tag{2.5}$$

and

$$\mathrm{d}_A^+\alpha = \partial_A\alpha' + \bar\partial_A\alpha'' + \tfrac{1}{2}(\mathrm{d}_A^{\mathbb{C}*}\alpha) \otimes \omega \tag{2.6}$$

(ii) *with respect to a self-dual $\mathscr{G}_P$-valued 2-form $\phi = \phi^{2,0} + \phi^{0,2} + \phi^0 \otimes \omega$, $\phi^{2,0} \in \Omega^{2,0}(\mathfrak{g}_P^{\mathbb{C}})$, $\phi^{0,2} \in \Omega^{0,2}(\mathfrak{g}_P^{\mathbb{C}})$, $\phi^0 \in \Omega^0(\mathfrak{g}_P)$*

$$\mathrm{d}_A^{+*}\phi = \partial_A^*\phi^{2,0} + \bar\partial_A^*\phi^{0,2} + \mathrm{d}_A^{\mathbb{C}}\phi^0 \tag{2.7}$$

Proof of Proposition 2.1. We have from (2.4) $\mathrm{d}_A\mathrm{d}_A^*\alpha = (\partial_A + \bar\partial_A)(\partial_A^*\alpha' + \bar\partial_A^*\alpha'') = \partial_A\partial_A^*\alpha' + \partial_A\bar\partial_A^*\alpha'' + \bar\partial_A\partial_A^*\alpha' + \bar\partial_A\bar\partial_A^*\alpha''$ and we operate d_A^{+*} to $\mathrm{d}_A^+\alpha$ by the aid of (2.7) to get $\partial_A^*\partial_A\alpha' + \bar\partial_A^*\bar\partial_A\alpha'' + \mathrm{d}_A^{\mathbb{C}}(\tfrac{1}{2}\mathrm{d}_A^{\mathbb{C}*}\alpha) = \partial_A^*\partial_A\alpha' + \bar\partial_A^*\bar\partial_A\alpha'' + \tfrac{1}{2}(\partial_A\partial_A^*\alpha' - \partial_A\bar\partial_A^*\alpha'' - \bar\partial_A\partial_A^*\alpha' + \bar\partial_A\bar\partial_A^*\alpha'')$. Thus (2.3) follows from this.

Proof of Lemma 2.2. (i) We have for $d_A\phi = \partial_A\phi + \bar\partial_A\phi$ $\langle d_A\phi, \alpha\rangle_M = \langle\langle\partial_A\phi, \alpha'\rangle\rangle_M + \langle\langle\bar\partial_A\phi, \alpha''\rangle\rangle_M = \langle\langle\phi, \partial_A^*\alpha' + \bar\partial_A^*\alpha''\rangle\rangle_M$. Since $\partial_A^*\alpha' + \bar\partial_A^*\alpha''$ is real, this is just equal to $\langle\phi, \partial_A^*\alpha' + \bar\partial_A^*\alpha''\rangle_M$. Similarly we get (2.5). For $\alpha \in \Omega^1(\mathfrak{g}_P)$ the self-dual part $d_A^+\alpha$ of $d_A\alpha$ is written as $d_A^+\alpha = \partial_A\alpha' + \bar\partial_A\alpha'' + \frac{1}{2}\langle\bar\partial_A\alpha' + \partial_A\alpha'', \omega\rangle \otimes \omega$, because we have that for any real 2-form $\theta = \theta^{2,0} + \theta^{0,2} + \theta^{1,1}$ its self-dual part θ^+ is given by $\theta^+ = \theta^{2,0} + \theta^{0,2} + \frac{1}{2}\langle\theta^{1,1}, \omega\rangle \otimes \omega$ [7]. Since $\omega = i\sum g_{\mu\bar\nu}\, dz^\mu \wedge dz^{\bar\nu}$ and $\bar\partial_A\alpha' = \sum \tilde\nabla_{\bar\nu}\alpha_\mu\, dz^{\bar\nu} \wedge dz^\mu$, $\partial_A^*\alpha' = -\sum g^{\bar\nu\mu}\tilde\nabla_{\bar\nu}\alpha_\mu$ $(\tilde\nabla_\nu\cdot = \nabla_\nu\cdot + [A_\nu, \cdot], \tilde\nabla_{\bar\nu}\cdot = \nabla_{\bar\nu}\cdot + [A_{\bar\nu}, \cdot])$, we have $\langle\bar\partial_A\alpha' + \partial_A\alpha'', \omega\rangle = d_A^{\mathbb{C}*}\alpha$. Thus (2.6) is obtained.

To get (2.7) we use the following formula

$$\langle\phi, \psi\rangle = \langle\langle\phi^{2,0}, \psi^{2,0}\rangle\rangle + \langle\langle\phi^{0,2}, \psi^{0,2}\rangle\rangle + 2\langle\phi^0, \psi^0\rangle$$

for self-dual 2-forms ϕ and ψ. Then we obtain

$$\begin{aligned}\langle d_A^+\alpha, \phi\rangle_M &= \langle\langle\partial_A\alpha', \phi^{2,0}\rangle\rangle_M + \langle\langle\bar\partial_A\alpha'', \phi^{0,2}\rangle\rangle_M \\ &\quad + 2\int_M \langle -i/2(\partial_A^*\alpha' - \bar\partial_A^*\alpha''), \phi^0\rangle\, dv \\ &= \langle\langle\alpha', \partial_A^*\phi^{2,0}\rangle\rangle_M + \langle\langle\alpha'', \bar\partial_A^*\phi^{0,2}\rangle\rangle_M \\ &\quad + \langle\langle -i(\partial_A^*\alpha' - \bar\partial_A^*\alpha''), \phi^0\rangle\rangle_M.\end{aligned}$$

Here the third term reduces to

$$\int_M \{\langle\langle\alpha', i\partial_A\phi^0\rangle\rangle + \langle\langle\alpha'', -i\bar\partial_A\phi^0\rangle\rangle\}\, dv$$

from which we derive (2.7).

Proposition 2.3. *Let A be a connection. Then for each $\psi \in \Omega^{2,0}(\mathfrak{g}_P^{\mathbb{C}})$*

$$\square_A\psi = \nabla_A^*\nabla_A\psi - 2i[F^0, \psi] + 2S\psi \tag{2.8}$$

where S is the scalar curvature of the metric g.

Proof. The component of $\partial_A\psi$ is given by $\tilde\nabla_\mu\psi_{\nu\lambda} + \tilde\nabla_\nu\psi_{\lambda\mu} + \tilde\nabla_\lambda\psi_{\mu\nu}$. Then $\partial_A^*\partial_A\psi$ reduces to $-\sum g^{\bar\sigma\tau}\tilde\nabla_{\bar\sigma}(\tilde\nabla_\tau\psi_{\mu\nu} + \tilde\nabla_\mu\psi_{\nu\tau} + \tilde\nabla_\nu\psi_{\tau\mu})$. We have similarly

$$\partial_A\partial_A^*\psi = -\sum g^{\bar\sigma\tau}(\tilde\nabla_\mu\tilde\nabla_{\bar\sigma}\psi_{\tau\nu} - \tilde\nabla_\nu\tilde\nabla_{\bar\sigma}\psi_{\tau\mu}).$$

Then

$$(\Box_A\psi)_{\mu\nu} = -\sum g^{\bar\sigma\tau}\tilde\nabla_{\bar\sigma}\tilde\nabla_\tau\psi_{\mu\nu} - \sum g^{\bar\sigma\tau}[\tilde\nabla_\mu, \tilde\nabla_{\bar\sigma}]\psi_{\tau\nu}$$
$$+ \sum g^{\bar\sigma\tau}[\tilde\nabla_\nu, \tilde\nabla_{\bar\sigma}]\psi_{\tau\mu}.$$

By making use of the Ricci formulae we reduce this to

$$-\sum g^{\bar\sigma\tau}\tilde\nabla_{\bar\sigma}\tilde\nabla_\tau\psi_{\mu\nu} - \sum g^{\bar\sigma\tau}[F_{\mu\bar\sigma}, \psi_{\tau\nu}] + \sum g^{\bar\sigma\tau}[F_{\nu\bar\sigma}, \psi_{\tau\mu}]$$
$$+ \sum (R^\varepsilon_\mu\psi_{\varepsilon\nu} - R^\varepsilon_\nu\psi_{\varepsilon\mu}).$$

Thus (2.8) is derived, since the base manifold is two dimensional and $S = \frac{1}{2}\Sigma g^{\bar\sigma\tau}R_{\tau\bar\sigma}$ and $F^0 = \frac{1}{2}\langle F_A, \omega\rangle = -i/2\,\Sigma g^{\bar\sigma\tau}F_{\tau\bar\sigma}$.

The following gives a vanishing theorem relative to a harmonic (2, 0)-form.

PROPOSITION 2.4. (i) *Let M be a compact complex Kähler surface with positive scalar curvature. If a connection A on a G-principal bundle over M satisfies $F^0 = 0$, then in the space $\Omega^{2,0}(\mathfrak{g}_P^{\mathbb{C}})$* Ker $\Box_A = 0$.
(ii) *Let M be either a two dimensional complex flat torus or a $K3$ surface with a Ricci flat Kähler metric. If a connection A is irreducible and satisfies $F^0 = 0$ then* Ker $\Box_A = 0$.

Proof. (i) This is obvious because we have from (2.8)

$$\langle\langle\Box_A\psi, \psi\rangle\rangle_M = \int_M (|\nabla_A\psi|^2 + 2S|\psi|^2\,dv. \tag{2.9}$$

(ii) From the above formula each harmonic (2, 0)-form ψ is ∂_A-covariant constant. It is also d_A-covariant constant since $\bar\nabla_A^*\bar\nabla_A\psi = \nabla_A^*\nabla_A\psi + 2i[F^0, \psi]$. By the way the bundle $\Lambda^{2,0}$ is trivial and admits a covariant constant section $(\neq 0)$. Hence ψ vanishes from the irreducibility of the connection A.

3. Yang–Mills connection with harmonic curvature

Let P be a G-principal bundle over a compact Kähler surface M. Let A be a connection on P. Decompose the self-dual part F^+ of the curvature into $F^+ = F^{2,0} + F^{0,2} + F^0 \otimes \omega$. We first show the following proposition.

PROPOSITION 3.1. *The following conditions are equivalent;*

(i) *A is Yang–Mills, that is,* $\mathrm{d}_A F^+ = 0$,

(ii) $\bar{\partial}_A F^{2,0} = -\partial_A(F^0 \otimes \omega)$, (3.1)

(iii) $\partial_A^* F^{2,0} = -i\partial_A F^0$. (3.2)

Proof. Suppose that A is a Yang–Mills connection. Then $\mathrm{d}_A F^+ = (\partial_A + \bar{\partial}_A)(F^{2,0} + F^{0,2} + F^0 \otimes \omega) = 0$. Hence the (2, 1)-component of this equation vanishes also. Conversely, assume (3.1). By a reality condition on F_A the complex conjugate of (3.1) holds, that is, $\partial_A F^{0,2} = -\bar{\partial}_A(F^0 \otimes \omega)$. Thus $\mathrm{d}_A F^+ = 0$ by the aid of the Bianchi identity (2.1). Assume (3.1) again. Since $\partial_A^* = -* \circ \bar{\partial}_A \circ *$ and $\bar{\partial}_A^* = -* \circ \partial_A \circ *$ [11], we have $\partial_A^* F^{2,0} = -\bar{\partial}_A^*(F^0 \otimes \omega)$. Here we get moreover $(\bar{\partial}_A^*(F^0 \otimes \omega))_\mu = \Sigma\, g^{\bar{\sigma}\tau}\tilde{\nabla}_\tau(F^0 \otimes \omega)_{\mu\bar{\sigma}} = i\tilde{\nabla}_\mu F^0$. Hence (3.2) is obtained. From (3.2) together with the conjugate formula $\bar{\partial}_A^* F^{0,2} = i\bar{\partial}_A F^0$, we have by the aid of (2.7) $\mathrm{d}_A^+{}^* F^+ = \partial_A^* F^{2,0} + \bar{\partial}_A^* F^{0,2} + \mathrm{d}_A^{\mathrm{C}} F^0 = \partial_A^* F^{2,0} + \bar{\partial}_A^* F^{0,2} + i(\partial_A F^0 - \bar{\partial}_A F^0) = 0$. Then $\mathrm{d}_A F^+ = 0$.

As a direct consequence of Proposition 3.1 we have

COROLLARY 3.2 [5]. *Let A be a Yang–Mills connection. If A is irreducible and its curvature is of type* (1, 1), *then it is anti-self-dual.*

The following is a Weitzenböck formula with respect to the (2, 0)-component $F^{2,0}$ of the curvature.

PROPOSITION 3.3. *Let A be a Yang–Mills connection. Then we have*

$$\Box_A F^{2,0} = i[F^0 \wedge F^{2,0}] \tag{3.3}$$

and equivalently

$$\nabla_A^* \nabla_A F^{2,0} - 3i[F^0 \wedge F^{2,0}] + 2SF^{2,0} = 0 \tag{3.4}$$

Proof. From Proposition 3.1 we have $\Box_A F^{2,0} = \partial_A \partial_A^* F^{2,0} + \partial_A^* \partial_A F^{2,0} = -i\partial_A \partial_A F^0 = -i[F^{2,0}, F^0]$. Here we made use of Ricci formula. We apply further Proposition 2.3 to (3.3) to get (3.4).

PROPOSITION 3.4. *Let M be a compact Kähler surface with positive scalar curvature. Let A be an irreducible Yang–Mills connection. If it satisfies one of*

the following, then it is anti-self-dual;
(i) $[F^{2,0} \wedge F^0] = 0$,
(ii) $[F^{2,0} \wedge F^{0,2}] = 0$.

Proof. We assume (i). Then from Proposition 3.3 $\nabla_A^*\nabla_A F^{2,0} + 2SF^{2,0} = 0$ and hence

$$\int_M (|\nabla_A F^{2,0}|^2 + 2S|F^{2,0}|^2)\,\mathrm{d}v = 0.$$

Since the scalar curvature is positive, $F^{2,0}$ vanishes. Then by Corollary 3.2 we have that A is anti-self-dual. Now assume (ii). We take the inner product of (3.4) with $F^{2,0}$. Then

$$\int_M (|\nabla_A F^{2,0}|^2 + 2S|F^{2,0}|^2 + \langle\!\langle -3i[F^0 \wedge F^{2,0}], F^{2,0}\rangle\!\rangle)\,\mathrm{d}v = 0$$

Thus A is anti-self-dual, since $\langle\!\langle i[F^0 \wedge F^{2,0}], F^{2,0}\rangle\!\rangle\,\mathrm{d}v = \langle F^0, i[F^{2,0} \wedge \overline{F^{2,0}}]\rangle$.

Remark. From this proposition we observe that if the base manifold is as above and A is an irreducible Yang–Mills connection satisfying $F^0 = 0$, then A is anti-self-dual.

There is a constant $c > 0$ depending on the Lie algebra $\mathfrak{g}$ and the inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that $\|[X, \bar{X}]\| \leqq c|X|^2$, $X \in \mathfrak{g}^{\mathbb{C}}$. Then $|\langle\!\langle[F^0 \wedge F^{2,0}], F^{2,0}\rangle\!\rangle|\,\mathrm{d}v = |\langle F^0, [F^{2,0} \wedge \overline{F^{2,0}}]\rangle| \leqq c|F^0|\,|F^{2,0}|^2\,\mathrm{d}v$. Applying the technique of Sobolev inequalities appeared in [12] we obtain

COROLLARY 3.5. *Let M be as above. Let A be an irreducible Yang–Mills connection. If $\|F^0\|_{L^2} < c_1$ for some constant $c_1 > 0$ depending on the inner product on $\mathfrak{g}$ and the Riemannian structure of M, then A is anti-self-dual.*

In the case that the scalar curvature $S = 0$ we have

PROPOSITION 3.6. *Let M be a two dimensional complex flat torus or a $K3$ surface with a Ricci flat metric. Then an irreducible Yang–Mills connection satisfying either $[F^0 \wedge F^{2,0}] = 0$ or $[F^{2,0} \wedge F^{0,2}] = 0$ is anti-self-dual.*

This proposition is easily verified by (ii), Proposition 2.4.

4. Weakly stable Yang–Mills connection

We relax the positive scalar curvature condition of the base manifold and then argue a relation between the anti-self-duality and the weakly stability for a Yang–Mills connection with harmonic curvature. We will show here Theorem 3.

Proof of Theorem 3. The Hessian $\mathscr{H}$ of the Yang–Mills functional is given by the formula [14]

$$\mathscr{H}(\alpha, \beta) = \int_M \{\langle d_A^+ \alpha, d_A^+ \beta\rangle + \langle F^+, [\alpha \wedge \beta]^+\rangle\} \, dv,$$

where $[\alpha \wedge \beta]^+$ is the self-dual part of the 2-form $[\alpha \wedge \beta]$. If the connection is irreducible, then the weakly stability is equivalent to the following bilinear form

$$\bar{\mathscr{H}}(\alpha, \beta) = \mathscr{H}(\alpha, \beta) + \frac{1}{2}\int_M \langle d_A^* \alpha, d_A^* \beta\rangle \, dv \tag{4.1}$$

being positive semi definite over $\Omega^1(\mathfrak{g}_P)$ [14]. By the way $F^{2,0}$ is harmonic from (3.3), since $\langle\langle \square_A F^{2,0}, F^{2,0}\rangle\rangle = -\langle\langle i[F^{2,0} \wedge F^0], F^{2,0}\rangle\rangle = \langle iF^0, [F^{2,0} \wedge \overline{F^{2,0}}]\rangle$. Hence F^0 is covariant constant and then vanishes. Therefore by Proposition 2.1 (4.1) is written as for $\beta = \alpha$

$$\bar{\mathscr{H}}(\alpha, \alpha) = \int_M \{\langle\langle \square_A \alpha', \alpha'\rangle\rangle + \langle\langle \bar{\square}_A \alpha'', \alpha''\rangle\rangle + \langle\langle F^{2,0}, [\alpha' \wedge \alpha']\rangle\rangle + \langle\langle F^{0,2}, [\alpha'' \wedge \alpha'']\rangle\rangle\} \, dv \tag{4.2}$$

We have also for $J\alpha$

$$\bar{\mathscr{H}}(J\alpha, J\alpha) = \int_M \{\langle\langle \square_A \alpha', \alpha'\rangle\rangle + \langle\langle \bar{\square}_A \alpha'', \alpha''\rangle\rangle - \langle\langle F^{2,0}, [\alpha' \wedge \alpha']\rangle\rangle - \langle\langle F^{0,2}, [\alpha'' \wedge \alpha'']\rangle\rangle\} \, dv. \tag{4.3}$$

On the other hand the dimension dim Ker $\square_A$ is given by dim Ker $\square_A^1 =$ dim Ker $\square_A^2 + 4k - 3p_a(M)$, where $\square_A^1$ and $\square_A^2$ are the complex Laplacians associated to the sequence; $\Omega^0(\mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{\partial_A} \Omega^{1,0}(\mathfrak{g}_P^{\mathbb{C}}) \xrightarrow{\partial_A} \Omega^{2,0}(\mathfrak{g}_P^{\mathbb{C}})$, because the index of this sequence dim Ker $\square_A^0$ − dim Ker $\square_A^1$ + dim Ker $\square_A^2$ is represented by $-c_2(\mathfrak{g}_P^{\mathbb{C}}) + 3p_a(M)$ [2], and Ker $\square_A^0 = 0$ and $c_2(\mathfrak{g}_P^{\mathbb{C}}) = 4c_2(E)$.

Assume now that A is not anti-self-dual. Then $F^{2,0}$ is a non zero harmonic 2-form. Hence $\dim \ker \square_A^1 \geqq 1 + 4k - 3p_a(M) > 0$. For any non zero $\alpha' \in \mathrm{Ker}\, \square_A^1$ we have

$$\mathscr{H}(\alpha, \alpha) = \langle\langle F^{2,0}, [\alpha' \wedge \alpha']\rangle\rangle_M + \langle\langle F^{0,2}, [\alpha'' \wedge \alpha'']\rangle\rangle_M. \tag{4.4}$$

Since $\square_A$ and $\overline{\square}_A$ commute with J, $\mathscr{H}(J\alpha, J\alpha) = -\mathscr{H}(\alpha, \alpha)$ for $\alpha' \in \mathrm{Ker}\, \square_A^1$. It follows then from the weakly stability that $\mathscr{H}(\alpha, \alpha) = \mathscr{H}(J\alpha, J\alpha) = 0$ and hence $\mathscr{H}(\alpha, \beta) = 0$ for all $\beta \in \Omega^1(\mathfrak{g}_P)$, namely

$$\langle\langle F^{2,0}, [\alpha' \wedge \beta']\rangle\rangle_M = 0, \tag{4.5}$$

$\beta' \in \Omega^{1,0}(\mathfrak{g}_P^{\mathbb{C}})$. Hence we have $[F_{12}, \bar{\alpha}_1] = [F_{12}, \bar{\alpha}_2] = 0$, where $F^{2,0} = F_{12}\, dz^1 \wedge dz^2$ and $\alpha' = \alpha_1\, dz^1 + \alpha_2\, dz^2$. Since A is not anti-self-dual and the structure group is $SU(2)$, $\bar{\alpha}_1$, $\bar{\alpha}_2$ are scalar multiples of F_{12}. Then $[\alpha'' \wedge \alpha''] = [\bar{\alpha}' \wedge \bar{\alpha}'] = 0$ holds on an open subset. $[\alpha'' \wedge \alpha'']$ is a polynomial function of a solution to the elliptic equations $\overline{\square}_A \alpha'' = 0$. From a uniqueness theorem on continuation of a solution [1] $[\alpha'' \wedge \alpha''] = 0$ holds over M. We show now that each $A + \alpha$ is Yang–Mills and is with harmonic curvature and satisfies $F^+_{A+\alpha} = F^+_A$. Since $\partial_A \alpha' = [\alpha' \wedge \alpha'] = 0$, we have $F^{2,0}_{A+\alpha} = F^{2,0}_A$ and then $F^{0,2}_{A+\alpha} = F^{0,2}_A$. Moreover $F^{1,1}_{A+\alpha} = F^{1,1}_A + \bar{\partial}_A \alpha' + \partial_A \alpha'' + [\alpha' \wedge \alpha'']$. But we get $[\alpha' \wedge \alpha''] = 0$, since it is proportional to $[F^{2,0} \wedge F^{0,2}] = 0$. Then $F^0_{A+\alpha} = F^0_A + i/2(\Sigma\, g^{\bar{\nu}\mu} \tilde{\nabla}_{\bar{\nu}} \alpha_\mu - \Sigma\, g^{\bar{\nu}\mu} \nabla_\mu \alpha_{\bar{\nu}}) = F^0_A = 0$, because $\alpha' \in \mathrm{Ker}\, \square_A$ and $\alpha'' \in \mathrm{Ker}\, \overline{\square}_A$. Since $[\alpha'' \wedge F^{2,0}] = 0$ we have that $\bar{\partial}_{A+\alpha} F^{2,0}_{A+\alpha} = \bar{\partial}_A F^{2,0}_A$. Hence $A + \alpha$ is a Yang–Mills connection from Proposition 3.1.

References

1. N. Aronszajn: A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pure et Appl.* 35 (1957) 235–249.
2. M.F. Atiyah, N.J. Hitchin and I.M. Singer: Self-duality in four dimensional Riemannian geometry. *Proc. Roy. Soc. London, Ser. A* 362 (1978) 425–461.
3. J.P. Bourguignon and H.B. Lawson: Stability and isolation phenomena for Yang–Mills fields. *Commun. Math. Phys.* 79 (1981) 189–230.
4. S.K. Donaldson: An application of gauge theory to four dimensional topology. *J. Differential Geometry* 18 (1983) 279–315.
5. S.K. Donaldson: Anti self-dual Yang–Mills connections over complex algebraic surfaces and stable vector bundles, *Proc. London Math. Soc.* (3) 50 (1985) 1–26.
6. D.S. Freed and K. Uhlenbeck: *Instantons and four-manifolds*. New York: Springer-Verlag (1984).
7. M. Itoh: On the moduli space of anti-self-dual Yang–Mills connections on Kähler surfaces. *Pub. R.I.M.S. Kyoto* 19 (1983) 15–32.

8. M. Itoh: The moduli space of Yang–Mills connections over a Kähler surface is a complex manifold. *Osaka J. Math.* 22 (1985) 845–862.
9. S. Kobayashi: Curvature and stability of vector bundles. *Proc. Japan Acad. Ser. A, Math. Sci.* 58 (1982) 158–162.
10. S. Kobayashi, Y. Ohnita and M. Takeuchi: On instability of Yang–Mills connections. *Math. Z.* 193 (1986) 165–189.
11. K. Kodaira and J. Morrow: *Complex manifolds*. Holt, Rinehalt and Winston (1971).
12. Min-Oo: An L_2-isolation theorem for Yang–Mills fields. *Comp. Math.* 47 (1982) 153–163.
13. C. Shen: The gap phenomena of Yang–Mills fields over the complete manifold. *Math. Zeit.* 189 (1982) 69–77.
14. C. Taubes: Stability in Yang–Mills theories. *Commun. Math. Phys.* 91 (1983) 235–263.