KUMIKO NISHIOKA

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KUMIKO NISHIOKA
Department of Mathematics, Nara Women's University, Kita-Uoya Nishimachi, Nara 630, Japan

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In what follows, $K$ denotes an algebraic number field. Let $f(z) = \sum_{k=0}^{\infty} a_k z^{e_k}$ be a power series with nonzero coefficients $a_k \in K$, the convergence radius $R > 0$ and increasing nonnegative integers $e_k$ satisfying the condition

$$\lim_{k \to \infty} \left( e_k + \log M_k + \log A_k \right) / e_{k+1} = 0 \quad (1)$$

where $A_k = \max\{1, \lceil a_0 \rceil, \ldots, \lceil a_k \rceil\}$ and $M_k$ is positive integer such that $M_k a_0, \ldots, M_k a_k$ are algebraic integers. By [Cijsouw and Tijdeman, 1973], the number $f(\alpha)$ is transcendental for any algebraic $\alpha$ with $0 < |\alpha| < R$. Moreover by [Bundschuh and Wylegala, 1980], the numbers $f(\alpha_1), \ldots, f(\alpha_n)$ are algebraically independent for any algebraic numbers $\alpha_1, \ldots, \alpha_n$ with $0 < |\alpha_1| < \cdots < |\alpha_n| < R$. In this paper, we shall establish necessary and sufficient conditions for algebraic independence of the values $f(\alpha_1), \ldots, f(\alpha_n)$ at algebraic numbers $\alpha_1, \ldots, \alpha_n$. Definition. We say the algebraic numbers $\alpha_1, \ldots, \alpha_s$ are $(e_k)$-dependent if there exist a number $\gamma$, roots of unity $\zeta_i$ $(1 \leq i \leq s)$ and algebraic numbers $d_1, \ldots, d_s$ not all zero such that

$$\alpha_i = \zeta_i \gamma$$

$$\sum_{i=1}^{s} d_i \zeta_i^{e_k} = 0$$

for any sufficiently large $k$.

Then we have the following theorem.

**Theorem 1.** Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers with $0 < |\alpha_i| < R(1 \leq i \leq n)$. Then the following three properties are equivalent:

i) $f^{(l)}(\alpha_i)$ $(1 \leq i \leq n, 0 \leq l)$ are algebraically dependent over the rationals, where $f^{(l)}(z)$ denotes the $l$th derivative of $f(z)$. 

ii) There is a non-empty subset \( \{ \alpha_{i_1}, \ldots, \alpha_{i_t} \} \) of \( \{ \alpha_1, \ldots, \alpha_n \} \) such that \( \alpha_{i_1}, \ldots, \alpha_{i_t} \) are \( \{ e_k \} \)-dependent.

iii) \( 1, f(\alpha_1), \ldots, f(\alpha_n) \) are linearly dependent over the algebraic numbers.

**Example 1.** Let \( f(z) = \sum_{k=1}^{\infty} z^{k!} \) and \( \alpha_1, \ldots, \alpha_n \) be algebraic numbers with \( 0 < |\alpha_i| < 1 (1 \leq i \leq n) \). Then \( f^{(l)}(\alpha_i) (1 \leq i \leq n, 0 \leq l \) are algebraically independent if and only if \( \alpha_i/\alpha_j \) is not a root of unity for \( i \neq j \). This result was conjectured to be true by Masser.

**Example 2.** Let \( f(z) = \sum_{k=1}^{\infty} z^{k!+k} \) and \( \alpha_1, \ldots, \alpha_n \) be algebraic numbers with \( 0 < |\alpha_i| < 1 (1 \leq i \leq n) \). Then \( f^{(l)}(\alpha_i) (1 \leq i \leq n, 0 \leq l \) are algebraically independent if and only if \( \alpha_i \neq \alpha_j \) for \( i \neq j \).

**Proof of Theorem 1.** Obviously the property ii) implies the property iii) and the property iii) implies the property i). We prove the property i) implies the property ii). Suppose the property i) is satisfied. We may assume \( f^{(l)}(\alpha_i) (1 \leq i \leq n, 0 \leq l \leq L) \) are algebraically dependent and \( f^{(l)}(\alpha_i) \) are algebraically independent for any subset of \( n-1 \) numbers \( \alpha_i \). Changing the indices of the \( \alpha \)'s, we may suppose

\[
|\alpha_{11}| = \cdots = |\alpha_{1s_1}| = |\alpha_{21}| = \cdots = |\alpha_{2s_2}| = \cdots = |\alpha_{ts_t}| > |\alpha_{t+1,1}| \geq \cdots \geq |\alpha_{t+1,s_{t+1}}|,
\]

\[
s_1 + s_2 + \cdots + s_{t+1} = n,
\]

\[
\alpha_{iq}/\alpha_{i1} (1 \leq i \leq t, 1 \leq q \leq s_i) \text{ are roots of unity,}
\]

\[
\alpha_{i1}/\alpha_{j1} (1 \leq i < j \leq t) \text{ are not roots of unity.}
\]

Define \( U \) and \( U_m \in \mathbb{C}^{n(L+1)} \) by

\[
U = \left( \alpha_{iq} f^{(l)}(\alpha_{iq}) \right)_{1 \leq i \leq t+1, 1 \leq q \leq s_i, 0 \leq l \leq L},
\]

\[
U_m = \left( \sum_{k=0}^{m-1} P_l(e_k) a_k \alpha_{iq}^{s_i} \right)_{1 \leq i \leq t+1, 1 \leq q \leq s_i, 0 \leq l \leq L},
\]

where \( P_0(X) = 1, P_{l}(X) = X(X-1) \ldots (X-l+1) \). Then \( \lim_{m \to \infty} U_m = U \) and there is a nonzero polynomial \( F \in \overline{\mathbb{Q}}[(y_{i,q}^{(l)})_{1 \leq i \leq t+1, 1 \leq q \leq s_i, 0 \leq l \leq L}] \) such that \( F(U) = 0 \). We may assume \( F \) has the least total degree among such polynomials and algebraic integer coefficients. By the assumption on the minimality of \( n \), for any \( i \) and \( q \) \( (1 \leq i \leq t+1, 1 \leq q \leq s_i) \) there exists an integer \( l \) \( (0 \leq l \leq L) \)
such that \( \partial F / \partial y_{i q}(l) \neq 0 \). Since the total degree of \( \partial F / \partial y_{i q}(l) \) is less than the total degree of \( F \), \( \partial F / \partial y_{i q}(l)(U) \neq 0 \). By Taylor expansion, we have

\[
-F(U_m) = F(U) - F(U_m) = \sum_{|J| > 1} J!^{-1} \partial^{\left| J \right|} F / \partial y^{J}(U_m)(U - U_m)^{J},
\]

where \( J = (j_{iq}(l))_{1 \leq i \leq l+1, 1 \leq q \leq s, 0 \leq t \leq L} \) with \( j_{iq}(l) \) being nonnegative integers and \( |J|, J! / \partial y^{J} \) and \( (U - U_m)^{J} \) are defined in the usual way. There is a positive number \( \theta < 1 \) such that \( e_{m}^{L} |a_{m}| |\alpha_{11}|^{e_{n}} = O(\theta^{e_{n}}) \). (In what follows, the constants implicit in the symbol \( O \) and positive constants \( c_{1}, c_{2}, \ldots \) depend only on \( K, f(z), \alpha_{1}, \ldots, \alpha_{n}, \) and \( F \).) Then we have

\[
\sum_{k = m}^{\infty} P_{l}(e_{k}) a_{k}^{(e_{n})} = P_{l}(e_{m}) a_{m}^{(e_{n})} + O(\theta^{e_{n}+1}).
\]

By the fundamental inequality: for any algebraic \( \alpha \neq 0 \), \( \log |\alpha| \geq -[Q(\alpha) : Q]\{ \log |\alpha| + \log(\text{den} \alpha) \} \), we have \( \log |a_{m}| \geq -[K : Q]\{ \log M_{m} + \log A_{m} \} \). Therefore by (1), \( \theta^{e_{n}+1} = O( |a_{m}| |\alpha_{11}|^{e_{n}} \theta^{e_{n}}) \) and

\[
-F(U_m) = \sum_{l=1}^{t} \sum_{l=0}^{L} \sum_{q=1}^{s_{i}} \partial F / \partial y_{i q}(l)(U_m) P_{i}(e_{m}) a_{m}^{(e_{n})} + O(e_{m}^{L} |a_{m}| |\alpha_{r+1,1}|^{e_{n}}) + O(e_{m}^{L} |a_{m}| |\alpha_{11}|^{e_{n}} \theta^{e_{n}})
\]

\( = O(\theta^{e_{n}}) \). (2)

Let \( g \) be the total degree of \( F \) and \( d \) be a positive integer with \( \alpha_{d} \) (\( 1 \leq i \leq n \)) being algebraic integers. Then \( |F(U_m)| = O((A_{m-1}^{e_{n}-1})^{g}) \) and \( (M_{m-1}^{e_{n}-1})^{g}F(U_m) \) is an algebraic integer. Hence by (1), (2) and the fundamental inequality, we have \( F(U_m) = 0 \) for sufficiently large \( m \). By (2),

\[
\sum_{i=1}^{t} \sum_{l=0}^{L} P_{l}(e_{m}) \sum_{q=1}^{s_{i}} \partial F / \partial y_{i q}(l)(U_m) a_{i q}^{(e_{n})} = O(A^{e_{n}}),
\]

where \( A \) is a number with max \( (|\alpha_{t+1,1}|, |\alpha_{11}| \theta) < A < |\alpha_{11}| \). For each \( i, l(1 \leq i \leq t, 0 \leq l \leq L) \), let \( Q_{i}^{(l)} \) be a maximal subset of \( \{1, 2, \ldots, s_{i}\} \) such that \( \partial F / \partial y_{i q}(l) \) (\( q \in Q_{i}^{(l)} \)) are linearly independent over the algebraic numbers. Then we have

\[
\sum_{q=1}^{s_{i}} \partial F / \partial y_{i q}(l)(U_m) a_{i q}^{(e_{n})} = \sum_{q \in Q_{i}^{(l)}} \partial F / \partial y_{i q}(l)(U_m) \sum_{p=1}^{s_{i}} d_{i q}^{(l)} a_{i q}^{(e_{n})},
\]

where \( Q_{i}^{(l)} = (d_{i q}^{(l)}) \neq (0, \ldots, 0) \). By assumption, for any \( i \) there exists \( l \) \( (0 \leq l \leq L) \) with \( Q_{i}^{(l)} \) being non-empty. Let \( \alpha_{i q} = \xi_{i q} \gamma_{i} \) (\( 1 \leq q \leq s_{i} \)), \( \xi_{s_{i} q} = 1 \)
(1 ≤ i ≤ t, 1 ≤ q ≤ s_i), and take a triplet (i, l, q) with 1 ≤ i ≤ t, 0 ≤ l ≤ L and
q ∈ Q_i^{(l)}. We assert that \( \sum_{p=1}^{s_i} d_{ip}^{(l)} \neq 0 \) for any sufficiently large \( k \). This
implies the property ii). On the contrary, suppose that there is an integer \( a \)
\((0 ≤ a < N)\) such that \( e_m \equiv a (\text{mod } N) \) for infinitely many \( m \) and \( \sum_{p=1}^{d_{ip}^{(l)} q_p} \neq 0 \). Define

\[
D_{iq}^{(l)} = \sum_{p=1}^{s_i} d_{ip}^{(l)} q_p
\]

for each \( i, l, q \) with \( 1 ≤ i ≤ t, 0 ≤ l ≤ L, q ∈ Q_i^{(l)} \), and

\[
B = \{ (i, l, q) \mid D_{iq}^{(l)} \neq 0 \}.
\]

Then \( B \) is not empty. Let \( D \) be a positive integer with \( DD_{iq}^{(l)} ((i, l, q) ∈ B) \)
being algebraic integers, and define

\[
E_{iq}^{(l)}(m) = DM_{m-1} P_i(e_m) \partial F/\partial y_{iq}^{(l)}(U_m) D_{iq}^{(l)}
\]

for each \((i, l, q) ∈ B\). Since \( \lim_{m \to \infty} U_m = U \), there is a positive constant \( M \) such
that \( E_{iq}^{(l)}(m) \neq 0 \) for any \((i, l, q) ∈ B\) and any \( m > M \). By (3) and (4), if \( e_m \equiv a (\text{mod } N) \), then

\[
\sum_{(i, l, q) ∈ B} E_{iq}^{(l)}(m) y_{iq}^{e_m} = O(A^{e_m} M_{m-1}^g).
\]

We may assume \( \alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_l, D_{iq}^{(l)} \) and the coefficients of \( F \) are in \( K \),
by extending \( K \) if necessary.

Before proceeding, we must here explain Evertse’s theorem, which will play
an important role to prove our theorem. By a prime on \( K \) we mean an
equivalence class of non-trivial valuations on \( K \). We denote by \( S_K \) the set of
all primes on \( K \) by \( S_{\infty} \) the set of all infinite primes on \( K \). For every prime \( v \)
on \( K \) lying above a prime \( p \) on \( \mathbb{Q} \), we choose a valuation \( \| \cdot \|_v \) such that

\[
\| \alpha \|_v = | \alpha |_{p, \mathbb{Q}} \quad \text{for all } \alpha ∈ \mathbb{Q}.
\]

Then we have the product formula:

\[
\prod_{v ∈ S_K} \| \alpha \|_v = 1 \quad \text{for all } \alpha ∈ K, \quad \alpha \neq 0.
\]

For \( X = (x_0 : x_1 : \ldots : x_n) ∈ P^n(K) \), Put
By the product formula, this height is well-defined. Put

\[ h_K(\alpha) = h(\alpha) = H(1: \alpha) \quad \text{for } \alpha \in K. \]

Then we have the following fundamental inequality:

\[ -\log h(\alpha) \leq \sum_{v \in S} \log \|\alpha\|_v \leq \log h(\alpha) \quad \text{for } \alpha \in K, \quad \alpha \neq 0, \]

where \( S \) is any set of primes on \( K \). Let \( S \) be a finite set of primes on \( K \), enclosing \( S_\infty \), and \( c, d \) be constants with \( c > 0, d \geq 0 \). A projective point \( X \in P^n(K) \) is called \((c, d, S)\)-admissible if its homogeneous coordinates \( x_0, x_1, \ldots, x_n \) can be chosen such that

i) all \( x_k \) are \( S \)-integers, i.e. \( \|x_k\|_v \leq 1 \) if \( v \notin S \), and

ii) \[ \prod_{v \in S} \prod_{k=0}^n \|x_k\|_v \leq cH(x)^d. \]

The following theorem is due to [Evertse, 1984]: let \( c, d \) be constants with \( c > 0, 0 \leq d < 1 \) and let \( n \) be a positive integer. Then there are only finitely many \((c, d, S)\)-admissible projective points \( X = (x_0 : x_1 : \ldots : x_n) \in P^n(K) \) satisfying

\[ x_0 + x_1 + \cdots + x_n = 0 \]

but

\[ x_{i_0} + x_{i_1} + \cdots + x_{i_s} \neq 0 \]

for each proper, non-empty subset \( \{i_0, i_1, \ldots, i_s\} \) of \( \{0, 1, \ldots, n\} \).

Let \( S \) be a finite set of primes on \( K \) which includes \( S_\infty \) and all prime divisors of \( \alpha_1, \ldots, \alpha_n \). Then \( E_{ij}(m)\gamma_{ij}^{e_{ij}}((i, l, q) \in B, \ m > M) \) are nonzero \( S \)-integers and

\[ h\left(E_{ij}(m)\right) \leq c_2^{\log \epsilon_m + \epsilon_{m-1} + \log M_{m-1} + \log A_{m-1}}. \tag{6} \]

**Proposition 1.** Let \((i_j, l_j, q_j) \in B(j = 1, 2), i_1 \neq i_2 \) and \( m_1 > m_2 > M \). If \( m_1 \) is sufficiently large, then

\[ \left(E_{ij_1}(m_1)\gamma_{ij_1}^{e_{ij_1}} : E_{ij_2}(m_1)\gamma_{ij_2}^{e_{ij_2}}\right) \neq \left(E_{ij_1}(m_2)\gamma_{ij_1}^{e_{ij_1}} : E_{ij_2}(m_2)\gamma_{ij_2}^{e_{ij_2}}\right). \]

**Proof.** Suppose the proposition is not true. Then

\[ \left(\gamma_{ij_1}^{e_{ij_1}} : \gamma_{ij_2}^{e_{ij_2}}\right) = E_{ij_1}(m_1)E_{ij_1}(m_2)E_{ij_1}(m_1)^{-1}E_{ij_2}(m_2)^{-1}. \]
hence
\[ h\left(\gamma_{i_1}/\gamma_{i_2}\right)^{e_{m_1} - e_{m_2}} \leq e_2^{4\left(\log e_{m_1} + e_{m_1-1} + \log M_{m_1-1} + \log A_{m_1-1}\right)} \]
for infinitely many \(m_1\). Since \(\gamma_{i_1}/\gamma_{i_2}\) is not a root of unity and so \(h(\gamma_{i_1}/\gamma_{i_2}) > 1\), this contradicts the equality (1).

**Proposition 2.** Let \(B_0\) be any non-empty subset of \(B\). If \(m\) is sufficiently large, then
\[ \sum_{(i, l, q) \in B_0} E^{(l)}_{i q}(m) \gamma_i^{e_m} \neq 0. \]

**Proof.** We prove the proposition by induction on the cardinal number \(|B_0|\) of \(B_0\). If \(|B_0| = 1\), then the proposition is true. Let \(|B_0| \geq 2\) and suppose that
\[ \sum_{(i, l, q) \in B_0} E^{(l)}_{i q}(m) \gamma_i^{e_m} = 0 \tag{7} \]
for infinitely many \(m\). First we consider the case where \(i_0\) is the only integer such that \((i_0, l, q) \in B_0\) for some \(q\) and \(l\). Define
\[ l_0 = \max\{l | (i_0, l, q) \in B_0 \text{ for some } q\}. \]
Then
\[ P_{l_0}(e_m) \sum_{(i_0, l_0, q) \in B_0} \frac{\partial F}{\partial y_{i_0 q}^{(l_0)}} (U_m) D_{i_0 q}^{(l_0)} \]
\[ = - \sum_{(i_0, l, q) \in B_0} P_{l_0}(e_m) \frac{\partial F}{\partial y_{i_0 q}^{(l_0)}} (U_m) D_{i_0 q}^{(l_0)} \]
for infinitely many \(m\). When we divide both sides by \(P_{l_0}(e_m)\) and let \(m\) tend to infinitely, we have
\[ \sum_{(i_0, l_0, q) \in B_0} \frac{\partial F}{\partial y_{i_0 q}^{(l_0)}} (U) D_{i_0 q}^{(l_0)} = 0. \]
Since the total degrees of \(\partial F/\partial y_{i_0 q}^{(l_0)} (q \in Q_{i_0}^{(l_0)})\) are less than the total degree of \(F\),
\[ \sum_{(i_0, l_0, q) \in B_0} \frac{\partial F}{\partial y_{i_0 q}^{(l_0)}} D_{i_0 q}^{(l_0)} = 0. \]
This contradicts that $\partial F/\partial y^{(l_0)}_{i_0 q}(q \in Q^{(l_0)})$ are linearly independent over the algebraic numbers. Next we suppose there exist two triplets $(i_j, l_j, q_j) \in B_0$ ($j = 1, 2$) with $i_1 \neq i_2$. Let $\epsilon$ be any positive number $< 1$. By Proposition 1 and the induction hypothesis on $|B_0|$, applying Evertse's theorem to the equality (7) as $c = 1$ and $d = 1 - \epsilon$, we have

$$
\prod_{\nu \in \mathfrak{S}} \prod_{(i, l, q) \in B_0} || E^{(l)}_{i q}(m) y^{e_m}_i || \nu > H(... : E^{(l)}_{i q}(m) y^{e_m}_i : ...)^{1-\epsilon}
$$

for infinitely many $m$. By the facts that $\prod_{\nu \in \mathfrak{S}} || y_i || \nu = 1$ and there exists a prime $v$ on $K$ such that $|| y_i/y_j || \nu > 1$, we have

$$
c_s \sum_{l \in \mathfrak{S}} \log e_m + e_m - \log M_{m-1} - \log A_{m-1} > || E^{(l)}_{i q_1}(m)/E^{(l)}_{i q_2}(m) || v^{1-\epsilon} || y_i/y_j || v^{e_m(1-\epsilon)}.
$$

This is a contradiction.

Now we can complete the proof of Theorem 1. For infinitely many $m$ with $e_m \equiv a (\text{mod } N)$, we have, by (5),

$$
\sum_{(i, l, q) \in B} E^{(l)}_{i q}(m) y^{e_m}_i + \delta_m = 0, \quad \delta_m = O(A^{e_m M_{m-1}})
$$

where $E^{(l)}_{i q}(m) y^{e_m}_i$ and $\delta_m$ are $S$-integers. First we consider the case where $i_0$ is the only integer such that $(i_0, l, q) \in B$ for some $l$ and $q$. Define

$$
l_0 = \max \{ l \mid (i_0, l, q) \in B \text{ for some } q \}.
$$

Then

$$
P_{i_0}(e_m) \sum_{(i_0, l_0, q) \in B} \partial F/\partial y^{(l_0)}_{i_0 q}(U_m) D^{(l_0)}_{i_0 q} y^{e_m}_{i_0} = - \sum_{(i_0, l, q) \in B} P_l(e_m) \partial F/\partial y^{(l)}_{i_0 q}(U_m) D^{(l)}_{i_0 q} y^{e_m}_i + O(A^{e_m}).
$$

When we divide the both sides by $P_{i_0}(e_m) y^{e_m}_{i_0}$ and make $m$ tend to infinity, we have

$$
\sum_{(i_0, l_0, q) \in B} \partial F/\partial y^{(l_0)}_{i_0 q}(U) D^{(l_0)}_{i_0 q} = 0,
$$

since $A < |\alpha_{11}| = |y_{i_0}|$. This is a contradiction. Next we suppose that there are two triplets $(i_j, l_j, q_j) \in B$ ($j = 1, 2$) with $i_1 \neq i_2$. Let $\epsilon$ be any positive number $< 1$. We may assume $K$ is not a real field and $|| \cdot ||^2 = || \cdot ||_{v_0}$ for some infinite prime $v_0$ on $K$. By Proposition 1 and Proposition 2, applying the
Evertse's theorem to the equality (8), we have

\[
\left( \prod_{v \in S} \prod_{(i, l, q) \in B} \| E_{iq}^{(l)}(m) \|_v \right) \prod_{v \in S} \| \delta_m \|_v > H\left( \ldots : E_{iq}^{(l)}(m) \gamma_i \gamma_m : \ldots : \delta_m \right)^{1-\epsilon} \tag{9}
\]

for sufficiently large \( m \) with \( e_m \equiv a \pmod{N} \). By (6) and (8), the left hand side of the equality (9) is not greater than

\[
c_4^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}} \left( \prod_{v \in S} \max_{(i, l, q) \in B} \| \gamma_i \|_v \right)^{e_m} A^{2e_m}.
\]

On the other hand, taking a triplet \((i_0, l_0, q_0) \in B\), we have

\[
H\left( \ldots : E_{iq}^{(l)}(m) \gamma_i \gamma_m : \ldots \right)
\]

\[
\geq H\left( \ldots : E_{iq}^{(l)}(m) \gamma_i \gamma_m : \ldots \right)
\]

\[
= H\left( \ldots : E_{iq}^{(l)}(m) E_{iq_{l_0}}^{(l_0)}(m)^{-1} (\gamma_i / \gamma_{i_0}) \gamma_m: \ldots \right)
\]

\[
\geq \prod_{v \in S} \max_{(i, l, q) \in B} \| E_{iq}^{(l)}(m) E_{iq_{l_0}}^{(l_0)}(m)^{-1} \|_v \| \gamma_i / \gamma_{i_0} \|_{\gamma_m}^{e_m}
\]

\[
\geq c_5^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}} \left( \prod_{v \in S} \max_{(i, l, q) \in B} \| \gamma_i / \gamma_{i_0} \|_v \right)^{e_m}.
\]

Since

\[
\prod_{v \in S} \max_{(i, l, q) \in B} \| \gamma_i / \gamma_{i_0} \|_v = \prod_{v \in S} \max_{(i, l, q) \in B} \| \gamma_i / \gamma_{i_0} \|_v = \prod_{v \in S} \max_{(i, l, q) \in B} \| \gamma_i \|_v = \prod_{v \in S} \max_{(i, l, q) \in B} \| \gamma_i \|_v,
\]

the right hand side of the equality (9) is not less than

\[
c_5^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}} \left( \prod_{v \in S} \max_{(i, l, q) \in B} \| \gamma_i \|_v \right)^{(1-\epsilon)e_m}.
\]

Then we have

\[
c_6^{\log e_m + e_{m-1} + \log M_{m-1} + \log A_{m-1}} A^{2e_m}
\]
infinitely many \( m \). As \( m \) tends to infinity, we have

\[ 2 \log A \geq 2(1-\epsilon) \log |\alpha_{11}| - \epsilon \log \left( \prod_{\nu \neq \nu_0} \max_{(i, l, q) \in B} \| \gamma_i \|_\nu \right). \]

Since \( \epsilon \) is any positive number \( < 1 \), this implies \( \log A > \log |\alpha_{11}| \). This contradicts the fact \( A < |\alpha_{11}| \). This completes the proof of Theorem 1.

In [Nishioka, to appear in J. Number Theory], the \( p \)-adic analogue of Example 1 is proved. Here we have the following theorem. Let \( p \) be a prime number and denote by \( C_p \) the \( p \)-adic completion of \( \mathbb{Q}_p \) with respect to the valuation \( |\cdot|_p \). Let the convergence radius \( R_p \) of \( f(z) \) in \( C_p \) be positive. For an algebraic number \( \alpha \) with \( 0 < |\alpha|_p < R_p \) we denote by \( f(\alpha)_p \) the value of \( f(z) \) at \( z = \alpha \) in \( C_p \).

**THEOREM 2.** Let \( \alpha_1, \ldots, \alpha_n \) be algebraic numbers with \( 0 < |\alpha_i|_p < R_p (1 \leq i \leq n) \). Then the following three properties are equivalent.

i) \( f(\alpha_1)_p, \ldots, f(\alpha_n)_p \) are algebraically dependent over the rationals.

ii) There is a non-empty subset \( \{ \alpha_{i_1}, \ldots, \alpha_{i_s} \} \) of \( \{ \alpha_1, \ldots, \alpha_n \} \) such that \( \alpha_{i_1}, \ldots, \alpha_{i_s} \) are \( \{ e_k \} \)-dependent.

iii) \( 1, f(\alpha_1)_p, \ldots, f(\alpha_n)_p \) are linearly dependent over the algebraic numbers.

Theorem 2 is proved in the same way as Theorem 1, but ignoring the derivatives of \( f(z) \) (i.e. \( L = 0 \)).

**References**


