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Constructions of finitely generated submodules of constructively Noetherian modules

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Abstract

In constructive ring theory we define a commutative ring to be Noetherian if each ascending chain of finitely generated ideals iterates. Similarly, a module is Noetherian if each ascending chain of finitely generated submodules iterates. These definitions together with coherence are strong enough to allow for a substantial constructive theory of Noetherian rings and modules. On the other hand it is weak enough to have many models, including the ring of integers. This theory, however, does not imply that each ideal of a Noetherian ring is finitely generated. Therefore, if we want an ideal or submodule to be finitely generated, we have to give a constructive proof for the existence of these generators. We present another technique to show that in many cases we have this existence. We shall use that result to give constructive proofs of the Artin-Rees Lemma and of Krull's Intersection Theorem.

1. Preliminaries

Some definitions:

a) A commutative ring R is Noetherian if for each ascending chain $I_1 \subset I_2 \subset \ldots$ of finitely generated ideals in R there is n such that $I_n = I_{n+1}$. Similarly, a module M is Noetherian if for each chain $N_1 \subset N_2 \subset \ldots$ of finitely generated submodules there is n such that $N_n = N_{n+1}$.
b) A commutative ring R is coherent if all finitely generated ideals in R are finitely related. Similarly, a module M is coherent if all finitely generated submodules are finitely related.
c) Let X be a subset of Y. Then X is detachable from Y if for all $y \in Y$ we can decide whether or not $y \in X$.

Above conditions are straightforward generalizations of conditions a), b) and c) of [Seidenberg, 1974b].

A commutative ring R has detachable ideals if all finitely generated ideals of R are detachable from R. Similarly, a module M over R has detachable submodules if all finitely generated submodules are detachable from M.

A set X is discrete if for all x, $y \in X$ either $x = y$ or $x \neq y$. Hence a commutative ring R is discrete if and only if (0) is detachable from R. Similarly, a module M is discrete if and only if (0) is detachable from M.

The assumption of detachability is an inattractive one: it implies that the structures are discrete, thereby considerably limiting the number of models. However, if we assume detachability, then the Axiom of Dependent Choice can be avoided. This is of importance in case we want to work with topoi: the logic of topoi essentially is intuitionistic type theory with full comprehension, but without any choice principles. Also, the Lasker-Noether decomposition theorem as proved in [Seidenberg, 1984] assumes detachability from the outset. Examples of theorems where the proofs in the literature essentially use Dependent Choice unless detachability is assumed include: finitely presented modules over coherent commutative Noetherian rings are coherent and Noetherian, and Hilbert's Basis Theorem for coherent Noetherian rings.

In the next section we will make frequent use of the following theorem: let K, L be finitely generated submodules of a coherent module M over a commutative ring R. Then $K \cap L$ is a finitely generated submodule of M, and $K : L$ is a finitely generated ideal of R. □

For proofs, see [Seidenberg, 1974b], pp. 57–59, and [Richman, 1974], p. 438.

2. Artin-Rees Lemma, Krull's Intersection Theorem

Let M be a module over a commutative ring R. The tensor product $R[X] \otimes_R M$ is a module over $R[X]$. Its elements can be written as $\Sigma_i X^i \otimes m_i$. Let $M[X]$ be the module over $R[X]$ of formal expressions $\Sigma_{i=0}^{n} X^i m_i$ with scalar multiplication defined in the obvious way. One easily verifies that $R[X] \otimes_R M$ and $M[X]$ are isomorphic as modules over $R[X]$.

Lemma 1

Let M be a finitely presented module over a commutative ring R. Then $M[X]$ is a finitely presented module over $R[X]$. □

Let M be a module over a commutative ring R, and m an integer. Define $M[X]_m = \{ f \in M[X] \mid \deg(f) < m \}$. Note that $M[X]_m \cong M^m$ as modules over R. So if R is coherent (with detachable ideals) and M finitely presented, then $M[X]_m$ is a finitely presented coherent module (with detachable submodules) over R. If R is a coherent Noetherian ring (with detachable ideals) and M is finitely presented, then $M[X]_m$ is a finitely presented coherent Noetherian module (with detachable submodules) over R.

Lemma 2

Let M be a finitely presented module over a coherent commutative Noetherian ring R (with detachable ideals). Let N be an $R[X]$-submodule of $M[X]$

generated by $f_1, \ldots, f_s$. If $f_i \in M[X]_n$ for each i, then $P = N \cap M[X]_n$ is a finitely generated module over R such that $N \cap M[X]_m = \sum_{i=0}^{m-n} X^i P$ for each $m \geqslant n$.

Proof

We construct a chain $P_1 \subset P_2 \subset \ldots$ of finitely generated submodules of $N \cap M[X]_n$ as follows. Let $P_1 = Rf_1 + \ldots + Rf_s$, and let $P_{k+1} = P_k + XP_k \cap M[X]_n$. As $M[X]_{n+1}$ is coherent, the modules P_k are finitely generated; as $M[X]_n$ is Noetherian there is k such that $P_k = P_{k+1}$. Set $P = P_k$. Note that $XP \cap M[X]_n \subset P$.

As $P \subset N \cap M[X]_n$ we have $\sum_{i=0}^{m-n} X^i P \subset N \cap M[X]_m$. To show the reverse inclusion, suppose $f \in N \cap M[X]_m$. Write $f = \sum_{i=1}^{s} g_i f_i$, where $g_i \in R[X]_d$ for each i, and proceed by induction on d. If $d = 1$, then $f \in P_1$ and we are done. If $d > 1$, define $h_i \in R[X]$ by $g_i = g_i(0) + Xh_i$ and set $f^* = \sum_{i=1}^{s} h_i f_i \in N$. Note that $h_i \in R[X]_{d-1}$. Then $f = Xf^* + \sum_{i=1}^{s} g_i(0) f_i$, so $Xf^* \in N \cap M[X]_m$ whence $f^* \in M[X]_{m-1}$. If $m = n$, then induction on d gives $f^* \in P$ so $Xf^* \in XP \cap M[X]_n \subset P$ whereupon $f \in P_1 + P = P$. If $m > n$, then induction on d gives $f^* \in \sum_{i=0}^{m-1-n} X^i P$, so $f \in P_1 + X\sum_{i=0}^{m-1-n} X^i P \subset \sum_{i=0}^{m-n} X^i P$. □

Note that in above Lemma, if R has detachable ideals, then the ring $R[X]$ has detachable ideals, and N, $N[X]$ and $N \cap M[X]_m$ have detachable submodules.

Corollary 3

Let M be a finitely presented module over a coherent commutative Noetherian ring R (with detachable ideals). If N is a finitely generated $R[X]$-submodule of $M[X]$, then $N \cap M[X]_m$ is a finitely presented R-module, for each m. In particular, $N \cap M$ is a finitely presented R-submodule of M. □

A morphism ψ from an R-module M to an S-module N consists of a map $\psi: M \to N$ of abelian groups and a ring morphism $\varphi: R \to S$, such that $\psi(am) = \varphi(a)\psi(m)$ for all $a \in R$ and $m \in M$. If we consider N as a module over R, via $a \cdot n = \varphi(a)n$, then ψ is just an R-module homomorphism. The kernel $\mathrm{Ker}(\psi)$ is an R-submodule of M. Let M be a module over a commutative ring R, and let $\varphi: R[\overline{X}] \to R[\overline{Y}]$ be a ring morphism between the polynomial extensions $R[\overline{X}]$ and $R[\overline{Y}]$ of R such that φ is the identity on R. Then the canonical extension $\psi: M[\overline{X}] \to M[\overline{Y}]$ of φ from the $R[\overline{X}]$-module $M[\overline{X}]$ to the $R[\overline{Y}]$-module $M[\overline{Y}]$ is defined by $\psi(fm) = \varphi(f)m$ for all $f \in R[\overline{X}]$ and $m \in M$. We easily verify that ψ is a map in the above sense.

Following a suggestion of Fred Richman, morphisms as described above can also be considered as ring morphisms using Nagata's 'Principle of idealization': pass to the rings $R \oplus M$ and $S \oplus N$, where $mm' = 0$ for m, $m' \in M$, and $nn' = 0$ for n, $n' \in N$ (compare [Jacobson, 1974], p. 149). Then $\bar{\psi}$: $R \oplus M \to S \oplus N$ is simply a ring morphism extending φ.

Lemma 4

Let M be a module over a commutative ring R. Define $\overline{X}=(X_1,\ldots,X_m)$, and let $\varphi: R[\overline{X}]\to R$ be a map such that φ is the identity on R. Let $\psi: M[\overline{X}]\to M$ be the canonical extension of φ. Then $\mathrm{Ker}(\psi)=(X_1-\varphi(X_1),\ldots,X_m-\varphi(X_m))\cdot M[\overline{X}]$.

Proof

It suffices to prove the case for $\overline{X}=(X)$. Let $\psi(\Sigma_i f_i(X)m_i)=\Sigma_i f_i(\varphi(X))m_i=0$. By the remainder theorem there are $g_i(X)$ such that $f_i(X)=(X-\varphi(X))g_i(X)+f_i(\varphi(X))$. So $\Sigma_i f_i(X)m_i=\Sigma_i(X-\varphi(X))g_i(X)m_i+\Sigma_i f_i(\varphi(X))m_i=\Sigma_i(X-\varphi(X))g_i(X)m_i$. Hence $\Sigma_i f_i(X)m_i\in(X-\varphi(X))\cdot M[X]$. □

The key lemma that we will use (to prove the Artin-Rees Lemma, etc.) is the following:

Lemma 5

Let M be a finitely presented module over a coherent commutative Noetherian ring R (with detachable ideals). Define $\overline{X}=(X_1,\ldots,X_m)$ and $\overline{Y}=(Y_1,\ldots,Y_n)$. Let $\varphi: R[\overline{X}]\to R[\overline{Y}]$ be a map such that φ is the identity on R, and let $\psi: M[\overline{X}]\to M[\overline{Y}]$ be the canonical extension to M. Then ψ reflects finitely generated submodules, that is, if N is a finitely generated $R[\overline{Y}]$-submodule of $M[\overline{Y}]$, then $\psi^{-1}(N)$ is a finitely generated $R[\overline{X}]$-submodule of $M[\overline{X}]$.

Proof

Extend ψ to a map $\psi^+: M[\overline{X},\overline{Y}]\to M[\overline{Y}]$ by defining $\psi^+(Y_i m)=Y_i\psi(m)$ and $\varphi^+(Y_i)=Y_i$. Then ψ^+ is surjective, and $\mathrm{Ker}(\psi^+)=(X_1-\varphi(X_1),\ldots,X_m-\varphi(X_m))\cdot M[\overline{X},\overline{Y}]$ is a finitely generated module over $R[\overline{X},\overline{Y}]$. Let N be a finitely generated submodule of $M[\overline{Y}]$. Then $(\psi^+)^{-1}(N)$ is a finitely generated $R[\overline{X},\overline{Y}]$-submodule of $M[\overline{X},\overline{Y}]$. So after repeated application of Corollary 3 we have $\psi^{-1}(N)=(\psi^+)^{-1}(N)\cap M[\overline{X}]$ is a finitely generated $R[\overline{X}]$-module. □

If $M=R$, then above lemma specializes to:

Proposition 6

Let R be a coherent commutative Noetherian ring (with detachable ideals) and let S be a finitely generated extension of R such that $R\subset S\subset R[Y_1,\ldots,Y_n]$. Then S is a coherent Noetherian ring (with detachable ideals) such that the embedding $\sigma: S\to R[Y_1,\ldots,Y_n]$ reflects finitely generated ideals.

Proof

There is a map $\varphi: R[X_1,\ldots, X_m] \to R[Y_1,\ldots, Y_n]$ such that $im(\varphi) = S$. By Lemma 5 $\mathrm{Ker}(\varphi)$ is finitely generated, so $S \cong R[X_1,\ldots, X_m]/\mathrm{Ker}(\varphi)$ is a coherent Noetherian ring (with detachable ideals). Let I be a finitely generated ideal of $R[Y_1,\ldots, Y_n]$. Then by Lemma 5 $\sigma^{-1}(I) = \varphi(\varphi^{-1}(I))$ is finitely generated. □

Theorem 7 (Artin-Rees Lemma)

Let R be a coherent commutative Noetherian ring (with detachable ideals), $I \subset R$ a finitely generated ideal. Let $N \subset M$ be finitely presented modules over R. Then there is k such that for all $n \geqslant k$ we have

$$I^{n-k}(I^k M \cap N) = I^n M \cap N.$$

Proof

Let $I = (b_1,\ldots, b_m)$. Let $\varphi: R[X_1,\ldots, X_m] \to R[Y]$ be such that φ is the identity on R and $\varphi(X_i) = b_i Y$, and let $\psi: M[X_1,\ldots, X_m] \to M[Y]$ be the canonical extension of φ. By Lemma 5 $\mathrm{Im}(\psi) = M[IY]$ is a finitely presented module over the coherent Noetherian ring (with detachable ideals) $\mathrm{Im}(\varphi) = R[IY]$. The submodule $N[Y]$ is a finitely generated $R[Y]$-submodule of $M[Y]$, so $M[IY] \cap N[Y] = \psi(\psi^{-1}(N[Y]))$ is a finitely generated $R[IY]$-submodule of $M[IY]$. There is k such that $M[IY] \cap N[Y] = \Sigma_{i=0}^{\infty}(I^i M \cap N)Y^i$ is generated by $\Sigma_{i=0}^{k}(I^i M \cap N)Y^i$ as module over $R[IY]$. So for all $n \geqslant k$ we have $I^{n-k}(I^k M \cap N) = I^n M \cap N$. □
The ring $R[IY]$ in the proof of above theorem is known as the Rees ring.

Theorem 8 (Krull's Intersection Theorem)

Let M be a finitely presented module over a coherent commutative Noetherian ring R (with detachable ideals), and let I be a finitely generated ideal of R. Put $I^\infty M = \cap_n I^n M$. Then $I(I^\infty M) = I^\infty M$.

Proof

It suffices to prove $IN = N$ for each finitely generated module $N \subset I^\infty M$: by Theorem 7 there is k such that for all $n \geqslant k$ we have $I^n M \cap N = I^{n-k}(I^k M \cap N)$, so $N = I^{n-k}N$. Put $n = k+1$. □

By Nakayama's Lemma we have that if $IM = M$ for a finitely generated module M and an ideal I over a commutative ring R, then $I + Ann_R M = R$. This is equivalent to: there is $b \in I$ such that $(1 + b)M = 0$.

Proposition 9

Let M be a finitely presented module over a coherent commutative Noetherian ring R (with detachable ideals), and let I be a finitely generated ideal contained in the Jacobson radical rad(R). Then $I^{\infty}M = \cap_n I^n M = 0$. In particular, $I^{\infty} = 0$.

Proof

Let $N \subset I^{\infty}M$ be a finitely generated module. Then by Theorem 8 we have $IN = N$, so $(1 + b)N = 0$ for some $b \in I \subset \text{rad}(R)$. But $1 + b$ is invertible, so $N = 0$. Hence $I^{\infty}M = 0$. Put $M = R$. Then $I^{\infty} = 0$. □

Corollary 10

Let R be a coherent commutative Noetherian ring (with detachable ideals) such that the Jacobson radical rad(R) is finitely generated. Then $\text{rad}(R)^{\infty} = \cap_n \text{rad}(R)^n = 0$. □

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By personal communication we learned that Theorem 8 was also proved by Gabriel Stolzenberg, in 1984. His proof uses a more explicit approach like in [Seidenberg, 1974a].

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