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## More on embedding subspaces of $L_p$ in $l_r^n$

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Given two normed spaces  $X, Y$  and a real number  $1 \leq K < \infty$ , we say that  $X$   $K$ -embeds into  $Y$  (denoted  $X \overset{K}{\hookrightarrow} Y$ ) if there is a one to one linear operator,

$$T: X \rightarrow T(X) \subseteq Y \quad \text{with} \quad \|T\| \|T^{-1}\| \leq K.$$

We are concerned here mostly with the situation where  $Y$  is one of the sequence spaces  $l_r^n \left( = \left\{ x \in \mathbb{R}^n; \|x\|_r = \left( \sum_{i=1}^n |x_i|^r \right)^{1/r} < \infty \right\} \right)$  and  $X$  is a general  $m$ -dimensional subspace of one of the function spaces  $L_p(0, 1) (= \{f; \|f\|_p = \left( \int_0^1 |f(t)|^p dt \right)^{1/p} < \infty \})$ .

The expressions  $\|x\|_r, \|f\|_p$  are norms only for  $r, p \geq 1$ . We shall need, however, to use these expressions also for  $r$  or  $p$  smaller than 1. We shall continue to refer to them as norms also in this situation. The notion ' $X$   $K$ -embeds into  $Y$ ' has meaning, with the same definition, also in this case  $\left( \text{e.g. } \|T\| = \sup \left\{ \frac{\|Tx\|_r}{\|x\|_p}; x \in X, x \neq 0 \right\} \right)$ .

We continue here the investigation of the following question: fixing  $K, p, r$  and  $m$ , how small can we take  $n$  to be? Following is a sample of some of the results of this paper:

- i) For  $0 < r \leq p < 2$ ,  $X \overset{1+\epsilon}{\hookrightarrow} l_r^n$ , where  $n \leq C(p, r, \epsilon)m^{1+r/p}$ .
- ii) For  $2 < r = p < \infty$ ,  $X \overset{1+\epsilon}{\hookrightarrow} l_p^n$ , where  $n \leq C(p, \epsilon)m^{1+r/2}$ .

Changing the small constant,  $1 + \epsilon$ , in (i) with a large one, we get a much better estimate on the relation between  $n$  and  $m$  for  $r < p$ .

- iii) For  $0 < r < p < 2$ , there exists a  $K = K(p, r)$  such that  $X \overset{K}{\hookrightarrow} l_r^n$  for  $n \leq C m(\log m)^4$ ,  $C$  absolute.

The proofs here are much simpler than in the related papers [Johnson and Schechtman, 1982; Pisier, 1983; Schechtman, 1984/85, 1985].

For  $1 = r < p$ , i) is an important special case of Theorem 1 of [Schechtman, 1985] (except for a missing log factor. What is special here is that the range space is  $l_1^n$  rather than more general spaces).

The case  $r = 1 = p$  in i) is an improvement of Theorem 2 in [Schechtman, 1985].

We refer the reader to [Milman and Schechtman, 1986] for the background to the subjects discussed here.

The main results are contained in Theorems 5 and 6. Proposition 4 is the main tool in proving these theorems. We begin with three lemmas, versions of which were used also in [Schechtman, 1985]. The first two are versions of Lemma 1 in [Schechtman, 1985]. Note again that  $\|x\|_r$  denotes the homogeneous ‘norm’ in  $L_r(0, 1)$  or  $l_r^n$  also for  $0 < r < 1$   $\left( \left( \int_0^1 |x|^r \right)^{1/r}$  or  $\left( \sum_{i=1}^n |x_i|^r \right)^{1/r} \right)$ . The Banach-Mazur distance,  $d(X, Y)$ , between a subspace  $X$  of  $L_r$  and a subspace  $Y$  of  $L_s$  is defined, as for normed spaces,

$$d(X, Y) = \inf \left\{ ab; a^{-1} \|x\|_r \leq \|Tx\|_s \leq b \|x\|_r, T: X \xrightarrow[\text{onto}]{} Y, T \text{ linear} \right\}.$$

LEMMA 1. Let  $0 < r \leq 2$  and let  $Z$  be an  $m$ -dimensional subspace of  $L_r(0, 1)$ , then

a) there exist a probability measure  $\mu$  on  $[0, 1]$  and a subspace  $W$  of  $L_r(\mu)$  isometric to  $Z$  and satisfying

$$\sup \{ \|w\|_\infty; \|w\|_r \leq 1, w \in W \} \leq em^{1/2} d(Z, l_2^m)$$

b)  $d(Z, l_2^m) \leq e^{(2/r)-1} m^{(1/r)-(1/2)}$ .

Consequently,

$$\sup \{ \|w\|_\infty; \|w\|_r \leq 1, w \in W \} \leq e^{2/r} m^{1/r}.$$

Proof. As in [Schechtman, 1985], let  $x_1, \dots, x_m$  be a basis for  $Z$  satisfying

$$a^{-1} \left( \sum_{i=1}^m a_i^2 \right)^{1/2} \leq \left\| \sum_{i=1}^m a_i x_i \right\|_r \leq b \left( \sum_{i=1}^m a_i^2 \right)^{1/2}$$

with  $ab = d(Z, l_2^m)$ .

Define

$$d\mu = \left[ \left( \sum_{i=1}^m x_i^2 \right)^{r/2} / \int_0^1 \left( \sum_{i=1}^m x_i^2 \right)^{r/2} \right] dt$$

and  $T: Z \rightarrow L_r(\mu)$  by

$$Tx = \frac{x}{\left(\sum_{i=1}^m x_i^2\right)^{1/2}} \left( \int \left( \sum_{i=1}^m x_i^2 \right)^{r/2} \right)^{1/r}.$$

$T$  is clearly an isometry. Let  $W = TZ$ . For  $w = T(\sum a_i x_i)$  of norm one, we have

$$|w| = \frac{\left| \sum_{i=1}^m a_i x_i \right|}{\left(\sum_{i=1}^m x_i^2\right)^{1/2}} \left( \int \left( \sum_{i=1}^m x_i^2 \right)^{r/2} \right)^{1/r}. \tag{1}$$

Now,

$$\frac{\left| \sum_{i=1}^m a_i x_i \right|}{\left(\sum_{i=1}^m x_i^2\right)^{1/2}} \leq \left( \sum_{i=1}^m a_i^2 \right)^{1/2} \leq a \left\| \sum_{i=1}^m a_i x_i \right\|_r = a, \tag{2}$$

and, with  $g_i$  being independent standard gaussian variables,

$$\begin{aligned} \left( \int \left( \sum_{i=1}^m x_i^2 \right)^{r/2} \right)^{1/r} &= \left( E \int \left| \sum_{i=1}^m x_i g_i \right|^r \right)^{1/r} / (E |g_1|^r)^{1/r} \\ &\leq b \left( E \left( \sum_{i=1}^m g_i^2 \right)^{r/2} \right)^{1/r} / (E |g_1|^r)^{1/r} \\ &\leq b\sqrt{m} / (E |g_1|^r)^{1/r}. \end{aligned} \tag{3}$$

To evaluate  $(E |g_1|^r)^{1/r}$  from below, use integration by parts to get

$$E |g_1|^r = \frac{1}{r+1} E |g_1|^{r+2} \geq \frac{1}{r+1}.$$

Thus,

$$(E |g_1|^r)^{1/r} \geq \left( \frac{1}{r+1} \right)^{1/r} \geq \frac{1}{e}.$$

Combining this with (1), (2) and (3), we get a).

To prove b), notice that, for  $w \in W$ ,

$$\|w\|_r \leq \|w\|_2 \leq \|w\|_\infty^{1-(r/2)} \|w\|_r^{r/2} \leq (e m^{1/2} d(Z, l_2^m))^{1-(r/2)} \|w\|_r.$$

Consequently,

$$d(Z, l_2^m) = d(W, l_2^m) \leq (e m^{1/2} d(Z, l_2^m))^{1-(r/2)}.$$

Rearranging we get b).  $\square$

LEMMA 2. Let  $2 < r < \infty$  and let  $Z$  be a  $m$ -dimensional subspace of  $L_r(0, 1)$ . Then there exist a probability measure  $\mu$  on  $[0, 1]$  and a subspace  $W$  of  $L_r(\mu)$  isometric to  $Z$  such that

$$\{ \|w\|_\infty; \|w\|_r = 1, w \in W \} \leq m^{1/2}.$$

*Proof.* By Theorem 1 in [Lewis, 1978] there is a probability measure  $\mu (= f^r dt$  in the notation of [Lewis, 1978]) and a basis  $(x_i)_{i=1}^m (x_i = f_i/f)$  of a space  $W \subseteq L_r(\mu)$ , isometric to  $Z$  ( $W = f^{-1}Z$ ), such that

$$a) \left\| \sum_{i=1}^m a_i x_i \right\|_2 = \frac{\left( \sum_{i=1}^m a_i^2 \right)^{1/2}}{\sqrt{m}} \quad \text{for all } a_1, \dots, a_m \in \mathbb{R}$$

$$b) \left( \sum_{i=1}^m x_i^2 \right)^{1/2} \equiv 1.$$

Now, for all  $a_1, \dots, a_m \in \mathbb{R}$ ,

$$\begin{aligned} \left\| \sum_{i=1}^m a_i x_i \right\|_\infty &\leq \left( \sum_{i=1}^m a_i^2 \right)^{1/2} \left\| \left( \sum_{i=1}^m x_i^2 \right)^{1/2} \right\|_\infty \\ &= \sqrt{m} \left\| \sum_{i=1}^m a_i x_i \right\|_2 \leq \sqrt{m} \left\| \sum_{i=1}^m a_i x_i \right\|_r. \quad \square \end{aligned}$$

The next lemma is a standard large deviation inequality for sums of independent random variables. We give a proof for completeness.

LEMMA 3. Let  $(d_i)_{i=1}^n$  be independent random variables with

$$E |d_i| \leq A, \quad E d_i = 0, \quad \|d_i\|_\infty \leq B, \quad i = 1, \dots, n,$$

then

$$P\left(\left|\sum_{i=1}^n d_i\right| > C\right) \leq 2 \exp\left(\frac{-C^2}{4eABn}\right)$$

for all  $C \leq 2eAn$ .

*Proof.* First notice that for all  $p \geq 2$

$$E |d_i|^p \leq E |d_i| \|d_i\|_\infty^{p-1} \leq AB^{p-1}, \quad i = 1, \dots, n.$$

For all  $\lambda \geq 0$  and all  $i = 1, \dots, n$

$$\begin{aligned} E e^{\lambda d_i} &\leq 1 + \sum_{k=2}^{\infty} \frac{\lambda^k E |d_i|^k}{k!} \leq 1 + \sum_{k=2}^{\infty} \frac{\lambda^k AB^{k-1}}{k!} \leq 1 + \lambda^2 AB \sum_{k=0}^{\infty} \frac{\lambda^k B^k}{k!} \\ &\leq \exp(\lambda^2 AB e^{\lambda B}). \end{aligned}$$

Independence implies

$$E e^{\lambda \sum_{i=1}^n d_i} \leq \exp(\lambda^2 ABn e^{\lambda B}).$$

Consequently, for  $0 < \lambda \leq B^{-1}$ ,

$$P\left(\sum_{i=1}^n d_i > C\right) \leq e^{\lambda \sum_{i=1}^n d_i - \lambda C} \leq \exp(\lambda^2 ABne - \lambda C).$$

Choosing  $\lambda = \frac{1}{2} \frac{C}{ABne}$  ( $\leq \frac{1}{B}$  for  $C \leq 2eAn$ ), we get

$$P\left(\sum_{i=1}^n d_i > C\right) \leq \exp\left(-\frac{C^2}{4eABn}\right).$$

The same inequality holds for  $-\sum_{i=1}^n d_i$  and we get the desired result.  $\square$

**PROPOSITION 4.** Let  $X$  be an  $m$ -dimensional subspace of  $L_r(\Omega, \mathcal{F}, \mu)$  for some probability space  $(\Omega, \mathcal{F}, \mu)$  and some  $0 < r < \infty$ . Assume

$$M = \sup\{\|x\|_\infty; x \in X, \|x\|_r = 1\} < \infty.$$

Then, for all  $\epsilon > 0$ ,  $X \overset{1+\epsilon}{\hookrightarrow} l_r^n$  for some

$$n \leq C(\epsilon, r)mM^r.$$

Moreover, for some absolute constant  $C$ ,

$$C(\epsilon, r) \leq \frac{C \log \frac{1}{r\epsilon}}{r^3 \epsilon^2} \quad \text{for } 0 < r < 1$$

and

$$C(\epsilon, r) \leq \frac{C \log \frac{1}{\epsilon}}{\epsilon^2} \quad \text{for } r > 1.$$

*Proof.* For  $t \in [0, 1]^n$  and  $x \in X$  define  $x_i(t) = x(t_i)$ . Then  $x_i$  are independent random variables and for all  $t$  the map

$$x \rightarrow (x_1(t), \dots, x_n(t))$$

is linear. Define, for  $t \in [0, 1]^n$ , an operator

$$T_t: X \rightarrow l_r^n$$

by

$$T_t x = \frac{1}{n^{1/r}} \sum_{i=1}^n x_i(t) e_i$$

( $(e_i)_{i=1}^n$  is the canonical basis of  $l_r^n$ ). Then

$$E \|T_t x\|_r^r = \|x\|_r^r$$

( $E$  denotes expectation with respect to  $P$  – the product measure on  $[0, 1]^n$ ).

For  $x \in X$  with  $\|x\|_r = 1$ ,

$$\|T_t x\|_r^r - 1 = \frac{1}{n} \sum_{i=1}^n (|x_i(t)|^r - 1).$$

Each of the summands  $y_i = |x_i(t)|^r - 1$  is bounded by  $M^r$  and satisfies  $E |y_i| \leq 2$ . Plugging these estimates in Lemma 3, with  $d_i = \frac{y_i}{n}$ ,  $A = \frac{2}{n}$  and  $B = \frac{M^r}{n}$ , we get, for  $0 < \eta < \frac{1}{3}$  and an absolute constant  $\delta > 0$ ,

$$P(|\|T_t x\|_r^r - 1| > \eta) \leq 2 e^{-\delta \eta^2 n / M^r}. \tag{4}$$

We now distinguish between the two cases  $0 < r < 1$  and  $1 \leq r < \infty$ . If

$0 < r < 1$  choose an  $\eta$ -net,  $N$ , in the sphere of  $X$  in the metric  $d(x, y) = \|x - y\|_r^r$ . One can do that with

$$|N| \leq \left(1 + \frac{2}{\eta}\right)^{m/r} \leq e^{(m/r)\log(2/\eta)}$$

(the proof is standard, see e.g. [Johnson and Schechtman, 1982] Lemma 2). Using (4) we get in this case that if

$$m \leq c(\eta)rn/M^r \tag{5}$$

( $c(\eta) \approx \eta^2/\log\frac{1}{\eta}$ ), then, for some  $t$

$$1 - \eta \leq \|T_t x\|_r^r \leq 1 + \eta$$

for all  $x \in N$ . Using a standard successive approximation argument (see e.g. [Johnson and Schechtman, 1982] Lemma 3) we get that

$$\frac{1 - 3\eta}{1 - \eta} \leq \|T_t x\|_r^r \leq \frac{(1 + \eta)^2}{(1 - \eta)} \tag{6}$$

for all  $x \in X$ ,  $\|x\|_r = 1$ . This concludes the proof in the case  $0 < r < 1$  except for the evaluation of the constant  $C(\epsilon, r)$ . Given  $0 < r < 1$  and  $0 < \epsilon < 1$  choose a  $\delta$  such that  $(1 + \delta)^{1/r} = 1 + \epsilon$  ( $\delta \approx \epsilon^r$  with absolute constants) then choose an  $0 < \eta < \frac{1}{3}$  such that  $\frac{(1 + \eta)^2}{(1 - 3\eta)} = 1 + \delta$  ( $\eta \approx \delta$  with absolute constants). Then, in the construction above,  $\|T_t\| \|T_t^{-1}\| \leq (1 + \delta)^{1/r} = 1 + \epsilon$  and by (5) we may choose

$$n \approx C(\epsilon, r)mM^r$$

where

$$C(\epsilon, r) = \frac{1}{c(\eta)r} \approx \frac{\log\frac{1}{\eta}}{\eta^2 r} \approx \frac{\log\frac{1}{\epsilon r}}{\epsilon^2 r^3}.$$

The proof for  $r > 1$  is very similar. Here we work with an  $\eta$ -net  $N$  in the metric given by the norm. Its size is

$$|N| \leq e^{m\log(2/\eta)}$$

(see e.g. [Figiel, Lindenstrauss and Milman, 1977]) and we get that if

$$m \leq c(\eta)n/M^r$$



$(c(\eta) \approx \eta^2 / \log \frac{1}{\eta})$  then there exists a  $t$  such that

$$\|T_t\| \|T_t^{-1}\| \leq \left[ \frac{(1+\eta)^2}{(1-3\eta)} \right]^{1/r} \leq \frac{(1+\eta)^2}{(1-3\eta)}.$$

Taking  $\eta$  of order  $\epsilon$  we get the desired result.  $\square$

**THEOREM 5.**

a) For  $0 < r \leq p < 2$  any  $m$ -dimensional subspace  $X$  of  $L_p(0, 1)$   $(1 + \epsilon)$ -embeds into  $l_r^n$  for some

$$n \leq K(\epsilon, r) m^{1+(r/p)}.$$

b) For  $0 < r \leq 1$  any  $m$ -dimensional normed subspace  $X$  of  $L_r(0, 1)$   $(1 + \epsilon)$ -embeds into  $l_r^n$  for some

$$n \leq K(\epsilon, r) m^{1+r}.$$

c) For  $2 < r < \infty$  any  $m$ -dimensional subspace  $X$  of  $L_r(0, 1)$   $(1 + \epsilon)$ -embeds into  $l_r^n$  for some

$$n \leq K(\epsilon) m^{1+(r/2)}.$$

The constants  $K(\epsilon, r)$  in a) and b) are dominated by  $10C(\epsilon, r)$  of Proposition 4. The constant in c) depends only on  $\epsilon$ .

*Proof.* Since  $L_p(0, 1) \xrightarrow{1} L_r(0, 1)$ ,  $0 < r < p < 2$ , we may assume in all three cases that  $X \subseteq L_r(0, 1)$ . By Lemmas 1 and 2, we may assume in addition that, putting

$$M = \sup \{ \|x\|_\infty; x \in X, \|x\|_r = 1 \},$$

$$M \leq e^{2/p} m^{1/p} \quad \text{in a)}$$

$$M \leq e m \quad \text{in b)}$$

$$M \leq m^{1/2} \quad \text{in c)}.$$

Now apply Proposition 4.  $\square$

*Remarks*

i) In a) and b) nothing is known about lower bounds (i.e. the case may be that  $n$  can be chosen proportional to  $m$ ). In c) there is a lower bound:  $n \geq k(\epsilon, r) m^{r/2}$  (see [Bennett *et al.*, 1977] or [Milman and Schechtman, 1986]).

ii) Following the proofs of Proposition 4 and Theorem 5, one can easily prove:

Let  $F$  be a finite set in  $L_r$ ,  $1 \leq r < 2$ ,  $|F| = m$ . Then for each  $\epsilon > 0$  and  $n \geq c(\epsilon)m \log m$  there exists a function  $f: F \rightarrow f(F) \subset l_r^n$  with

$$\|f\|_{L_r} \|f^{-1}\|_{L_r} \leq 1 + \epsilon$$

$$\left( \|f\|_{L_r} = \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|}, c(\epsilon) \text{ depends only on } \epsilon. \right)$$

This should be compared with a result of [Ball, 1984]: For  $\epsilon = 0$ ,  $n$  must be of order at least  $m^2$  (and  $n \approx m^2$  is always enough).

We suspect that the right order of  $n$  (for  $(1 + \epsilon)$ -Lipschitz embeddings) is some power of  $\log m$ .

**THEOREM 6**

- a) Given  $0 < q < p < 2$  there exists a  $K = K(q, p)$  such that any  $m$ -dimensional subspace  $X$  of  $L_p(0, 1)$   $K$ -embeds into  $l_q^n$  for some  $n \leq C m (\log m)^3 \log(\log m)$ ,  $C$  absolute.
- b) For any  $0 < r < q < 1$ , any  $m$ -dimensional normed subspace  $X$  of  $L_r(0, 1)$   $K(q)$ -embeds into  $l_q^n$  for some  $n \leq C m (\log m)^3 \log(\log m)$ ,  $C$  absolute and  $K(q)$  depends only on  $q$  (and not on  $r$ ).

*Proof.* In both cases  $X$  can be considered as a subspace of  $L_s(0, 1)$  for any  $0 < s \leq r$ . Thus, by Theorem 5,  $X$  2-embeds into  $l_s^n$  for

$$n \leq \frac{C \log \frac{1}{s}}{s^3} m^{1+(s/p)} \text{ in a)}$$

and

$$n \leq \frac{C \log \frac{1}{s}}{s^3} m^{1+s} \text{ in b).}$$

The choice  $s = \frac{p}{\log m}$  in a) and  $s = \frac{1}{\log m}$  in b) gives that  $X$  2-embeds into  $l_s^n$  for

$$n \leq C(p)(\log m)^3 (\log(\log m)) m$$

for some constant  $C(p)$ , depending only on  $p$ , in a), and for

$$n \leq C(\log m)^3 (\log(\log m)) m$$

for some absolute constant  $C$  in b).

Now apply one of Maurey's factorization theorems, Theorem 2 of [Maurey, 1974], to get an embedding of  $X$  into  $l_q^n$  via a change of measure. One should notice that  $s$  does not affect the constants.  $\square$

There are several problems which suggest themselves naturally. We shall mention explicitly only one with a possible way of attack (toward a negative solution).

**PROBLEM 7.** *Is there a function  $C(\epsilon)$ ,  $\epsilon > 0$  such that any  $m$ -dimensional subspace  $X$  of  $L_1(0, 1)$   $(1 + \epsilon)$ -embeds into  $l_1^n$  for some  $n \leq C(\epsilon)m$ ?*

Denote by  $R(Y)$  the  $K$ -convexity constant of  $Y$  [Maurey and Pisier, 1976], that is, the norm of the projection  $R \otimes I$  in  $L_2(Y)$ , where  $R$  is the orthogonal projection onto the span of the Rademacher functions. As is well known  $R(X) \leq C \sqrt{\log m}$  for any  $m$ -dimensional subspace  $X$  of  $L_1(0, 1)$ . Inspecting the proof of this fact one easily gets an estimate on  $C$

$$R(X) \leq (\sqrt{2} + o(1))\sqrt{\log m}, \quad \dim X = m \rightarrow \infty. \tag{7}$$

In particular,

$$R(l_1^m) \leq (\sqrt{2} + o(1))\sqrt{\log m}, \quad m \rightarrow \infty. \tag{8}$$

We are mainly interested in whether the numerical constants in (7) and (8) are the same or different. Indeed if for some  $\alpha > 1$  and  $\epsilon > 0$

$$\limsup_{m \rightarrow \infty} \left[ \sup_{\dim X = m} R(X) / R(l_1^{m^\alpha}) \right] \geq 1 + \epsilon$$

then for some  $m$  there exists an  $m$ -dimensional subspace  $X$  of  $L_1$  which does not  $1 + \epsilon$  embed into  $l_1^{m^\alpha}$ .

**PROBLEM 8.** *What are the best numerical constants in (7) and (8)?*

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**Added in proof:** J. Bourgain, J. Lindenstrauss and V.D. Milman (private communication) improved recently the results of this paper. They proved that for an  $m$ -dimensional subspace,  $X$ , of  $L_p$ ,

$$X \overset{1+\epsilon}{\hookrightarrow} l_1^n \quad \text{for } n \leq C(p, \epsilon)m \quad \text{if } 1 < p \leq 2$$

$$X \overset{1+\epsilon}{\hookrightarrow} l_1^n \quad \text{for } n \leq C(\epsilon)m(\log m)^5 \quad \text{if } p = 1$$

$$X \overset{1+\epsilon}{\hookrightarrow} l_p^m \quad \text{for } n \leq C(p, \epsilon)m^{p/2+\epsilon} \quad \text{if } 2 < p < \infty.$$

Their proof is based on the results and method developed here.