COMPOSITIO MATHEMATICA

PIERRE LIARDET

Regularities of distribution

Compositio Mathematica, tome 61, n° 3 (1987), p. 267-293
<http://www.numdam.org/item?id=CM_1987__61_3_267_0>

Compositio Mathematica 61: 267–293 (1987)

Regularities of distribution

PIERRE LIARDET
U.E.R. de Mathématiques, Université de Provence, place V. Hugo F-13331 Marseille, Cedex 3, France and Institut für Mathematik, Universität in Salzburg Petersbrunnstrasse 19, A-5020 Salzburg, Austria

Received 30 October 1985; accepted 1 April 1986

dedicated to Jean Coquet †

Résumé. Une partie A de X, espace compact métrisable, est dite à restes bornés pour une suite donnée $x:\mathbb{N}\rightarrow X$ s'il existe a, $0\leqslant a\leqslant 1$, tel que la suite $n\rightarrow \operatorname{card}\{m<n;\ x_m\in A\}-na$ soit bornée. L'étude de ces ensembles est étroitement liée aux propriétés spectrales du flot associé à x. Les cas particuliers des suites $(n\alpha)$ dans $\mathbb{T}^d$, des suites de Weyl et des suites multiplicatives en base g sont examinés plus en détail.

Abstract. A subset A of a compact metrizable space X is said to be a bounded remainder set for a given sequence x: $\mathbb{N}\rightarrow X$, if there exists a, $0\leqslant a\leqslant 1$, such that the sequence $n\rightarrow \operatorname{card}\{m<n;\ x_m\in A\}-na$ is bounded. The study of such sets is closely related to spectral properties of the flow associated with x. We particularly investigate sequences $(n\alpha)$ in $\mathbb{T}^d$, Weyl sequences and multiplicative sequences to base g.

I. Introduction

I.1.

The aim of this paper is to study regularities of distribution for sequences $x=(x_n)_{n\geqslant 0}$ with values in a compact metrizable space X. One way to deal with regularities or as well irregularities of distribution is to describe *bounded remainder sets,* abbreviated B.R.S., in X for x. These are subsets A for which there exists a in [0, 1] such that the sequence of remainders $N\rightarrow r_N(A;\ a)$ defined by

$$r_N(A;\ a)=\operatorname{card}\{n<N;\ x_n\in A\}-Na$$

is bounded. In this case a is called an *admissible frequency* of x in A.

The first result is about the familiar sequence $(n\alpha)$ on the torus $\mathbb{T}$, with α irrational. It was proved by [Hecke, 1922] that for all arcs I in $\mathbb{T}$ of length $|I|>0$ in $\alpha\mathbb{Z}+\mathbb{Z}$, one has

$$|r_n(I;\ |I|)|\leqslant|h|,$$

where h is the unique integer such that $|I| - h\alpha \in \mathbb{Z}$. The proof derives from the relation

$$1_{[u,v[}(x) - (v-u) = \langle x - v\rangle - \langle x - u\rangle,$$

where 1_I denotes the characteristic function of I in $[0, 1[$ and $\langle y\rangle$ denotes the fractional part of the real number y; here $0 \leqslant x < 1$ and $0 \leqslant u \leqslant v < 1$. The converse, conjectured by [Erdös, 1964] was first proved by [Kesten, 1973]. Proofs and generalisations of this theorem in the framework of both topological dynamics and ergodic theory were given by [Furstenberg, Keynes and Shapiro, 1973; Petersen, 1973; Oren, 1982]. The method of Oren is purely topological. The proof of Petersen is ergodic and shows that the condition $\beta \in \alpha\mathbb{Z} + \mathbb{Z}$ is equivalent to

$$\sum_{m\neq 0}\left(\frac{\sin(m\beta\pi)}{m\sin(m\alpha\pi)}\right)^2 < +\infty. \tag{1}$$

I.2.

In some cases the sequence (x_n) is derived from iteration of a map T: $X \to X$. The initial point x_0 is given and

$$x_n = T(x_{n-1}), \quad n \geqslant 1. \tag{2}$$

The fact that $A(\subset X)$ is a B.R.S. depends on x_0, but in many examples this initial value is irrelevant. The subset A will be called *T-admissible* if A is a B.R.S. with the same admissible frequency for all sequences satisfying (2).

In the general case we look at a sequence as a dynamical system. Let S be the one-sided shift on the compact metrizable product space $X^{\mathbb{N}}$, given by

$$S(t_0, t_1, t_2, \ldots) = (t_1, t_2, \ldots).$$

An X-valued sequence x is viewed as a point of $X^{\mathbb{N}}$ and we denote by K_x the orbit closure of x with respect to the shift. We have $S(K_x) \subset K_x$, so that the restriction T of S on K_x gives the flow $\mathscr{K}(x) = (T;\ K_x)$. Now let $I(x)$ be the set of Borel probability measures on $X^{\mathbb{N}}$ which are accumulation points of the sequence

$$N \to \frac{1}{N}\sum_{n<N}\delta_{S^n x}$$

with respect to the weak topology. To each λ in $I(x)$ we associate the measured flow $\mathscr{K}(x;\lambda) = (T;\ K_x, \lambda)$. We shall be concerned with spectral properties of such a flow.

I.3.

In part II we give the ergodic method following the work of [Petersen, 1973] and we derive some general results on B.R.S. called *regular* (Theorem 2). In part III we find all *blocks* $A = \prod_{k=1}^{d} I_k$ with bounded remainders $r_n(A; a)$ for sequences $n \to (n\alpha_1, \dots, n\alpha_d)$ in $\mathbb{T}^d$ (Theorem 3). We also give examples of cylinders which are B.R.S. for these sequences (Theorem 4), generalizing earlier examples of [Szüsz, 1954] and [Rauzy, 1983–1984]. In part IV it is proved in particular that any arc A of $\mathbb{T}$ which is B.R.S. for a *Weyl sequence* of degree $d \geqslant 2$ is in fact trivial, namely is of length 0 or 1. The next part deals with *q-multiplicative sequences* such as $n \to e^{2i\pi\alpha s_g(n)}$, where α is an irrational number and $s_g(n)$ the sum of digits of n to the base g. Spectral properties of these sequences obtained in [Coquet *et al.,* 1977; Queffelec, 1979] are used to describe the related measured flows. In the case of *strongly* q-multiplicative sequences z with additional properties, Theorem 6 says that arcs which are B.R.S. are trivial. This extends a first result of [Queffelec, 1984] about sequences $(\alpha s_g(n))$.

II. Coboundaries

II.1.

Let E be a locally convex linear space (L.C.S.) over $\mathbb{R}$ or $\mathbb{C}$. The Schauder-Tychonoff theorem [Dunford and Schwartz, 1967] says that if K is a compact convex subset of E and if F is a continuous map from K into K, then F has a fixed point. Following the method of [Petersen, 1973] we give a coboundary theorem in a general form using the notion of quasi-complete L.C.R. [Bourbaki, 1964].

COBOUNDARY THEOREM. *Let E be a quasi-complete L.C.R. and let U: $E \to E$ be a linear map, continuous for the weak topology on E. Assume the E-valued sequence $n \to a_n$, $n \geqslant 0$, given by $a_0 = 0$ and*

$$a_n = a + U(a) + \dots + U^{n-1}(a)$$

is weakly relatively compact. Then there exists b in E such that

$$a = b - U(b).$$

a is called a U-coboundary and moreover for every continuous linear functional L,

$$|L(b)| \leqslant \sup_{n \geqslant 0} |L(a_n)|$$

holds.

Proof. Define F: $E \to E$ by $F(x) = a + U(x)$.

This map is weakly continuous. Let K be the convex (weak)-closure in E of the points a_n, $n \geqslant 0$. Since E is quasi-complete we derive from a theorem of Krein [Bourbaki, 1964] that K is weakly compact. Now we note that for all n-tuples $(\lambda_0, \ldots, \lambda_{n-1})$ of real numbers $\lambda_i \geqslant 0$ with $|\lambda| = \sum_{i<n} \lambda_i = 1$, one has $F(\sum_{i<n} \lambda_i a_i) = \sum_{i<n} \lambda_i a_{i+1}$. Hence $F(K) \subset K$ follows from the continuity of F and consequently there exists a fixed point b of F in K. Moreover for any continuous linear functional L,

$$|L(b)| \leqslant \sup_{|\lambda|=1} \sum_{i<n} \lambda_i |L(a_i)| \leqslant \sup_{n \geqslant 0} |L(a_n)|. \qquad \square$$

II.2.

The above theorem will be used particularly for $E = L^2$ and U a linear isometry arising from a measure-preserving transformation but we shall need an improvement of it. Let H be a Hilbert space endowed with the scalar product $(\,.\,|\,.\,)$, the norm $\|\,.\,\|$ and let U be a linear isometry of H. We do not require U to be unitary. Let x be an element of H. The map

$$(m, n) \to (U^m x \,|\, U^n x)$$

defined on $\mathbb{N} \times \mathbb{N}$ has a constant value $\gamma_x(a)$ on the half-line $m - n = a$, a given in $\mathbb{Z}$, and so $a \to \gamma_x(a)$ is defined on $\mathbb{Z}$. The map γ_x is the well known *correlation function* of x with respect to U; it is a positive-definite function. From the Bochner-Herglotz theorem, it follows that γ_x is the Fourier transform of a Borel measure λ_x on the torus $\mathbb{T}$, called the *spectral measure* of x with respect to U and moreover, if h denotes the Haar measure on $\mathbb{T}$:

$$\lambda_x(\mathrm{d}t) = * - \lim_{N:\infty} \frac{1}{N} \left\| \sum_{n<N} \mathrm{e}^{-2i\pi nt} U^n x \right\|^2 h(\mathrm{d}t)$$

where the limit is taken in the space $\mathscr{M}_*(\mathbb{T})$ of complex measures on $\mathbb{T}$, dual of $\mathscr{C}(\mathbb{T})$, endowed with the weak topology. We recall that if $\mathscr{M}(\mathbb{T})$ denotes the dual Banach space of $\mathscr{C}(\mathbb{T})$, then the map $x \to \lambda_x$ from H to $\mathscr{M}(\mathbb{T})$ is continuous. By means of the above weak limit one gets $\lambda_{cx} = |c|^2 \lambda_x$ for any complex number c and $\lambda_{x+y} \leqslant 2\lambda_x + 2\lambda_y$ for any x and y in H.

THEOREM 1. *Let U be a linear isometry on the Hilbert space H and let x be in H, λ_x its spectral measure with respect to U. Then the following are equivalent*:

i) The sequence $n \to x + U(x) + \ldots + U^{n-1}(x)$ is bounded in H.

ii) *x is a U-coboundary*; *there exist y in H such that* $x = y - U(y)$.
iii) *The map*

$$t \to \frac{1}{\sin^2 \pi t}$$

from $\mathbb{T}$ *into* $\overline{\mathbb{R}}$ (*with the value* $+\infty$ *at* $t=0$) *is* λ_x-*integrable*.

Proof. The unit ball of H is weakly compact according to the theorem of Alaoglu, therefore *i*) and *ii*) are equivalent by the coboundary theorem.

ii) ⇒ *iii*): Assume $x = y - Uy$, then a straightforward computation gives $\gamma_x(n) = 2\gamma_y(n) - \gamma_y(n-1) - \gamma_y(n+1)$ $(n \in \mathbb{Z})$ and using spectral measures we obtain

$$\gamma_x(n) = \int_{\mathbb{T}} 4 \sin^2(\pi t)\, e^{2i\pi nt} \lambda_y(dt)$$

in other words

$$\lambda_x(dt) = 4 \sin^2(\pi t) \lambda_y(dt).$$

Therefore $\lambda_x(\{0\}) = 0$ since λ_x is bounded and we get iii) with the following value:

$$\int_{\mathbb{T}} \frac{1}{4 \sin^2(\pi t)} \lambda_x(dt) = \lambda_y(\mathbb{T}) = \| y \|^2.$$

iii) ⇒ *i*): Assume that $\dfrac{1}{\sin^2(\pi t)}$ is λ_x-integrable. A straightforward computation gives

$$\lambda_{x_N}(dt) = \left| \frac{1 - e^{2i\pi Nt}}{1 - e^{2i\pi t}} \right|^2 \lambda_x(dt),$$

where $x_N := \sum_{n<N} U^n(x)$, $N \in \mathbb{N}$. In particular,

$$\lambda_{x_N}(\mathbb{T}) = \left\| \sum_{n<N} U^n(x) \right\|^2 \leq \int_{\mathbb{T}} \frac{1}{\sin^2(\pi t)} \lambda_x(dt).$$

This proves i). □

Note that the above proof gives

COROLLARY. *With the assumptions of Theorem* 1, *the equality* $x = y - U(y)$ *implies*:

$$\| y \| = \left\| \frac{1}{2 \sin \pi(\cdot)} \right\|_{L^2(\mathbb{T};\lambda_x)}.$$

II.3.

Now we quote a simple but typical fact about bounded remainder sets for sequences arising from regular processes. By a *process* $\mathscr{X}$ we shall mean in this paper a triplet $(T; X, \mu)$ where X is a compact metrizable space, T a Borel transformation from X to X and μ a Borel measure which is preserved by T. We recall that a map defined on X into a topological space is said to be μ-continuous if it is continuous at μ-almost every point of X. A subset Y of X is then said to be μ-continuous if its characteristic function $1Y$ is μ-continuous. The process $\mathscr{X}$ is called *regular* if T is μ-continuous. Let J be an infinite subset of $\mathbb{N}$. A point x in X is called $(\mathscr{X}, J)$-generic if for all continuous complex maps f: $X \to \mathbb{C}$, one has

$$\lim_{N \in J} \frac{1}{N} \sum_{n<N} f(T^n x) = \int_X f(t) \mu(\mathrm{d}t). \tag{3}$$

When the map f: $X \to \mathbb{C}$ is only μ-Riemann-integrable, that is to say bounded and μ-continuous, then (3) is still true whenever $\mathscr{X}$ is regular.

THEOREM 2. *Let $\mathscr{X} = (T; X, \mu)$ be a regular process and x a $(\mathscr{X}, J)$-generic point. If A is a μ-continuous B.R.S. for the sequence $\xi = (T^n x)_n$ of admissible frequency a then there exists F in $L^\infty(X; \mu)$ such that*

$$1_A - a = F \circ T - F \quad \mu - a.e. \tag{4}$$

Moreover $t \to \dfrac{1}{\sin^2(\pi t)}$ is integrable for the spectral measure λ_f of $f = 1_A - a$ with respect to the linear isometry given by T on $L^2(X; \mu)$.

Proof. Assume A is μ-continuous subset of X. Put $f = 1_A - a$, $f_0 = 0$ and $f_m = f + f \circ T + \dots + f \circ T^{m-1}$ for $m \geqslant 1$. The maps f_m are μ-Riemann-integrable and by assumption on x for all p in $[1, +\infty[$ one has:

$$\lim_{N \in J} \frac{1}{N} \sum_{n<N} | f_m(T^n x) |^p = \| f_m \|^p_{L^p(X;\mu)}.$$

Let A be a B.R.S. and let a be the corresponding admissible frequency. There exists $c \geqslant 0$ such that

$$\sup_m | f_m(x) | \leqslant c,$$

hence for all integers m, $n \geqslant 0$,

$$| f_m(T^n x) | \leqslant 2c.$$

This implies

$$\| f_m \|_{L^p(X;\mu)} \leqslant 2c$$

for all p, $1 \leqslant p < +\infty$ and consequently these inequalities also hold for $p = +\infty$. Now the sequence $m \to f_m$ is strongly bounded in $L^\infty(X; \mu)$, dual of $L^1(X; \mu)$. From the theorem of Alaoglu (f_m) is relatively compact for the weak topology $\sigma(L^\infty, L^1)$ and the coboundary theorem can be applied to $E = L^\infty(X, \mu)$ endowed with the weak topology. Therefore f is a U_T-coboundary where U_T is the linear isometry derived from the map $\varphi \to \varphi \circ T$ defined for complex maps φ: $X \to \mathbb{C}$. □

Remark 1. The equality $1_A - a = F \circ T - F$ μ.a.e. implies that for μ-almost every point y in X, the set A is a B.R.S. for the sequence $\eta = (T^n y)_n$. Conversely, if A is a Borel set and if there exist a Borel subset Y of X with $\mu(Y) > 0$ and A a B.R.S for all sequences $(T^n y)_n$ with $y \in Y$, then there exists F in $L^\infty(X, \mu)$ satisfying (4) with $a = \mu(A)$. This derives directly from the individual ergodic theorem. Note that in [Halász, 1976] Halász gets the same result but the remainder $r_N(y) = \sum_{n<N} 1_A(T^n y) - N\mu(A)$ is only assumed to be bounded below on Y.

Remark 2. Theorem 2 can be viewed as the metric version of the classical theorem of [Gottschalk and Hedlund, 1955].

III. The sequences $(n\alpha)_n$ in $\mathbb{T}^d$

III.1.

Let τ_α: $\mathbb{T}^d \to \mathbb{T}^d$ be the translation given by $\tau_\alpha(x) = x + \alpha$. It is well known that τ_α preserves the Haar measure h on $\mathbb{T}^d$ and so induces a unitary operator $U_\alpha: L^2(\mathbb{T}^d, h) \to L^2(\mathbb{T}^d, h)$ given by $U_\alpha(f) = f \circ \tau_\alpha$.

THEOREM 3. *Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be in $\mathbb{T}^d$ such that $\alpha_1, \ldots, \alpha_d$ are $\mathbb{Z}$-independent in $\mathbb{T}$. Let $P = I_1 \times \cdots \times I_d$ be a block with arcs I_j of length $|I_j| \neq 0$. Then P is a bounded remainder set for $(n\alpha)_n$ if and only if there exists an index k such that $|I_k| \in \mathbb{Z}\alpha_k + \mathbb{Z}$ and for all other indices $j \neq k$, one has $|I_j| = 1$.*

Proof. Assume that the given block P satisfies $|I_k| \in \mathbb{Z}\alpha_k + \mathbb{Z}$ for an index k and $|I_j| = 1$ for the other indices j. Clearly we can as well assume $I_j = \mathbb{T}$ so that P is a B.R.S. from Kesten's theorem. Note that the case $h(P) = 0$ is obvious.

Now suppose that P is a B.R.S. for $(n\alpha)_n$ with $h(P)=\prod_{n=1}^{d}|I_n|\neq 0$ and put $f=1_P-h(P)$. Since $(n\alpha)_n$ is uniformely distributed in $\mathbb{T}^d$ we derive as in Theorem 2 that the sequence

$$N\to f+f\circ\tau_\alpha+\cdots+f\circ\tau_\alpha^{N-1}$$

is bounded in $L^2(\mathbb{T}^d;\,h)$ (and also in L^∞). By Theorem 1, the map $\frac{1}{\sin^2\pi(\cdot)}$ is integrable for the spectral measure h_f of f with respect to U_α. Using the Fourier expansion of f we easily verify that

$$h_f=\sum_{\substack{m\in\mathbb{Z}^d\\ m\neq 0}}|(1_P|\mathrm{e}_m)|^2\delta_{(m|\alpha)},$$

where δ_a is the Dirac measure at the point a $(a\in\mathbb{T})$, $(m|\alpha)=m_1\alpha_1+\cdots+m_d\alpha_d$ for $m=(m_1,\ldots,m_d)$ in $\mathbb{Z}^d$ and e_m is the character on $\mathbb{T}^d$ given by $\mathrm{e}_m(t)=\mathrm{e}^{2i\pi(m|t)}$. The condition iii) in Theorem 1 is now equivalent to

$$\sum_{\substack{m\in\mathbb{Z}^d\\ m\neq 0}}\frac{|(1_P|\mathrm{e}_m)|^2}{(\sin\pi(m|\alpha))^2}<+\infty. \tag{5}$$

For $d=1$, (5) is the Petersen's condition (1). For $d\geqslant 2$ we get from (5)

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\frac{|(1_{I_j}|\mathrm{e}_m)|^2}{(\sin\pi m\alpha_j)^2}<+\infty,$$

hence

$$|I_j|\in\mathbb{Z}\cdot\alpha_j+\mathbb{Z}$$

for all indices j. Assume that k is an index such that $|I_k|\neq 1$. From (5) we derive now:

$$\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\left(\frac{\sin(\pi m|I_j|)\sin(\pi|I_k|)}{m\sin\pi(m\alpha_j+\alpha_k)}\right)^2<+\infty \tag{6}$$

for all indexes j, $j\neq k$. By a result of [Kronecker, 1884] there exists a constant $c>0$ such that

$$\liminf_{m:\infty}|m\sin\pi(m\alpha_j+\alpha_k)|\leqslant c.$$

In other words, there is a sequence $(m_n)_n$ of integers such that

$$\lim_{n:\infty} (m_n\alpha_j) = -\alpha_k \tag{7}$$

on the torus $\mathbb{T}$ and

$$\frac{\sin(\pi m_n |I_j|)}{m_n \sin \pi(m_n\alpha_j + \alpha_k)} \geqslant \frac{\sin(\pi m_n |I_j|)}{2c}.$$

But now (6) implies

$$\lim_{n:\infty} \sin(\pi m_n |I_j|) = 0.$$

Since there exists $p \in \mathbb{Z}$ with $|I_j| - p\alpha_j \in \mathbb{Z}$, it follows

$$\lim_{n:\infty} (m_n p\alpha_j) = 0$$

in $\mathbb{T}$ and (7) gives

$$p \cdot \alpha_k = 0,$$

so that $p = 0$, hence $|I_j| = 1$. □

III.2.

We now give a construction of bounded remainder sets in d-dimensional torus, $d \geqslant 2$, which are not blocks. To do this we need some definitions. Let $v = (v_1, \ldots, v_d)$ be in $\mathbb{R}^d$ with $v_d \neq 0$ and let ρ: $\mathbb{R}^d \to \mathbb{R}^d/\mathbb{Z}^d$ be the canonical map. τ_v is the translation by v modulo $\mathbb{Z}^d$ to which is associated the *cross translation* θ_v modulo $\mathbb{Z}^d$ given by

$$\theta_v(t_1, \ldots, t_{d-1}, 0) = \left(t_1 + \frac{v_1}{v_d}, \ldots, t_{d-1} + \frac{v_{d-1}}{v_d}, 0\right) \bmod \mathbb{Z}^d.$$

We consider $\mathbb{T}^d$ as $Q_d = \prod_{i=1}^{d} [0, 1[$. Now a subset Σ of Q_d will be called a *section* for v if for any point σ in Σ one has

$$\Sigma \cap (\sigma + \mathbb{R} \cdot v) = \{\sigma\}.$$

Put $H_0 = \{t \in \mathbb{R}^d;\ t_d = 0\}$ and define

$$\Sigma_0 := (\Sigma + \mathbb{R} \cdot v) \cap H_0.$$

Finally, a subset B in $\mathbb{R}^d$ will be called *simple* (with respect to $\mathbb{Z}^d$) if it is bounded and satisfies

$$\forall x,\ y \in B;\ x - y \in \mathbb{Z}^d \Rightarrow x = y.$$

The following theorem extends results of [Szüsz, 1954], its proof follows an idea of [Larcher, 1985].

THEOREM 4. *Let* $\alpha = (\alpha_1, \ldots, \alpha_d)$ *be in* Q_d *with* $1, \alpha_1, \ldots, \alpha_d$ $\mathbb{Z}$*-independent, let* v *in* $\mathbb{N}\alpha + \mathbb{Z}^d$, $v \notin \mathbb{Z}^d$ *and let* Σ *be a section for* v *such that* $\rho(\Sigma_0)$ *is* θ_v*-admissible. If the cylinder* $C = \Sigma + [0, 1[.\ v$ *is simple then* $\rho(C)$ *is* τ_α*-admissible.*

Proof. We first prove the theorem in a particular case. Assume $\Sigma = \Sigma_0$ and denote by $\langle x \rangle$ the point in Q_d congruent to x ($x \in \mathbb{R}^d$) modulo $\mathbb{Z}^d$. Let $p \in \mathbb{N}$, $p \neq 0$, and $v = \langle p\alpha \rangle$. For any x in Q_d we denote by $X_n(x)$ the point $x + n.v$ in $\mathbb{R}^d$ and we define the half-straight lines

$$D := \{ y \in \mathbb{R}^d;\ \exists \lambda \geqslant 0,\ y = \lambda X_1(0) \},$$

$$D_x := D + x.$$

Let $d_n(x)$ be the intersection point of D_x with the affine hyperplane $P_n = \{ y \in \mathbb{R}^d;\ y_d = n \}$, $n \in \mathbb{N}$. For any point ξ in Q_d, we have to prove that the sequence $(\xi_N)_N$ given by

$$\xi_N := \sum_{n<N} 1_B(\rho(\xi + n\alpha)) - \langle p\alpha_1 \rangle \cdot \lambda N$$

is bounded, where $B = \rho(C)$ and λ is the admissible frequency of $\Sigma_0 (= \Sigma)$ for the cross translation θ_v. Let k_N be the integer such that $X_{[(N/p)]}(0)$ belongs to the segment $[d_{k_N - 1}(0),\ d_{k_N}(0)]$, then

$$k_N = \frac{N}{p} \langle p\alpha_1 \rangle + \mathcal{O}(1)$$

and with $\Sigma' = \Sigma + \mathbb{Z}^d$:

$$\sum_{n<N} 1_B(\rho(\xi + n\alpha)) = \sum_{r=0}^{p-1} \sum_{k<k_N} c_{k,r} 1_{\Sigma'}\ (d_k(\langle \xi + r\alpha \rangle)) + \mathcal{O}(1)$$

where $c_{k,r}$ is the number of points $X_m(\langle \xi + r\alpha \rangle)$ on the segment $[d_k(\langle \xi + r\alpha \rangle),\ d_{k+1}(\langle \xi + r\alpha \rangle)]$ such that $m < \dfrac{N}{p}$ and

$$\langle \xi_d + r\alpha \rangle + m \langle p\alpha_d \rangle \in [k,\ k + \langle p\alpha_1 \rangle[.$$

But for $k \geqslant 1$, $t \in [0, 1[$ and $a + [0, 1[$ one has

$$\operatorname{card}\{m \in \mathbb{N};\ k \leqslant t + ma < k + a\} = 1$$

hence

$$\sum_{n<N} 1_B(\rho(\xi + n\alpha)) = \sum_{r=0}^{p-1} \sum_{k<k_N} 1_\Sigma \circ \theta_v^k(\langle \xi + r\alpha \rangle) + \mathcal{O}(1)$$

$$= p\lambda k_N + \mathcal{O}(1) = \langle p\alpha_1 \rangle \lambda N + \mathcal{O}(1),$$

this proves the particular case.

Now let A be any τ_α-admissible part of $\mathbb{T}^d$ and let B be a subset of A. Then for any β in $\mathbb{Z}\alpha$, the set $(A \setminus B) \cup (\tau_\beta B)$ is τ_α-admissible. From this we derive the theorem in its general form. □

IV. Weyl sequences

IV.1. Weyl flows

Let $P(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_d$ be a polynomial of degree $d \geqslant 2$ with real coefficients a_i. When a_0 is irrational P is said to be a *Weyl polynomial.* Put $\alpha = d!a_0$. Following [Furstenberg, 1981] we define the flow $W_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ by

$$W_\alpha(t_1, \ldots, t_d) = (t_1 + \alpha,\ t_2 + t_1, \ldots, t_d + t_{d-1}).$$

This flow is uniquely ergodic, preserving the Haar measure. Let Δ be the transformation defined on the space of real polynomials by

$$\Delta Q(X) = Q(X+1) - Q(X).$$

The following formula will be useful:

$$W_\alpha^m(\Delta^{d-1}p(0), \ldots, \Delta p(0),\ p(0)) = (\Delta^{d-1}p(m), \ldots, \Delta p(m),\ p(m)) \quad (8)$$

for all $m \in \mathbb{Z}$.

IV.2.

The next theorem derives from spectral properties of W_α.

THEOREM 5. *Let A be a h-continuous subset of $\mathbb{T}$ for the Haar measure h. Let P be a Weyl polynomial of degree $d \geqslant 2$. If A is a B.R.S. for the sequence $(p(n)$* mod $1)_n$ *then the Haar measure $h(A)$ of A is* 0 *or* 1.

Proof. Define $B := \{(t_1, \ldots, t_d) \in \mathbb{T}^d;\ t_d \in A\}$ and put $f := 1_B - h(B)$. By assumption and (8) there exists $C > 0$ such that

$$\left| \sum_{m<M} f \circ W_\alpha^m(\Delta^{d-1}p(0), \ldots, \Delta p(0), p(0)) \right| \leqslant C.$$

But any point x in $\mathbb{T}^d$ is $(W_\alpha, \mathbb{N})$ – generic hence by Theorem 2 the map $t \to \dfrac{1}{\sin^2 \pi t}$ on $\mathbb{T}$ is integrable for the spectral measure h_f of f with respect to W_α. The Fourier expansion

$$f = \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} (1_A | \mathrm{e}_m)\, \mathrm{e}_m$$

holds in $L^2(\mathbb{T}^d, h)$ with e_m viewed as the character

$$(t_1, \ldots, t_d) \to \mathrm{e}^{2i\pi m t_d}$$

on $\mathbb{T}^d$. By finite induction we easily obtain that for m in $\mathbb{Z}$, and n in $\mathbb{N}$ there exists a continuous map

$$\psi_{m,n} : \mathbb{T}^{d-2} \to \mathbb{C},$$

which is constant if $d = 2$ (and in this case, we put $\mathbb{T}^0 = \{0\}$) such that

$$\mathrm{e}_m \circ W_\alpha^n(t_1, \ldots, t_d) = \psi_{m,n}(t_1, \ldots, t_{d-2})\, \mathrm{e}_{mn}(t_{d-1})\, \mathrm{e}_m(t_d)$$

on $\mathbb{T}^d$. Hence

$$(\mathrm{e}_m \circ W_\alpha^n | \mathrm{e}_{m'}) = \begin{cases} 0 & \text{if } m \neq m' \quad \text{or } n \neq 0, \\ 1 & \text{if } m = m' \quad \text{and } n = 0, \end{cases}$$

so that

$$(f \circ W_\alpha^n | f) = \begin{cases} 0 & \text{if } n \neq 0, \\ \| f \|^2 & \text{if } n = 0. \end{cases}$$

It follows

$$h_f(\mathrm{d}t) = \| f \|^2 h(\mathrm{d}t).$$

Since $\| f \|^2 = h(A)(1 - h(A))$, the integrability condition iii) of Theorem 1 implies $h(A) = 0$ or 1. □

V. q-Multiplicative sequences

V.1.

Let $q=(q_n)_{n\geqslant 1}$ be a sequence of integers $q_n \geqslant 2$. We put $q_0=1$ and $p_n = q_0 q_1 \dots q_n$. A complex valued sequence $z:\mathbb{N}\to\mathbb{C}$ is said to be q-multiplicative if for all integers $b\geqslant 0$, $n\geqslant 0$ and $0\leqslant a<p_n$, one has

$$z(a+bp_n)=z(a)z(bp_n). \tag{9}$$

The special case where $z(bp_n)=\mathrm{e}^{2\mathrm{i}\pi b/p_{n+1}}$ for $0\leqslant b<q_{n+1}$ corresponds to a generalisation of Halton sequences. The set of arcs of bounded remainder for such sequences is almost completely determined by [Hellekalek, 1984]. Let g be an integer $\geqslant 2$ and assume that in place of (9) one has

$$z(a+bp_n)=z(a)z(b) \tag{10}$$

where $p_n=g^n$. Then z is said to be a *strongly multiplicative* sequence to base g. An example is provided by the sequence $n\to\mathrm{e}^{2\mathrm{i}\pi\alpha s_g(n)}$ given in the introduction and for which the corresponding dynamical system is known to be a uniquely ergodic flow [Kamae, 1977]. Another example is the Kakutani sequence studied in part V.4. below.

From now on we assume that the q-multiplicative sequences take their values in the group of complex numbers of modulus 1. This group will be denoted by $\mathbb{U}$.

THEOREM 6. *Let z be a strongly multiplicative ($\mathbb{U}$-valued) sequence to base g. The following hold:*

i) The associated flow $\mathscr{K}(z)$ is uniquely ergodic and z is well-distributed in the closed subgroup $\mathbb{U}(z)$ generated by all the values of z.

ii) Assume moreover that z takes a value which is not a root of unity (hence $\mathbb{U}(z)=\mathbb{U}$) and suppose there exists an integer $m\neq 0$ prime to g such that:

$$(*)\qquad \left|\sum_{j<g}(z(j))^m\right|>1.$$

Then an arc I of $\mathbb{U}$ is a B.R.S. for z if and only if the length of I is 0 or 2π.

Before we are going to prove Theorem 6 we will study q-multiplicative flows.

V.2. q-Multiplicative flows

Let z: $\mathbb{N}\to\mathbb{U}$ be a q-multiplicative sequence and let $\mathscr{K}(z;\, \nu)$ be the corresponding measured flow with $\nu\in I(z)$. For any integer $k\geqslant 0$ we define the sequences $q^{(k)}$, $p^{(k)}$ and $z^{(k)}$ respectively by

$$q_0^{(k)} = 1, \quad q_n^{(k)} = q_{n+k} \quad (n \geqslant 1)$$

$$p_n^{(k)} = q_0^{(k)} \dots q_n^{(k)}$$

and

$$z^{(k)}(n) = z(np_k) \quad (n \in \mathbb{N}).$$

Clearly $z^{(k)}$ is a $q^{(k)}$-multiplicative sequence. Finally for each $m = (m_0, \dots, m_s)$ in $\mathbb{Z}^{s+1}$ we associate the character χ_m on $\mathbb{U}^{\mathbb{N}}$ given by

$$\chi_m(u_0, u_1, u_2, \dots) := u_0^{m_0} \dots u_s^{m_s}$$

and for short, we put

$$\Omega := \mathbb{U}^{\mathbb{N}}$$

$$\chi_m(j) := \chi_m(S^j(z)),$$

$$|m| := m_0 + \cdots + m_s.$$

We prove the following extension of earlier results [Coquet *et al.,* 1977; Liardet, 1980; Queffelec, 1984].

THEOREM 7. *Let z: $\mathbb{N} \to \mathbb{U}$ be a q-multiplicative sequence. For any m in $\mathbb{Z}^{s+1}$ the spectral measure $\Lambda_{z,m}$ of χ_m with respect to $\mathscr{H}(z;\, \nu)$ does not depend on the choice of ν in $I(z)$. Moreover, let $P_{m,N}$ be the polynomial*

$$P_{m,N}(t) := \frac{1}{N}\left| \sum_{j<N} \chi_m(j)\, \mathrm{e}^{-2\mathrm{i}\pi jt} \right|^2,$$

then
i) $\Lambda_{z,m}$ is the weak limit of $\mu_{m,N}(\mathrm{d}t) = P_{m,N}(t)h(\mathrm{d}t)$.
ii) Let λ be a Borel measure on $\mathbb{T}$ and let p be an integer > 0.
We denote by λ^p the Borel measure given by

$$\int_{\mathbb{T}} f(t)\lambda^p(\mathrm{d}t) = \int_{\mathbb{T}} \left(\frac{1}{p} \sum_{l<p} f\left(\frac{u}{p} + \frac{l}{p} \right) \right) \lambda(\mathrm{d}u)$$

for all f in $\mathscr{L}(\mathbb{T})$. Then $\Lambda_{z,m}$ is the limit in the sense of total variance (i.e. in $\mathscr{M}(\mathbb{T})$) of the sequence of measures

$$\sigma_{m,k}(\mathrm{d}t) = P_{m,p_k}(t)(\Lambda_{z^{(k)},|m|})^{p_k}(\mathrm{d}t).$$

Proof. Let k be a given integer and define $\chi_m^{(k)}: \mathbb{N} \to \mathbb{U}$ by

$$\chi_m^{(k)}(j) := \left(z\left(\left[\frac{j}{p_k}\right] p_k\right)\right)^{|m|} \chi_m^*(j) \tag{11}$$

where $[x]$ is the integral part of x and $\chi_m^*(\cdot)$ is the periodic sequence of period p_k given by

$$\chi_m^*(j) = \chi_m(j)$$

for $j = 0, 1, \ldots, p_k - 1$. By construction $\chi_m^{(k)}(j) = \chi_m(j)$ for all integers j which do not belong to

$$A_k = \bigcup_{l=1}^{\infty} ([lp_k - s,\ lp_k[\, \cap \mathbb{N}).$$

From now on we assume $s < p_k$ so that A_k has the density s/p_k in $\mathbb{N}$.

Let J be an infinite part of $\mathbb{N}$ such that ν is the weak limit of $N \to \frac{1}{N} \sum_{j<N} \delta_{S^j z}$ $(N \in J)$. Hence

$$\lim_{N \in J} \frac{1}{N} \sum_{j<N} \chi_m(j) = \int_\Omega \chi_m(u) \nu(\mathrm{d}u).$$

Assume $|m| = 0$, then

$$\lim_{N:\infty} \sup \frac{1}{N} \sum_{j<N} |\chi_m(j) - \chi_m^*(j)| \leqslant \frac{s}{p_k}$$

and χ_m^* has the mean

$$M_k = \frac{1}{p_k} \sum_{j<p_k} \chi_m(j).$$

This proves that

$$\lim_{k:\infty} M_k = \int_\Omega \chi_m(u) \nu(\mathrm{d}u)$$

and implies finally:

LEMMA 1. *If* $|m| = 0$, *then for any* ν *in* $I(z)$:

$$\int_\Omega \chi_m(u) \nu(\mathrm{d}u) = \lim_{N:\infty} \frac{1}{N} \sum_{j<N} \chi_m(j).$$

Now let γ_m be the correlation function

$$\gamma_m(n) := \int_\Omega \chi_m(S^n u)\overline{\chi_m(u)}\nu(\mathrm{d}u) = \lim_{N\in J} \frac{1}{N} \sum \chi_m(j+n)\overline{\chi_m(j)}.$$

On the other hand there exist $s' \in \mathbb{N}$ and $m' \in \mathbb{Z}^{s'+1}$ such that $|m'| = 0$ and $\chi_m(j+n)\overline{\chi_m(j)} = \chi_{m'}(j)$ for all j, hence from the above Lemma $\gamma_m(n)$ does not depend on ν and the same is true for $\Lambda_{z,m}$ with Fourier coefficients given by

$$\hat{\Lambda}_{z,m}(n) = \lim_{N:\infty} \frac{1}{N} \sum_{j<N} \chi_m(j+n)\overline{\chi_m(j)}.$$

For n fixed, a straightforward computation leads to

$$\int_{\mathbb{T}} P_{m,N}(t)\, \mathrm{e}^{2i\pi nt}\, \mathrm{d}t = \frac{1}{N} \sum_{\substack{u,\, v<N \\ u=n+v}} \chi_m(u)\overline{\chi_m(v)}$$

$$= \frac{1}{N}\left(\sum_{v<N} \chi_m(n+v)\overline{\chi_m(v)} + o(1)\right)$$

hence i) holds.

Set for short

$$\sigma_k(\mathrm{d}t) = P_{m,p_k}(t)\left(\Lambda_{z^{(k)},|m|}\right)^{p_k}(\mathrm{d}t)$$

where $\Lambda_{z^{(k)},|m|}$ is the spectral measure of the $q^{(k)}$-multiplicative sequence

$$n \to (z(np_k))^{|m|} = \chi_{|m|}(np_k).$$

By a classical result [Coquet *et al.*, 1977], $\Lambda_{z^{(k)},|m|}$ is the weak limit of the sequence

$$N \to \frac{1}{N}\left| \sum_{j<N} \chi_{|m|}(jp_k)\, \mathrm{e}^{-2i\pi jt}\right|^2 \mathrm{d}t.$$

We claim that, in the sense of total variance (i.e. for the norm $|||\cdot|||$ in the dual of $\mathscr{C}_{\mathbb{C}}(\mathbb{T})$) the inequality

$$|||\, \sigma_k - \Lambda_{z,m}\, ||| \leqslant 4\left(\frac{s}{p_k} + \left(\frac{s}{p_k}\right)^{1/2}\right) \tag{12}$$

holds. In fact, put $P_{m,N}^{(k)}(t)=\frac{1}{N}\left|\sum_{j<N}\chi_m^{(k)}(j)\ \mathrm{e}^{-2i\pi jt}\right|^2$, so that σ_k is the weak limit of

$$P_{m,Np_k}^{(k)}(t)\ \mathrm{d}t=\frac{1}{N}P_{m,p_k}(t)\left|\sum_{j<N}\chi_{|m|}(jp_k)\ \mathrm{e}^{-2i\pi jp_k t}\right|^2\ \mathrm{d}t,$$

then from i):

$$|||\,\sigma_k-\Lambda_{z,m}\,|||\leqslant\liminf_{N:\infty}\int_{\mathbb{T}}|\,P_{m,Np_k}(t)-p_{m,Np_k}^{(k)}(t)\,|\mathrm{d}t.$$

The equality

$$\chi_m(j)=\chi_m(j')\chi_{|m|}(lp_k)(=\chi_m^{(k)}(j))$$

with $j=j'+lp_k$, $0\leqslant j'<p_k$ is true for all j in $\mathbb{N}\backslash A_k$, therefore, using the inequality

$$|\,|\alpha|^2-|\beta|^2\,|\leqslant|\alpha-\beta|^2+2|\alpha|\,|\alpha-\beta|$$

satisfied for all complex numbers α, β one gets:

$$\begin{aligned}&|\,P_{m,N}(t)-P_{m,N}^{(k)}(t)\,|\\&\quad\leqslant\frac{1}{N}\left|\sum_{\substack{j<N\\ j\in A_k}}\left(\chi_m(j)-\chi_m^{(k)}(j)\right)\mathrm{e}^{-2i\pi jt}\right|^2\\&\qquad+\frac{2}{N}\left|\sum_{j<N}\chi_m(j)\ \mathrm{e}^{-2i\pi jt}\right|\left|\sum_{j<N}\left(\chi_m(j)-\chi_m^{(k)}(j)\right)\mathrm{e}^{-2i\pi jt}\right|.\end{aligned}$$

Integration on the torus and Schwarz inequality lead to

$$\begin{aligned}&\int_{\mathbb{T}}|\,P_{m,N}(t)-P_{m,N}^{(k)}(t)\,|\mathrm{d}t\\&\quad\leqslant\frac{1}{N}\sum_{\substack{j<N\\ j\in A_k}}|\chi_m(j)-\chi_m^{(k)}(j)|^2\\&\qquad+2N^{-1/2}\left(\int_{\mathbb{T}}P_{m,N}(t)\ \mathrm{d}t\right)^{1/2}\left(\sum_{\substack{j<N\\ j\in A_k}}|\chi_m(j)-\chi_m^{(k)}(j)|^2\right)^{1/2},\end{aligned}$$

but $\int_{\mathbb{T}} P_{m,N}(t)\,\mathrm{d}t = 1$, hence we get in fact

$$\lim_{N:\infty} \sup \int_{\mathbb{T}} | P_{m,N} - P_{m,N}^{(k)} | \mathrm{d}t \leqslant 4\left(\frac{s}{p_k} + \left(\frac{s}{p_k} \right)^{1/2} \right).$$

Taking Np_k in place of N above we derive (12). □

Remark 3. Assume $|m| = 0$, then $\Lambda_{z^{(k)},0} = \delta_0$ (the Dirac measure at 0 on $\mathbb{T}$) and

$$P_{m,p_k}(t) \cdot (\delta_0)^{p_k}(\mathrm{d}t) = \frac{1}{p_k} \sum_{l<p_k} P_{m,p_k}\left(\frac{l}{p_k} \right) \delta_{l/p_k},$$

hence: *if* $|m| = 0$ *then*:

$\Lambda_{z,m}$ *is discrete, supported by the group* $G_q = \bigcup_{k=0}^{\infty} \left(\frac{1}{p_k}\mathbb{Z} + \mathbb{Z} \right)$ modulo 1.

Remark 4. In the general case for any α in $\mathbb{T}$:

$$\sigma_{m,k}(\{\alpha\}) = \frac{1}{p_k} P_{m,p_k}(\alpha) \Lambda_{z^{(k)},|m|}(\{p_k\alpha\}),$$

and

$$\left(\Lambda_{z^{(k)},|m|}(\{p_k\alpha\}) \right)^{1/2} = \lim_{l:\infty} \frac{1}{p_l^{(k)}} \left| \sum_{u<p_l^{(k)}} \chi_{|m|}(p_k u)\, \mathrm{e}^{-2\mathrm{i}\pi u p_k \alpha} \right|$$

by a classical result in [Coquet *et al.,* 1977], therefore

$$\left(\sigma_{m,k}(\{\alpha\}) \right)^{1/2} = \lim_{l:\infty} \frac{1}{p_l} \left| \sum_{j<p_l} \chi_m^{(k)}(j)\, \mathrm{e}^{-2\mathrm{i}\pi j \alpha} \right|$$

and since $\chi_m^{(k)} = \chi_m$ on $\mathbb{N} \setminus A_k$ we get

$$\left(\Lambda_{z,m}(\{\alpha\}) \right)^{1/2} = \lim_{l:\infty} \frac{1}{p_l} \left| \sum_{j<p_l} \chi_m(j)\, \mathrm{e}^{-2\mathrm{i}\pi j \alpha} \right|$$

$$= \lim_{N:\infty} \sup \frac{1}{N} \left| \sum_{j<N} \chi_m(j)\, \mathrm{e}^{-2\mathrm{i}\pi j \alpha} \right|, \tag{13}$$

the last equality following from a result of J.P. Bertrandias [Bertrandias, 1964; Coquet *et al.,* 1977]. The next result supplies to a misstatement in ([Liardet, 1980], thm 4) and furnishes a proof.

THEOREM 8. *Let* $z\colon \mathbb{N} \to \mathbb{U}$ *be a q-multiplicative sequence. The flow* $\mathscr{K}(z)$ *is uniquely ergodic if and only if for all integers* $s \geqslant 0$ *and* m *in* $\mathbb{Z}^{s+1}$ *one has*:

$$\Lambda_{z,m}(\{0\}) > 0 \Rightarrow \lim_{k:\infty}\left(\sup_{j\geqslant 0}|(z(p_k j))^{|m|} - 1|\right) = 0. \tag{14}$$

Proof. Since the functions χ_m, $m \in \mathbb{Z}^{s+1}$, $s \in \mathbb{N}$ generate a dense subspace in $\mathscr{C}_{\mathbb{C}}(\mathbb{U}^{\mathbb{N}})$, the unique ergodicity of $\mathscr{K}(z)$ is equivalent to the uniform convergence in n of the following sequences of means:

$$N \to \frac{1}{N}\sum_{j<N}\chi_m(j+n).$$

Assume this condition holds. Let $m = (m_0, \ldots, m_s)$ be in $\mathbb{Z}^{s+1}$. For $\epsilon > 0$ given there exists N_0 such that

$$\left|\sum_{j<N}\chi_m(j+n) - \sum_{j<N}\chi_m(j)\right| \leqslant \epsilon N \tag{15}$$

for all $N \geqslant N_0$ and all $n \in \mathbb{N}$. In particular, take $n = p_k l$, $l \in \mathbb{N}$ with $p_k \geqslant N + |m|$, then (15) gives

$$|\chi_{|m|}(p_k l) - 1|\left|\sum_{j<N}\chi_m(j)\right| \leqslant \epsilon N.$$

Suppose $\Lambda_{z,m}(\{0\}) = c_m > 0$ and choose $N = p_s$ large enough such that by (13):

$$\left|\sum_{j<p_s}\chi_m(j)\right| \geqslant \tfrac{1}{2}p_s(c_m)^{1/2},$$

then

$$|\chi_{|m|}(p_k l) - 1| \leqslant 2\epsilon(c_m)^{1/2}$$

therefore

$$\sup_l |\chi_{|m|}(p_k l) - 1| \leqslant 2\epsilon(c_m)^{1/2}$$

for any k such that $p_k \geqslant p_s + |m|$. This proves (14).

Conversely, assume that (14) holds. Let $\chi_m^{(k)}(\cdot)$ be defined as in Theorem 7. By a straightforward computation one has

$$\left|\sum_{j<N}(\chi_m(j+n) - \chi_m^{(k)}(j+n)\right| \leqslant \left(\frac{N}{p_k} + 1\right)s$$

for all integers $n \geqslant 0$. Let n, p_k and N be given and let a, b integers defined by

$$ap_k \leqslant n < (a+1)p_k, \quad bp_k \leqslant n+N-1 < (b+1)p_k.$$

Since $\chi_m^{(k)}(r+lp_k) = \chi_{|m|}(lp_k)\chi_m(r)$ for $0 \leqslant r < p_k$,

one has

$$\left| \sum_{n \leqslant j < n+N} \chi_m^{(k)}(j) - \left(\sum_{r<p_k} \chi_m(r) \right)\left(\sum_{a \leqslant l < b} \chi_{|m|}(p_k l) \right) \right| \leqslant 2p_k$$

and then

$$\left| \sum_{n \leqslant u < n+N} \chi_m^{(k)}(u) - \sum_{0 \leqslant v < N} \chi_m^{(k)}(v) \right| \leqslant 4p_k + \left| \sum_{r<p_k} \chi_m(r) \right| (\Delta_m(a, b) + 1)$$

with

$$\Delta_m(a, b) = \left| \sum_{a \leqslant l < b} \chi_{|m|}(p_k l) - \sum_{0 \leqslant l' < b-a} \chi_{|m|}(p_k l') \right|,$$

hence

$$\left| \sum_{j<N} \chi_m(j+n) - \sum_{j<N} \chi_m(j) \right| \leqslant 2\left(\frac{N}{p_k} + 1 \right)s + 4p_k + p_k(\Delta_m(a, b) + 1). \tag{16}$$

Let $\epsilon > 0$ and assume $\Lambda_{z,m}(\{0\}) > 0$. By (14) there exists t such that $k \geqslant t$ implies $|\chi_{|m|}(p_k l) - 1| \leqslant \epsilon$ for all l, therefore

$$\Delta_m(a, b) \leqslant 2\epsilon(b-a) \leqslant 2\epsilon\left(\frac{N}{p_k} + p_k \right)$$

and (16) now gives

$$\left| \sum_{j<N} (\chi_m(j+n) - \chi_m(j)) \right| \leqslant N\left(\frac{2s}{p_k} + 2\epsilon \right) + 2\epsilon p_k^2 + 5p_k + 2s.$$

Choose $k \geqslant t$ such that $\frac{s}{p_k} \leqslant \epsilon$, then for

$$N \geqslant 2p_k^2 + \frac{5p_k + 2s}{\epsilon}$$

we get for all integers n:

$$\left| \frac{1}{N} \sum_{j<N} \left(\chi_m(j+n) - \chi_m(j)\right) \right| \leqslant 5\epsilon,$$

so that $(1/N) \sum_{j<N} \chi_m(j+n)$ converges uniformly in n.

Assume now that $\Lambda_{z,m}(\{0\}) = 0$, then with the above notations one has

$$\begin{aligned}
\left| \sum_{j<N} \chi_m(j+n) \right| &\leqslant 2p_k + \left| \sum_{ap_k \leqslant u < bp_k} \chi_m^{(k)}(u) \right| + \left(\frac{N}{p_k} + 1 \right) s \\
&\leqslant 2p_k + \left(\frac{N}{p_k} + 1 \right) s + \left| \sum_{r<p_k} \chi_m(r) \right| (b-a) \\
&\leqslant 2p_k + \left(\frac{N}{p_k} + 1 \right) s + \left(N + p_k^2 \right) \left| \frac{1}{p_k} \sum_{r<p_k} \chi_m(r) \right|.
\end{aligned}$$

Let k be such that $\frac{s}{p_k} \leqslant \epsilon$ and by (13)

$$\left| \frac{1}{p_k} \sum_{r<p_k} \chi_m(r) \right| \leqslant \epsilon.$$

Now with $N \geqslant p_k^2 + \frac{2p_k + s}{\epsilon}$ we obtain

$$\left| \frac{1}{N} \sum_{j<N} \chi_m(j+n) \right| \leqslant 3\epsilon$$

for all integers n. This finishes the proof. □

V.3. Proof of Theorem 6

We continue the study of flows $\mathscr{K}(z)$ but now we assume that z: $\mathbb{N} \to \mathbb{U}$ is a strong multiplicative sequence to base g.

V.3.1. $\mathscr{X}(z)$ is uniquely ergodic. Let m be a rational integer. By (13) one has

$$\Lambda_{z,m}(\{0\})^{1/2} = \lim_{k:\infty} \frac{1}{g^k}\left|\sum_{j<g^k}(z(j))^m\right| = \lim_{k:\infty}\left|\frac{1}{g}\sum_{j<g}(z(j))^m\right|^k$$

$$= \lim_{N:\infty} \sup\left|\frac{1}{N}\sum_{j<N}(z(j))^m\right|, \tag{17}$$

hence $\Lambda_{z,m}(\{0\})$ is 0 or 1 and the value 1 is taken if and only if z^m is the constant sequence $n \to 1$ since $z(0) = 1$.

Now let m be in $\mathbb{Z}^{s+1}$. By assumption on z, the equality $z^{(k)} = z$ holds for any integer k, so that by Theorem 7:

$$\sigma_{m,k}(\{0\}) = \left|\frac{1}{g^k}\sum_{j<g^k}\chi_m(j)\right|^2 \Lambda_{z,|m|}(\{0\}).$$

If $z^{|m|}$ is a constant sequence then the implication (14) of theorem 8 is obvious. If $z^{|m|}$ is not constant then $\sigma_{m,k}(\{0\}) = 0$ and by Theorem 7, part ii):

$$\Lambda_{z,m}(\{0\}) = 0, \tag{18}$$

hence (14) holds again. This proves the unique ergodicity of $\mathscr{X}(z)$. Let λ be the corresponding unique invariant measure on $\mathscr{X}(z)$. The projection $\lambda_{|1}$ of λ onto the first factor of $\Omega(= \mathbb{U}^{\mathbb{N}})$ is carried by $\mathbb{U}(z)$ and from (17) we easily derive that $\lambda_{|1}$ is the Haar measure on $\mathbb{U}(z)$, as expected to complete the proof of i), Theorem 6.

V.3.2. $\Lambda_{z,1}$ is g-invariant. For short we put $\Lambda_z := \Lambda_{z,1}$ and $P := P_{1,g}$ (see Theorem 7). The g-invariant property of Λ_z means that Λ_z is an invariant measure for the transformation $t \to g.t$ on the torus. This result is contained in [Queffelec, 1979], it derives from i) Theorem 7 and the identity

$$\frac{1}{g}\sum_{l<g} P\left(\frac{t}{g} + \frac{l}{g}\right) \equiv 1. \tag{19}$$

V.3.3. A spectral criterium. Define $t \to \| t \|$ on $\mathbb{T}$ by

$$\| t \| = \mathrm{Min}\{\,|\theta|;\quad t = \theta + \mathbb{Z}\}.$$

LEMMA 2

$$\left|\sum_{j<g} z(j)\right| > 1 \Rightarrow \int_{\mathbb{T}} \frac{1}{\| t \|^2}\Lambda_z(\mathrm{d}t) = +\infty.$$

Proof. We follow an idea of [Queffelec, 1984]. Let $I_k = \left[\frac{1}{g^k}, \frac{1}{g^{k-1}}\right[$ with $k \geqslant 2$. The g-invariant property of Λ_z implies

$$J_K := \int_{\mathbb{T}} 1_{I_k}(t) . \frac{1}{\| t \|^2} \Lambda_z(\mathrm{d}t) = \int_{\mathbb{T}} 1_{I_k}(gt) \frac{1}{\| gt \|^2} \Lambda_z(\mathrm{d}t)$$

$$= \int_{\mathbb{T}} \sum_{j<g} 1_{I_{k+1}}\left(u + \frac{j}{g}\right) \frac{1}{\| gu \|^2} \Lambda_z(\mathrm{d}u)$$

$$= \frac{1}{g^2} \int_{I_{k+1}} \frac{1}{\| t \|^2} \left(\sum_{j<g} \delta_{\{j/g\}} * \Lambda_z \right)(\mathrm{d}t).$$

But from i) Theorem 7 and (19) we get

$$\frac{1}{g} \sum_{j<g} \delta_{\{j/g\}} * \Lambda_z = \lim_{k:\infty} \left[\left(\frac{1}{g} \sum_{j<g} P\left(t + \frac{j}{g}\right)\right) P(gt) \dots P(g^k t) h(\mathrm{d}t) \right]$$

$$= (\Lambda_z)^g(\mathrm{d}t) = \frac{1}{P(t)} \Lambda_z(\mathrm{d}t),$$

hence

$$J_k = \int_{\mathbb{T}} 1_{I_{k+1}}(t) . \frac{1}{gP(t) \| t \|^2} \Lambda_z(\mathrm{d}t).$$

Assume that $gP(0) > 1$, then for k large enough the inequality $J_k \leqslant J_{k+1}$ holds and consequently the integral $\int_{\mathbb{T}} \frac{1}{\| t \|^2} \Lambda_z(\mathrm{d}t)$ diverges.

V.3.4. End of the proof. Let I be an arc of $\mathbb{U}$ and suppose that I is a B.R.S. for z satisfying the assumptions of ii). Define f: $\Omega \to \mathbb{C}$ by

$$f(\omega) = 1_I(\omega_0) - |I|$$

and let the above invariant measure λ on $\mathscr{K}(z)$ be viewed as a measure on Ω. By i) $\lambda_{|1}$ is equal to the Haar measure h on $\mathbb{U}$, hence the Fourier expansion of f in $L^2(\Omega; \lambda)$ is given by

$$f = \sum_{m \in \mathbb{Z}^*} (1_I | \chi_m) \chi_m$$

where $(1_I \mid \chi_m)$ is the ordinary Fourier coefficient $\int_I t^m h(\mathrm{d}t)$. On the other hand

$$(\chi_m \circ S^n \mid \chi_{m'}) = \lim_{N:\infty} \frac{1}{N} \sum_{k<N} (z(k+n))^m \overline{z(k)^{m'}}$$

$$= \int_\Omega \chi_M(t)\lambda(\mathrm{d}t) = \Lambda_{z,M}(\{0\})$$

with $M = (-m', 0, \ldots, 0, m)$ in $\mathbb{Z}^{n+1}$. Therefore $(\chi_m \circ S^n \mid \chi_{m'}) = 0$ by (18) if $m \neq m'$ and finally it follows

$$\lambda_f = \sum_{m \in \mathbb{Z}^*} |(1_I \mid \chi_m)|^2 \lambda_m$$

in $\mathscr{M}(\mathbb{T})$ where λ_m is set for the spectral measure $\Lambda_{z,m}$. Theorem 2 implies that $1/\sin^2\pi(\cdot)$, or equivalently $1/\|\cdot\|^2$, is integrable for λ_m if $(1_I \mid \chi_m) \neq 0$.

Assume $|I| \neq 0$ and $\neq 2\pi$. We are going to derive a contradiction proving that $1/\sin^2\pi(\cdot)$ is not integrable for λ_f. The functional equation (4) in Theorem 2 is satisfied with $a = |I|/2\pi$ since z is uniformly distributed in $\mathbb{U}$ and I is a B.R.S. Take the exponential of each member of (4), we obtain φ in $L^2(\Omega; \lambda)$, $\varphi \neq 0$, such that $U_s(\varphi) = \mathrm{e}^{\mathrm{i}|I|}\varphi$ and in terms of spectral measures $\lambda_\varphi = \delta_{\{a\}}$. Now let $\psi = \sum_{\chi \in F} c_\chi \chi$ be a finite sum over the characters χ on Ω, with complex coefficients c_χ, then by induction on the cardinal $|F|$ of F one has

$$\lambda_\psi \leqslant 2^{|F|} \sum_{\chi \in F} |c_\chi|^2 \lambda_\chi. \tag{20}$$

Since the subspace generated by characters χ is dense in $L^2(\Omega, \lambda)$, the continuity of $x \to \lambda_x$ from $L^2(\Omega, \lambda)$ to $\mathscr{M}(\mathbb{T})$ and (20) imply that there exists an integer $s \geqslant 0$ and M in $\mathbb{Z}^{s+1}$, $M \neq 0$, such that

$$\lambda_M(\{a\}) > 0.$$

If $|M| = 0$, the Remark 3 §V.2. means that there exists $k > 0$ and $q \in \{1, \ldots, g^k - 1\}$ with $a = \frac{q}{g^k}$. If $|M| \neq 0$, Theorem 7 implies that there exists $k > 0$ such that

$$\sigma_{M,k}(\{a\}) > 0$$

and Remark 4, §V.2. gives $\lambda_{|M|}(\{g^k a\}) > 0$. Put $\alpha = g^k a$ and use formula (13), then the infinite product

$$\prod_{n=0}^{\infty} \left| \frac{1}{g} \sum_{j<g} (z(j))^{|M|} e^{-2i\pi j g^n \alpha} \right|$$

converges to $\lambda_{|M|}(\{\alpha\})$ (> 0) so that

$$\lim_{n:\infty} \frac{1}{g} \left| \sum_{j<g} (z(j))^{|M|} e^{-2i\pi j g^n \alpha} \right| = 1.$$

It follows by convexity that

$$\lim_{n:\infty} (z(j))^{|M|} e^{-2i\pi j g^n \alpha} = 1$$

for $j = 1, \ldots, g-1$ (the case $j = 0$ being obvious), hence $(g^n \alpha)_n$ converges modulo 1 to β such that $g\beta \equiv \beta$ mod 1 and $(z(j))^{|M|} = e^{2i\pi j\beta}$. This leads to a contradiction, namely all the values of z are roots of unity. Finally we have just proved that $|I|/2\pi$ is the g-adic number q/g^k. Now let m, $m \neq 0$, be an integer prime to g such that $\left| \sum_{j<g} (z(j))^m \right| > 1$. Then $m|I|/2\pi \notin \mathbb{Z}$. By Lemma 2, $1/\sin^2 \pi(\cdot)$ is not integrable for λ_m and consequently is also not integrable for λ_f. □

For $g = 2$ there is an odd integer m satisfying (*). In the general case, it can only be proved there exists an integer $q \geq 1$ such that (*) holds for many integers $m = qm'$ with m' prime to g. From the above proof we derive:

Corollary. *For any strongly multiplicative* $\mathbb{U}$*-valued sequence to base* g, *the set of admissible frequencies is finite* (*reduced to the set* $\{0, 1\}$ *for* $g = 2$).

V.4. Applications

Let g be again an integer ≥ 2.

Corollary 2. *The only arcs* I *of* $\mathbb{T}$ *which are B.R.S. for the sequence* $n \to \alpha s_g(n)$ mod 1, *with a given irrational number* α, *are the trivial ones, that is to say* $|I| = 0$ *or* 1.

Proof. One has

$$\left| \sum_{j<g} e^{2i\pi\alpha jm} \right| = \left| \frac{\sin \pi g m \alpha}{\sin \pi m \alpha} \right|$$

and there exists a sequence (m_k) of integers such that g is prime to each m_k and $(m_k\alpha)_k$ converges to 0 modulo 1. In particular $\lim_k \left|\frac{\sin \pi g m_k \alpha}{\sin \pi m_k \alpha}\right| = g$ so that the required inequality (*) in Theorem 6 holds and the corollary follows. □

We apply the above method directly to the study of the Kakutani sequence ξ introduced in [Kakutani, 1967]. Let p be a prime number $\geqslant 3$, set $\varphi = e^{2i\pi/(p-1)}$ and let L be the logarithmic function which identifies the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$, with generator e, to the additive group $\mathbb{Z}/(p-1)\mathbb{Z}$ in such a way that $L(e) = 1$. Finally let $v_p(m)$ be the p-adic valuation of m. The sequence ξ is defined by

$$\xi(n) := \varphi^{L(\frac{n!}{p^{v_p(n!)}})}.$$

It is proved in [Coquet *et al.,* 1977] that ξ is a strongly multiplicative sequence to base p^2 and that all sequences ξ^j, for $j = 1, \ldots, p-2$, are pseudo-random. This implies that the measure $\Lambda_{\xi,m}$, for m in $\mathbb{Z}^{s+1}$, $s \geqslant 0$, is:

- *continuous if* $|m| \not\equiv 0 \bmod (p-1)$,
- *discrete, carried by* G_{p^2} *if* $|m| \equiv 0 \bmod (p-1)$.

Since ξ is uniformly distributed in $\mathbb{U}(\xi) = \{1, \varphi, \ldots, \varphi^{p-2}\}$, an admissible frequency for a B.R.S. has necessary the form $\frac{a}{p-1}$ with a in $\{0, 1, \ldots, p-1\}$. But measures $\delta_{\{(a)/p-1\}}$ with $a \in \{1, \ldots, p-2\}$ are not spectral measures for $\mathscr{X}(\xi)$, hence we have:

COROLLARY 3. *The only subsets A in $\mathbb{U}$ which are B.R.S. for the Kakutani sequence ξ are the trivial ones (i.e. either A contains all the $p-1$-th roots of unity or A contains none of these).*

Acknowledgements

The author wants to thank Peter Hellekalek whose interest in this work has been helpful during the one semester stay at the University of Salzburg.

References

Bertrandias, J.-P.: Suites pseudo-aléatoires et critères d'équirépartition modulo 1. *Compositio Math.* 16 (1964) 23–28.

Bourbaki, N.: *Espaces vectoriels topologiques,* ch III., *Actualités scientifiques et industrielles.* Hermann (1964).

Coquet, J., Kamae, T. and Mendès-France, M.: Sur la mesure spectrale de certaines suites arithmétiques. *Bull. Soc. Math. France* 105 (1977) 369–384.

Dunford, N. and Schwartz, J.T.: *Linear operator, part I, Pure and Applied Mathematics,* vol. VII. Interscience Publishers, New York (1967).

Erdös, P.: Problems and results on diophantine approximations. *Compositio Math.* 16 (1964) 52–65.

Furstenberg, H.: *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton Univ. Press (1981).

Furstenberg, H., Keynes, M. and Shapiro, L.: Prime flows in topological dynamics. *Israel J. Math.* 14 (1973) 26–38.

Gottschalk, W.H. and Hedlund, G.A.: *Topological Dynamics, Colloquium Publication* XXXVI. American Math. Soc., Providence, R.I. (1955).

Halász, G.: Remarks on the remainder in Birkhoff's ergodic theorem. *Acta Mathematica Acad. Sc. Hungaricae* 28 (1976) 389–395.

Hecke, E.: Analytische Funktionen und die Verteilung von Zahlen mod. eins. *Abh. Math. Semin. Hamburg Univ.* (1922) 54–76.

Hellekalek, P.: Regularities in the distribution of special sequences. *J. Number Theory* 18 (1984) 41–55.

Kakutani, S.: Ergodic theory of shift transformations, *Proceedings of the 5th Berkeley symposium on mathematical statistics and probability* [1965, Berkeley], vol. 2, part 2. University of California Press, Berkeley (1967) 405–414.

Kamae, T.: Mutual singularity of spectra of dynamical systems given by 'sum of digits' to different bases, Société Math. France. *Astérisque* 49 (1977) 109–114.

Kesten, H.: On a conjecture of Erdös and Szüsz related to uniform distribution mod 1. *Acta Arith.* 14 (1973) 26–38.

Kronecker, L.: *Berliner Sitzungsberichte,* 11 Déc. 1884. *Werke* 3 (1884) 49.

Larcher, G.: Private communication, Salzburg (May 1985).

Liardet, P.: Propriétés dynamiques de suites arithmétiques. *Journées Mathématiques SMF-CNRS, Théorie Analytique et élémentaire des nombres.* Caen (1980) 58–73.

Oren, I.: Admissible functions with multiple discontinuities. *Israel J.Math.* 42 (1982) 353–360.

Petersen, K.: On a series of cosecants related to a problem in ergodic theory. *Compositio Math.* 26 (1973) 313–317.

Queffelec, M.: Mesures spectrales associées à certaines suites arithmétiques. *Bull. Soc. Math. France* 107 (1979) 385–421.

Queffelec, M.: *Contribution à l'étude spectrale de suites arithmétiques.* Thèse d'Etat, Université Paris-Nord (1984).

Rauzy, G.: Ensembles à restes bornés. *Sém. Théorie des nombres.* Bordeaux exp. 24 (1983–1984) 1–12.

Szüsz, P.: Über die Verteilung der Vielfachen einer komplexen Zahl nach dem Modul des Einheitsquadrates. *Acta Math. Acad. Sci. Hungaricae* 5 (1954) 35–39.