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KURT BEHNKE

HORST KNÖRRER

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On infinitesimal deformations of rational surface singularities

KURT BEHNKE¹ & HORST KNÖRRER²

¹ *Mathematisches Seminar der Universität Hamburg, Bundesstrasse 55, 2000 Hamburg 13, West-Germany;*

² *Mathematisches Institut der Universität Bonn, Wegelerstrasse 10, 5300 Bonn 1, West-Germany*

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0. Introduction

This paper is concerned with the computation of the space T_X^1 of first order infinitesimal deformations of a two-dimensional rational singularity $(X, 0)$. For cyclic resp. dihedral quotient singularities the dimension of this space was determined in [Riemenschneider, 1974] [Pinkham, 1977] resp. [Behnke and Riemenschneider, 1977, 1978]. In these cases one obtains the formula

$$\dim T_X^1 = \dim T_{\tilde{X}}^1 + \text{emb}(X) - 4 \quad (0.1)$$

unless X is a rational double point. Here $\text{emb}(X)$ denotes the embedding dimension of X , $\pi: \tilde{X} \rightarrow X$ the minimal resolution of X , and $T_{\tilde{X}}^1 \cong H^1(\tilde{X}, \Theta_{\tilde{X}})$ is the space of first order infinitesimal deformations of $\tilde{X}$. The data at the right hand side of (0.1) can for many rational surface singularities be computed in terms of the resolution graph (see e.g. [Artin, 1966], Cor. 6 and [Laufer, 1973]).

For arbitrary two-dimensional quotient singularities C. Kahn recently gave a (computer-aided) proof of (0.1), based on invariant theoretic results worked out by Kahn, Riemenschneider and the authors (cf. [Behnke et al., in prep.] and [Kahn, 1984]).

On the other hand J. Wahl had found an example of a (non Gorenstein) rational surface singularity for which $\dim T_X^1 > \dim T_{\tilde{X}}^1 + \text{emb}(X) - 4$ (see [Behnke and Riemenschneider, 1977, 1978], p. 4 and Example 4.21. below). In a letter he also gave a proof of the inequality

$$\dim T_X^1 \geq \dim T_{\tilde{X}}^1 + \text{emb}(X) - 4 \quad (0.2)$$

for all rational surface singularities. We give his proof in an appendix to our paper.

In this article we prove (0.1) for a large class of two-dimensional rational singularities (see Theorem 4.10. below). We briefly sketch the method applied.

From Schlessinger's description of T_X^1 (cf. [Schlessinger, 1971] or Theorem 1.1. below) one concludes by local duality that the dual space $(T_X^1)^*$ can be computed as follows:

Let: $i: X \hookrightarrow \mathbb{C}^n$ be a closed embedding of a Stein representative, and let Ω_X^1 and $\Omega_{\mathbb{C}^n}^1$ be the sheaves of Kähler differentials and ω_X the canonical sheaf. By $X' = X - \{0\}$ we denote the smooth part of X . Then $(T_X^1)^*$ is isomorphic to the cokernel of the natural map

$$H^0(X', i^* \Omega_{\mathbb{C}^n}^1 \otimes \omega_X) \xrightarrow{(\mu' \otimes 1)} H^0(X', \Omega_X^1 \otimes \omega_X),$$

induced by the epimorphism $\mu': i^* \Omega_{\mathbb{C}^n}^1 \rightarrow \Omega_X^1$.

Let $f_1, \dots, f_n$ be a system of generators for the maximal ideal of $\mathcal{O}_{X,0}$. For a suitable trivialization $i^* \Omega_{\mathbb{C}^n}^1 \cong \mathcal{O}_{X'}^n$, $\mu': \mathcal{O}_{X'}^n \rightarrow \Omega_X^1$ is defined by $\mu'(g_1, \dots, g_n) = g_1 df_1 + \dots + g_n df_n$. This map can be studied using the resolution $\pi: \tilde{X} \rightarrow X$. Let $E = \pi^{-1}(0)$ be the exceptional set, Z the fundamental cycle, and let $\Omega_{\tilde{X}}^1 \langle \log E \rangle$ be the sheaf of meromorphic 1-forms with at most logarithmic poles along E . As above we have a map

$$\tilde{\mu}: \mathcal{O}_{\tilde{X}}^{\oplus n} \rightarrow \Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z),$$

$(g_1, \dots, g_n) \rightarrow g_1 df_1 + \dots + g_n df_n$, where now $f_1, \dots, f_n$ are considered as holomorphic functions on $\tilde{X}$. As $(X, 0)$ is a rational singularity and ω_X is reflexive there is a natural isomorphism between $H^0(X', \mathcal{O}_{X'}^{\oplus n} \otimes \omega_{X'})$ and $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^{\oplus n} \otimes \omega_{\tilde{X}})$. Using this isomorphism one sees that T_X^1 is dual to the cokernel of the following composite map

$$\begin{aligned} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}^{\oplus n} \otimes \omega_{\tilde{X}}) &\xrightarrow{(\tilde{\mu} \otimes 1)^*} H^0(\tilde{X}, \Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z) \otimes \omega_{\tilde{X}}) \\ &\hookrightarrow H^0(X', \Omega_{X'}^1 \otimes \omega_{X'}). \end{aligned}$$

The cokernel of the inclusion $H^0(\tilde{X}, \Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z) \otimes \omega_{\tilde{X}}) \hookrightarrow H^0(X', \Omega_{X'}^1 \otimes \omega_{X'})$ can be computed using results of J. Wahl [Wahl, 1975] (see Ch. 2). For the discussion of $(\tilde{\mu} \otimes 1)^*$ we have to make more restrictive assumptions (e.g. that the fundamental cycle is reduced) in order to be able to control the kernel and the cokernel of μ . This discussion is performed in Ch. 3 and Ch. 4 and leads to the proof of (0.1) for a large class of rational surface singularities. The precise results are stated in Theorem 4.8. and Example 4.13.

1. Schlessinger's description of T_X^1 and duality

Let $(X, 0)$ be a normal surface singularity. We recall a result of M. Schlessinger [Schlessinger, 1971] which gives a cohomological description of the space T_X^1 of infinitesimal deformations of X . Then we apply duality to obtain the description of $(T_X^1)^*$ which is basic for our paper.

Let $i: X \hookrightarrow \mathbb{C}^n$ be an embedding of a small Stein space representing the singularity $(X, 0)$. Denote by $X' = X - \{0\}$ the smooth part of X , by $\Omega_{\mathbb{C}^n}^1$ resp.

Ω_X^n the sheaves of Kähler differentials on $\mathbb{C}^n$ resp. X , and by $\Theta_{\mathbb{C}^n}$ resp. Θ_X their duals.

THEOREM 1.1.

([Schlessinger, 1971] §1, Lemma 2). *The module T_X^1 of first order infinitesimal deformations of $(X, 0)$ is the kernel of the map*

$$H^1(X', \Theta_{X'}) \rightarrow H^1(X', \Theta_{\mathbb{C}^n|X'})$$

which is induced by the natural inclusion of tangent sheaves $\Theta_{X'} \hookrightarrow \Theta_{\mathbb{C}^n|X'}$.

To apply local duality we remark that $H^1(X', \Theta_{X'})$ is canonically isomorphic to $H_{(0)}^2(X, \Theta_X)$, the second local cohomology group with support in the singular point 0. Similarly $H^1(X', \Theta_{\mathbb{C}^n|X'})$ is canonically isomorphic to $H_{(0)}^2(X, \Theta_{\mathbb{C}^n|X})$. Then we see by local duality that T_X^1 is dual to the cokernel of

$$\mathrm{Hom}_{\mathcal{O}_X}(\Theta_{\mathbb{C}^n|X}, \omega_X) \rightarrow \mathrm{Hom}_{\mathcal{O}_X}(\Theta_X, \omega_X).$$

As all these sheaves are reflexive we finally get

COROLLARY 1.2.

$(T_X^1)^$ is isomorphic to the cokernel of the map*

$$H^0(X', \Omega_{\mathbb{C}^n|X'}^1 \otimes \omega_{X'}) \rightarrow H^0(X', \Omega_{X'}^1 \otimes \omega_{X'})$$

induced by the restriction map $\Omega_{\mathbb{C}^n}^1 \otimes \mathcal{O}_X \rightarrow \Omega_X^1$.

REMARK 1.3.

We can make this result a little more explicit: observe that the restriction $\Omega_{\mathbb{C}^n|X}^1$ is generated as an $\mathcal{O}_X$ -module by the differentials $df_1, \dots, df_n$ of the coordinate functions f_i on $\mathbb{C}^n$. Equivalently we can take for $f_1, \dots, f_n$ any set of generators for the maximal ideal of $\mathcal{O}_{X,0}$. Let $\mu: \mathcal{O}_X^n \rightarrow \Omega_X^1$ be the surjection defined by $\mu(g_1, \dots, g_n) = \sum_{i=1}^n g_i df_i$. Then $(T_X^1)^*$ is isomorphic to the cokernel of the map

$$\mu \otimes 1: H^0(X', \omega_{X'}^{\oplus n}) \rightarrow H^0(X', \Omega_{X'}^1 \otimes \omega_{X'}).$$

In an invariant way the image of $(\mu' \otimes 1)$ can be characterized as the subspace of $H^0(X', \Omega_{X'}^1 \otimes \omega_{X'})$ generated by all elements of the form $\sum g_i \otimes dh_i$, $g_i \in H^0(X', \omega_{X'})$, $h_i \in H^0(X', \mathcal{O}_{X'})$.

2. The case of rational singularities

We keep our previous hypotheses and assume moreover that X is a rational singularity. Let $\pi: \tilde{X} \rightarrow X$ be the minimal good resolution of X , and let $E = \pi^{-1}(0)$ be the exceptional set. The irreducible components $E_1, \dots, E_r$ of E are nonsingular rational curves of selfintersection number $-b_i = E_i \cdot E_i \leq -2$.

Let $\Omega_{\tilde{X}}^1$ resp. $\omega_{\tilde{X}}$ be the sheaves of holomorphic 1- resp. 2-forms on $\tilde{X}$. Observe that by rationality $H^0(\tilde{X}, \omega_{\tilde{X}}) \cong H^0(X', \omega_{X'})$ (see e.g. [Pinkham, 1980], §15). We denote the pull backs to $\tilde{X}$ of the functions f_i of Remark 1.3. also by f_i . Their differentials are sections of $\Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z)$, where $\Omega_{\tilde{X}}^1 \langle \log E \rangle$ denotes the sheaf of meromorphic 1-forms on $\tilde{X}$ with logarithmic poles along E , and Z is the fundamental cycle of $\tilde{X}$. Again we define a sheaf map

$$\tilde{\mu}: \mathcal{O}_{\tilde{X}}^{\oplus n} \rightarrow \Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z)$$

by $\tilde{\mu}(g_1, \dots, g_n) = \sum_{i=1}^n g_i df_i$. This induces a map

$$(\tilde{\mu} \otimes 1)^*: H^0(\tilde{X}, \omega_{\tilde{X}}^{\oplus n}) \rightarrow H^0(\tilde{X}, \Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z) \otimes \omega_{\tilde{X}}).$$

Let ρ be the inclusion

$$\rho: H^0(\tilde{X}, \Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z) \otimes \omega_{\tilde{X}}) \rightarrow H^0(X', \Omega_{X'}^1 \otimes \omega_{X'}).$$

By Remark 1.3. we get

LEMMA 2.1.

$(T_X^1)^*$ is isomorphic to the cokernel of the composite map

$$\rho \circ (\tilde{\mu} \otimes 1)^*: H^0(\tilde{X}, \omega_{\tilde{X}}^{\oplus n}) \rightarrow H^0(X', \Omega_{X'}^1 \otimes \omega_{X'}). \quad \blacksquare$$

In this section we compute the cokernel of the map ρ . Let $\text{Der}_E(\tilde{X})$ be the locally free sheaf of logarithmic vectorfields on $\tilde{X}$ which is dual to $\Omega_{\tilde{X}}^1 \langle \log E \rangle$. Our result is

PROPOSITION 2.2.

The cokernel of the inclusion map

$$\rho: H^0(\tilde{X}, \Omega_{\tilde{X}}^1(\log E)(-Z) \otimes \omega_{\tilde{X}}) \rightarrow H^0(X', \Omega_{X'}^1, \otimes \omega_{X'})$$

has dimension

$$\dim H^1(\tilde{X}, \text{Der}_E(\tilde{X})) - 3 \cdot Z \cdot Z + Z \cdot E - 4.$$

As ρ is an injection this – together with Lemma 2.1. – implies

COROLLARY 2.3.

Let $\mathcal{R}$ resp. $\mathcal{C}$ be the kernel resp. cokernel of $\tilde{\mu}: \mathcal{O}_{\tilde{X}}^{\oplus n} \rightarrow \Omega_{\tilde{X}}^1(\log E)(-Z)$. Then

$$\begin{aligned} \dim T_X^1 &= \dim H^1(\tilde{X}, \text{Der}_E(\tilde{X})) + \dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) \\ &\quad + \dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}) - 3 \cdot Z \cdot Z + Z \cdot E - 4. \end{aligned}$$

REMARK 2.4.

Note that in the case of a reduced fundamental divisor this simplifies to

$$\begin{aligned} &\dim H^1(\tilde{X}, \text{Der}_E(\tilde{X})) + \dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) + \dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}) \\ &\quad + 2 \cdot \sum (b_i - 2), \end{aligned}$$

where we used that $-E^2 - 2 = \sum_{i=1}^r (b_i - 2)$.

For the proof of Proposition 2.2. we need the following vanishing result from [Wahl, 1975]:

THEOREM 2.5. (J. Wahl)

Let $\tilde{X}$ be the minimal good resolution of a normal surface singularity X . Then $H_E^1(\tilde{X}, \text{Der}_E(\tilde{X})) = 0$. ■

COROLLARY 2.6.

i) *On the minimal good resolution $\tilde{X}$ of a normal surface singularity X*

$$H^1(\tilde{X}, \Omega_{\tilde{X}}^1(\log E) \otimes \omega_{\tilde{X}}) = 0$$

ii) *If L is a positive cycle on the exceptional set of $\tilde{X}$, then $H^0(|L|, \text{Der}_E(\tilde{X}) \otimes \mathcal{O}_L(L)) = 0$.*

Proof

i) is deduced from 2.5. by Serre duality. For ii) observe that $H^0(|L|, \text{Der}_E(\tilde{X}) \otimes \mathcal{O}_L(L))$ injects into

$$H_E^1(\tilde{X}, \text{Der}_E(\tilde{X})) = \lim_{\vec{L}} H^0(|L|, \text{Der}_E(\tilde{X}) \otimes \mathcal{O}_L(L))$$

(see e.g. [Wahl, 1975], Proposition 2.2). ■

Now consider the first piece of the exact sequence of local cohomology (a locally free sheaf has no sections supported on E):

$$\begin{aligned} 0 \rightarrow H^0(\tilde{X}, \Omega_{\tilde{X}}^1(\log E)(-Z) \otimes \omega_{\tilde{X}}) &\rightarrow H^0(X', \Omega_{X'}^1(\log E)(-Z) \otimes \omega_{X'}) \\ &\rightarrow H_E^1(\tilde{X}, \Omega_{\tilde{X}}^1(\log E)(-Z) \otimes \omega_{\tilde{X}}) \rightarrow H^1(\tilde{X}, \Omega_{\tilde{X}}^1(\log E)(-Z) \otimes \omega_{\tilde{X}}). \end{aligned}$$

The cohomology group on the right hand side vanishes by 2.6. i) and the fact that $\mathcal{O}_{\tilde{X}}(-Z)$ is generated by global sections: there is an epimorphism of a direct sum of finitely many copies of $\Omega_{\tilde{X}}^1(\log E) \otimes \omega_{\tilde{X}}$ to $\Omega_{\tilde{X}}^1(\log E)(-Z) \otimes \omega_{\tilde{X}}$, and the functor $H^1(\tilde{X}, -)$ is right exact, since H^2 's vanish on $\tilde{X}$.

The restrictions to X' of the sheaves $\Omega_{\tilde{X}}^1 \otimes \omega_{\tilde{X}}$ and $\Omega_{\tilde{X}}^1(\log E)(-Z) \otimes \omega_{\tilde{X}}$ are isomorphic, hence the cokernel of ρ can be identified with $H_E^1(\tilde{X}, \Omega_{\tilde{X}}^1(\log E)(-Z) \otimes \omega_{\tilde{X}})$ which by Serre-duality has the same length as $H^1(\text{Der}_E(\tilde{X}) \otimes \mathcal{O}_{\tilde{X}}(Z))$.

From the exact sequence of sheaves

$$0 \rightarrow \text{Der}_E(\tilde{X}) \rightarrow \text{Der}_E(\tilde{X})(Z) \rightarrow \text{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z) \rightarrow 0$$

and 2.6. ii) we get

$$\begin{aligned} \dim H^1(\tilde{X}, \text{Der}_E(\tilde{X})(Z)) &= \dim H^1(\tilde{X}, \text{Der}_E(\tilde{X})) \\ &\quad + \dim H^1(|Z|, \text{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)). \end{aligned}$$

Again using 2.6. ii) we get the equality

$$\dim H^1(|Z|, \operatorname{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)) = -\chi(\operatorname{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)).$$

Let $Z_0 = E_{i_0}$, $Z_1 = Z_0 + E_{i_1}$, $\dots$ $Z_l = Z_{l-1} + E_{i_l} = Z$ be a sequence of effective divisors with $Z_{k-1} \cdot E_{i_k} = +1$ for all k . Such a sequence exists by rationality of X . From the exact sequences

$$0 \rightarrow \mathcal{O}_{Z_k}(Z_k) \rightarrow \mathcal{O}_{Z_{k+1}}(Z_{k+1}) \rightarrow \mathcal{O}_{E_{i_{k+1}}}(Z_{k+1}) \rightarrow 0$$

tensored with $\operatorname{Der}_E(\tilde{X})$, and the split exact sequences

$$0 \rightarrow \mathcal{O}_{E_i} \rightarrow \operatorname{Der}_E(\tilde{X}) \otimes \mathcal{O}_{E_i} \rightarrow \Theta_{E_i}(-t_i) \rightarrow 0,$$

t_i the number of components which meet the curve E_i , we get

$$\chi(\operatorname{Der}_E(\tilde{X}) \otimes \mathcal{O}_Z(Z)) = \sum_{k=0}^l (6 - t_{i_k} - 2b_{i_k}) - 2.$$

On the other hand, if K is a canonical divisor for $\tilde{X}$, since $\chi(\mathcal{O}_Z) = 1$ for the fundamental divisor of a rational singularity, we have $Z^2 = -2 - K \cdot Z = -2 + \sum_{k=0}^l (2 - b_{i_k})$. Together with $Z \cdot E = \sum_{k=0}^l (-b_{i_k} + t_{i_k})$ this proves what we want. ■

3. Computation of $H^0(\tilde{X}, \mathcal{E} \otimes \omega_{\tilde{X}})$

Recall that $\mathcal{E}$ was defined as the cokernel of

$$\tilde{\mu}: \mathcal{O}_{\tilde{X}}^{\oplus n} \rightarrow \Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z)$$

Let $\mathcal{F}$ be the image of $\tilde{\mu}$. Then $\mathcal{F}$ is a torsion free sheaf, and $\mathcal{E}$ is concentrated on E .

In this section we assume that the fundamental cycle meets every irreducible component of E – except possibly (-2) – curves-strictly negatively.

In order to compute $\mathcal{E}$ we will construct holomorphic functions on $\tilde{X}$ with prescribed divisors. We use the following observation of M. Artin ([Artin, 1966], proof of Theorem 4):

LEMMA 3.1.

Let $\pi: \tilde{X} \rightarrow X$ be the minimal good resolution of a rational surface singularity. Let D be an effective divisor on $\tilde{X}$ such that $D \cdot E_i = 0$ for every irreducible

component E_i of E . Then there is an open neighbourhood U of E in $\tilde{X}$ and a holomorphic function f on U such that $(f) = D \cap U$.

COROLLARY 3.2.

Let $\tilde{X}$ be the minimal resolution of a rational surface singularity with reduced fundamental cycle E . Let $E = E' + E''$ be a decomposition into effective divisors with connected E' . Denote by F the sum of irreducible components of E' which meet E'' , and write $E' = E'_0 + F$. Let D' be an effective divisor with support in E' , and let Δ be an effective divisor on a small neighbourhood U of E'_0 which has no components in common with E . Put $D := D' + \Delta$. Suppose that

- i) $D \cdot E_i = 0$ for all components E_i of E'_0
- ii) the multiplicity of a component E_i of F in D is greater or equal to

$$D \cdot E_i / (b_i - E_i \cdot E'').$$

Then there exists a holomorphic function f on $\tilde{X}$ such that $(f) \cap U = D$.

Proof

Let $E''_1, \dots, E''_k$ be the connected components of E'' , let F_i be the component of F meeting E''_i , and let m_i be its multiplicity in D . We put $C := D + \sum_{i=1}^k m_i E''_i$. Since $E''_i + F_i$ is the exceptional set of a rational singularity with reduced fundamental cycle it follows from ii) that $C \cdot E_i \leq 0$ for all irreducible

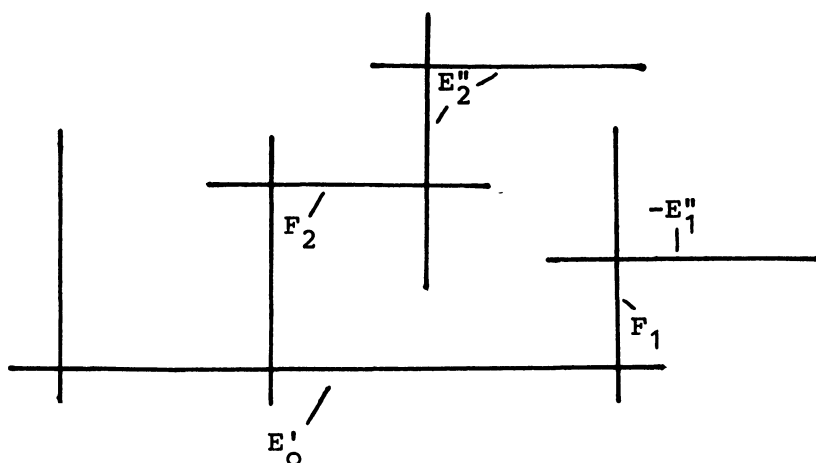


Fig. 1.

components of E . Obviously $C \cdot E_i = 0$ for all E_i contained in U , so we can modify C outside U to obtain an effective divisor $\tilde{C}$ with $\tilde{C} \cdot E_i = 0$ for all i . Applying (3.1) to $\tilde{C}$ we obtain the desired function f .

The next two lemmata give our description of $\mathcal{C}$. First we investigate $\mathcal{C}$ near curves with ‘high self-intersection number’.

LEMMA 3.3.

- i) Let p be a smooth point of E , and assume that $Z \cdot E_i < 0$ for the unique irreducible component E_i of E containing p . Then $\mathcal{C}_p = 0$.
- ii) Let p be a point, where two components E_i, E_j of E intersect, and assume that $Z \cdot E_i < 0$ and $Z \cdot E_j < 0$. Then $\mathcal{C}$ is a skyscraper sheaf near p and $\dim \mathcal{C}_p = 1$.

Proof

i) We can choose local coordinates (u, v) near p such that there is a holomorphic function f_1 on $\tilde{X}$ which in local coordinates is given by $f_1(u, v) = v^a$, a being the multiplicity of E_1 in the fundamental cycle Z . Locally $\Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z)$ is generated by $v^{a-1} dv$ and $v^a du$.

Let Δ_0 be the curve $\{u = 0\}$. As $Z \cdot E_i < 0$ we can choose other curves $\Delta_1, \dots, \Delta_l$ which are disjoint from Δ_0 and intersect the exceptional divisor transversally in smooth points such that

$$\left(Z + \sum_{k=0}^l \Delta_k \right) \cdot E_j = 0$$

for $j = 1, \dots, r$. By Lemma 3.1. there is a holomorphic function f_2 on $\tilde{X}$ with divisor $Z + \sum_{k=0}^l \Delta_k$. After changing the u -coordinate, this function can be written locally as $f_2 = uv^a$. Obviously, df_1 and df_2 generate $\Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z)$ near p .

ii) We proceed as before and choose smooth curves Δ_1 and Δ_2 through p such that $E_1, E_2, \Delta_1, \Delta_2$ are pairwise transversal in p . There are local coordinates u, v with $E_i = \{v = 0\}$, $E_j = \{u = 0\}$, and holomorphic functions f, g_1, g_2 on $\tilde{X}$ such that $f = u^a v^b$, $g_k = u^a v^b (\alpha_k u + \beta_k v + \text{higher order terms})$ with $\alpha_1 : \beta_1 \neq \alpha_2 : \beta_2$. Again a, b are the multiplicities of E_j, E_i resp. in Z .

Locally at p the sheaf $\Omega_{\tilde{X}}^1 \langle \log E \rangle (-Z)$ is generated by $u^{a-1} v^b du$ and $u^a v^{b-1} dv$, while $\mathcal{F}$ is generated by df, dg_1, dg_2 . A simple calculation now shows $\dim \mathcal{C}_p = 1$.

Now we restrict to the case $Z = E$, i.e. the fundamental divisor is reduced. We want to see, how $\mathcal{C}$ looks like on a linear chain of (-2) -curves which have intersection number 0 with E . So, let $E_0, \dots, E_{t+1}$ be irreducible components of E such that

$$E_1 \cdot E_1 = \dots = E_t \cdot E_t = -2$$

$$E_0 \cdot E_1 = E_1 \cdot E_2 = \dots = E_t \cdot E_{t+1} = 1,$$

for $i = 1, \dots, t$, E_i meets no other component but E_{i-1} and E_{i+1} , and $E \cdot E_0 < 0$, $E \cdot E_{t+1} < 0$. Let U be a small neighbourhood of $E_1 \cup \dots \cup E_t$. Since $E \cap U$ intersects $E_1, \dots, E_t$ trivially, E is a principal divisor on U (cf. [Artin, 1966]). The ideal sheaf $\mathcal{I}_{E|U}$ is generated by a single holomorphic function, say f_1 . It vanishes to first order along $E \cap U$.

Blowing down $E_1 \cup \dots \cup E_t$ yields a rational double point A_t . So f_1 can be extended to a minimal set f_1, f_2, f_3 of generators of the algebra of holomorphic functions on U . It is well-known that f_2 and f_3 can be chosen such that $f_1^{t+1} = f_2 f_3$ and such that they have the divisors

$$(f_2) = \sum_{i=1}^t i \cdot E_i + (t+1)(E_{t+1} \cap U)$$

$$(f_3) = \sum_{i=1}^t (t-i+1) \cdot E_i + (t+1)(E_0 \cap U).$$

REMARK 3.4.

$\mathcal{F}|_U$ is generated by $df_1, d(f_1 f_2), d(f_1 f_3)$.

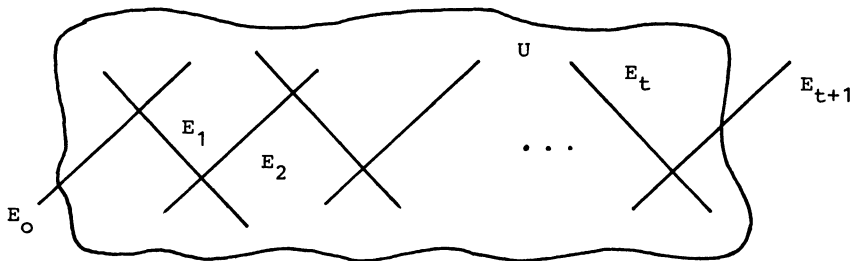


Fig. 2.

Proof

By Corollary 3.2. we see that $f_1, f_1 f_2, f_1 f_3$ can be chosen as restrictions of holomorphic functions on $\tilde{X}$. Conversely any holomorphic function on U which vanishes along $E \cap U$ is of the form $h \cdot f_1$, where h is in the ideal generated by f_1, f_2, f_3 .

We put

$$D := \sum_{i=1}^t \max(i, t-i+1) E_i$$

LEMMA 3.5.

- i) If t is odd, then $\mathcal{C}|_U \cong \mathcal{O}_D$
- ii) If t is even, say $t = 2k$, then $\mathcal{C}|_U$ has a torsion subsheaf τ of length 1, concentrated at $E_k \cap E_{k+1}$, and there is an exact sequence

$$0 \rightarrow \tau \rightarrow \mathcal{C}|_U \rightarrow \mathcal{O}_D \rightarrow 0$$

Proof

The sheaf $\Omega_{\tilde{X}}^1(\log E)(-E)|_U$ is free with generators $f_1 df_2/f_2$ and $f_1 df_3/f_3$. This assertion is easily checked via an explicit resolution of the A_t -singularity. Since $(t+1) df_1 = f_1 df_2/f_2 + f_1 df_3/f_3$ we see that $\mathcal{C}|_U$ is cyclic with generator $f_1 df_2/f_2 = -f_1 df_3/f_3$. The claim now follows from (3.4) by a simple calculation in local coordinates. ■

For a later use we note

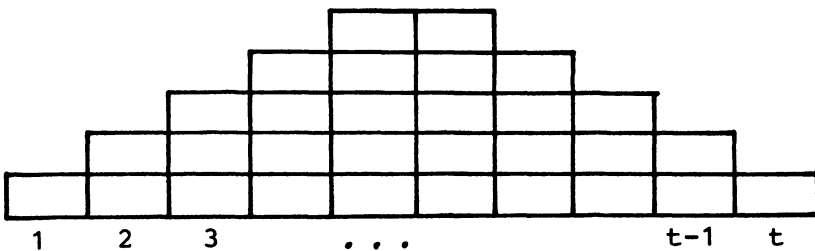


Fig. 3.

LEMMA 3.6.

$$\dim H^0(D, \mathcal{O}_D) = [(t+1)/2]$$

Proof

For an effective cycle C supported on the exceptional locus of a rational surface singularity one has $H^1(|C|, \mathcal{O}_C) = 0$ (cf. [Artin, 1966]). So it is sufficient to compute the holomorphic Eulercharacteristic $\chi(\mathcal{O}_D)$ of $\mathcal{O}_D$.

Consider the sequence of divisors

$$D_1 = E_1, \quad D_2 = E_1 + E_2, \quad \dots, \quad D_{t-1} = E_1 + \dots + E_{t-1},$$

$$D_t = E_1 + \dots + E_t, \quad D_{t+1} = E_1 + 2E_2 + E_3 + \dots, \dots$$

ending with D (cf. Fig. 3). Let E_{i_l} be the curve which is added to D_l to obtain D_{l+1} . Then the intersection number $D_l \cdot E_{i_l}$ is 1, if E_{i_l} does not start a new row, and it is 0 otherwise.

From the exact sequence

$$0 \rightarrow \mathcal{O}_{E_{i_l}}(-D_l) \rightarrow \mathcal{O}_{D_{l+1}} \rightarrow \mathcal{O}_{D_l} \rightarrow 0$$

we obtain

$$\chi(\mathcal{O}_{D_{l+1}}) = \chi(\mathcal{O}_{D_l}) + (1 - D_l \cdot E_{i_l}).$$

So

$$\chi(\mathcal{O}_D) = \sum_l (1 - D_l \cdot E_{i_l}).$$

By the discussion above this sum has precisely $[(t+1)/2]$ summands 1, and all other summands are zero. ■

4. Computation of $H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}})$

The most difficult part in formula (2.3) for $\dim T_X^1$ seems to be $H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}})$. Recall that we have the exact sequence

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{O}_{\tilde{X}}^{\oplus n} \xrightarrow{\tilde{\mu}} \Omega_{\tilde{X}}^1(\log E)(-Z) \rightarrow \mathcal{C} \rightarrow 0,$$

so by Hilbert's syzygy theorem $\mathcal{R}$ is locally free of rank $n-2$.

We will apply the results of Chapter 3, so we assume again that the fundamental cycle is reduced and meets all non- (-2) -curves strictly negatively. In other words: if an irreducible component E_i of E meets t_i other curves, then its self-intersection number $-b_i$ fulfills

$$b_i \geq t_i \quad \text{for } i = 1, \dots, r$$

$$b_i \geq t_i + 1 \quad \text{if } b_i \neq 2.$$

The restriction of the locally free sheaf $\mathcal{R}$ to E_i is a direct sum of line bundles (cf. [Grauert and Remmert, 1977] VII, Satz 5). We now give estimates for the degrees of these bundles.

PROPOSITION 4.1.

Let E_i be an irreducible component of E .

- i) If $b_i \geq t_i + 2$, then $\mathcal{R} \otimes \mathcal{O}_{E_i}$ decomposes into line bundles of degree at least -2 .
- ii) If $b_i = t_i + 1$, then all direct summands of $\mathcal{R} \otimes \mathcal{O}_{E_i}$ have degree at least -1 .
- iii) If $b_i = 2$, $t_i = 1$ and E_i meets a (-2) -curve, then $\mathcal{R} \otimes \mathcal{O}_{E_i}$ is trivial.

Proof

Consider the exact sequences

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{O}_{\tilde{X}}^{\oplus n} \xrightarrow{\tilde{\mu}} \mathcal{F} \rightarrow 0$$

$$0 \rightarrow \mathcal{F} \rightarrow \Omega_{\tilde{X}}^1 \langle \log E \rangle (-E) \rightarrow \mathcal{C} \rightarrow 0$$

The first one remains exact, when restricted to E_i :

$$0 \rightarrow \mathcal{R} \otimes \mathcal{O}_{E_i} \rightarrow \mathcal{O}_{E_i}^{\oplus n} \xrightarrow{\tilde{\mu}} \mathcal{F} \otimes \mathcal{O}_{E_i} \rightarrow 0 \quad (4.2)$$

But $\mathcal{F} \otimes \mathcal{O}_{E_i}$ is no longer torsion free, the second sequence gives

$$0 \rightarrow \mathcal{T}or_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{C}, \mathcal{O}_{E_i}) \rightarrow \mathcal{F} \otimes \mathcal{O}_{E_i} \rightarrow \Omega_{\tilde{X}}^1 \langle \log E \rangle (-E) \otimes \mathcal{O}_{E_i} \rightarrow \mathcal{C} \otimes \mathcal{O}_{E_i} \rightarrow 0. \quad (4.3)$$

So the torsion subsheaf of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ is concentrated in the points, where $\mathcal{C}$ is a skyscraper sheaf, and it has length 1 there (cf. (3.3) and (3.5)).

First we prove iii): In this case $\mathcal{T}or_1^{\mathcal{O}_{\tilde{X}}}(\mathcal{C}, \mathcal{O}_{E_i}) = 0$, while $\mathcal{C} \otimes \mathcal{O}_{E_i}$ is a

skyscraper sheaf of length 1 (see Lemma 3.5). Hence by (4.3) the Chern class of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ is zero. By (4.2) we see that $\mathcal{R} \otimes \mathcal{O}_{E_i}$ has Chern class zero. But a subsheaf of $\mathcal{O}_{E_i}^{\oplus n}$ has trivial Chern class, if and only if it is trivial.

We now concentrate on i) and ii). If we want to show that $\mathcal{R} \otimes \mathcal{O}_{E_i}$ splits into direct summands of degree at least -1 , it is sufficient to show the surjectivity of $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\tilde{\mu}} H^0(E_i, \mathcal{F} \otimes \mathcal{O}_{E_i})$. This follows from the cohomology sequence of (4.2) and the observation that $H^1(E_i, \mathcal{R} \otimes \mathcal{O}_{E_i})$ is never zero, if $\mathcal{R} \otimes \mathcal{O}_{E_i}$ has a line bundle summand of degree -2 or less. Similarly for the estimate -2 in (i) it suffices to prove the surjectivity of $H^0(E_i, \mathcal{O}_{E_i}(1)^{\oplus n}) \xrightarrow{\tilde{\mu}} H^0(E_i, \mathcal{F} \otimes \mathcal{O}_{E_i}(1))$.

We will discuss the torsion part and the non-torsion part of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ separately. For the torsion part we use

LEMMA 4.4.

Let E_i, E_j be two components of E which meet in a point p and for which $E \cdot E_i < 0$, $E \cdot E_j < 0$. Let f be a holomorphic function on $\tilde{X}$ whose zero divisor contains E_i with multiplicity 2, E_j with multiplicity 1, and no other curve passing through p . Then df represents a generator of the torsion part of $(\mathcal{F} \otimes \mathcal{O}_{E_i})_p$.

Proof

Let (u, v) be local coordinates around p with $E_i = \{v = 0\}$, $E_j = \{u = 0\}$. The computation in the proof of (3.3.ii) shows that locally $\Omega_{\tilde{X}}^1 \langle \log E \rangle (-E)$ is generated by $v du$ and $u dv$, while $\mathcal{F}$ is generated by $v du + u dv$, $u^2 dv$, $uv dv$, $uv du$, $v^2 du$. So the kernel of the map $\mathcal{F}/v\mathcal{F} \rightarrow \Omega_{\tilde{X}}^1 \langle \log E \rangle (-E)/v \cdot \Omega_{\tilde{X}}^1 \langle \log E \rangle (-E)$ is generated by $uv dv$. $\square$

COROLLARY 4.5.

Let E_i be a component of E such that $b_i \geq t_i + 1$. Then there are holomorphic functions on $\tilde{X}$ which vanish to order 2 along E_i and whose differentials generate the torsion of $\mathcal{F} \otimes \mathcal{O}_{E_i}$.

Proof

For each non- (-2) -curve E_j meeting E_i we find by (3.2) a holomorphic function on $\tilde{X}$ which vanishes to order 2 along E_i and all the curves $E_k \neq E_j$ that meet E_i .

The non-torsion part of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ is the image $\tilde{\mathcal{F}}_i$ of $\mathcal{F} \otimes \mathcal{O}_{E_i}$ in $\Omega_{\tilde{X}}^1 \langle \log E \rangle (-E) \otimes \mathcal{O}_{E_i}$. It is clear that the differential of a holomorphic

function on $\tilde{X}$ has non-vanishing image in $\tilde{\mathcal{F}}_i$ only if it vanishes to order 1 along E_i . In view of (4.5) it suffices for the proof of i) resp. ii) to show that the maps $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}(1)) \xrightarrow{\mu} H^0(E_i, \tilde{\mathcal{F}}_i(1))$ resp. $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\tilde{\mu}} H^0(E_i, \tilde{\mathcal{F}}_i)$ are surjective. Before doing this we note

LEMMA 4.6.

$\tilde{\mathcal{F}}_i$ has Chern class $2(b_i - t_i) - 2$ on E_i , and $H^1(E, \tilde{\mathcal{F}}_i) = 0$.

Proof

One can deduce from (2.6) that $\Omega_{\tilde{X}}^1 \langle \log E \rangle (-E) \otimes \mathcal{O}_{E_i} \cong (\omega_{E_i}(t_i) \oplus \mathcal{O}_{E_i})(-E \cdot E_i)$. So the claim on the degree of $\tilde{\mathcal{F}}_i$ follows from the exact sequence

$$0 \rightarrow \tilde{\mathcal{F}}_i \rightarrow \Omega_{\tilde{X}}^1 \langle \log E \rangle (-E) \otimes \mathcal{O}_{E_i} \rightarrow \mathcal{C} \otimes \mathcal{O}_{E_i} \rightarrow 0.$$

The sequence (4.2) shows that $H^1(E_i, \mathcal{F} \otimes \mathcal{O}_{E_i}) = 0$, hence also $H^1(E_i, \tilde{\mathcal{F}}_i) = 0$.

We now prove (4.1.i): as mentioned above it suffices to prove the surjectivity of $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}(1)) \rightarrow H^0(E_i, \tilde{\mathcal{F}}_i(1))$. The latter space has dimension $2(b_i - t_i + 1)$ by Lemma 4.6. Now choose a small curve Δ transversal to E_i which does not meet any other component of E . By Corollary 3.2. we find for $0 \leq k < b_i - t_i$ holomorphic functions f_k on $\tilde{X}$ whose zero divisor contains E_i and all components of E adjacent to E_i with multiplicity 1, and Δ with multiplicity k .

Choose local coordinates (u, v) around the point of $E_i \cap \Delta$ such that $E_i = \{v = 0\}$, $\Delta = \{u = 0\}$. Then $f_k = \epsilon_k \cdot u^k \cdot v$ with some unit ϵ_k . So

$$df_k = k \cdot u^{k-1} v \, du + u^k \, dv + \text{higher terms.}$$

If we take all linear combinations of $df_0, \dots, df_{b_i-t_i}$ with coefficients in $H^0(E_i, \mathcal{O}_{E_i}(1))$ (which means that we allow constants and $\frac{1}{u}$ as coefficients), we get $2(b_i - t_i + 1)$ linearly independent sections of $\tilde{\mathcal{F}}_i(1)$.

Finally we prove (4.1.ii): In this case $\dim H^0(E_i, \tilde{\mathcal{F}}_i) = 2$, and as above one constructs two independent holomorphic functions which vanish to first order along E_i . This shows that $H^0(E_i, \mathcal{O}_{E_i}^{\oplus n}) \xrightarrow{\tilde{\mu}} H^0(E_i, \tilde{\mathcal{F}}_i)$ is surjective. ■

As in Chapter 3 we also have to consider chains of (-2) -curves.

PROPOSITION 4.7.

Let $E_0, E_1, \dots, E_t, E_{t+1}$ be irreducible components of E such that $E_1, \dots, E_t$ form a chain of (-2) -curves, E_0 meets E_1 , E_{t+1} meets E_t , and there is no

intersection of $E_1, \dots, E_t$ with other components. Also assume that $E \cdot E_0 < 0$ and $E \cdot E_{t+1} < 0$.

Then on a sufficiently small neighbourhood U of $E_1 \cup \dots \cup E_t$ the vector bundle $\mathcal{R}$ splits into a trivial summand of rank $n - 3$ and a line bundle $\mathcal{L}$. The restriction of $\mathcal{L}$ to the irreducible components are

$$\mathcal{L} \otimes \mathcal{O}_{E_i} \cong \begin{cases} \mathcal{O}_{E_i} & \text{if } 1 \leq i \leq t, \begin{matrix} i \neq k, k+1 & \text{for } t = 2k \text{ even} \\ i \neq k & \text{for } t = 2k - 1 \text{ odd} \end{matrix} \\ \mathcal{O}_{E_k}(-2) & \text{if } i = k; t = 2k - 1 \\ \mathcal{O}_{E_i}(-1) & \text{if } i = k, k+1; t = 2k \end{cases}$$

Proof

The splitting of $\mathcal{R}|_U$ into a trivial summand and a line bundle follows from Remark 3.4. It remains to compute the Chern classes of $\mathcal{R} \otimes \mathcal{O}_{E_i}$ ($1 \leq i \leq t$). By (4.2) and (4.3) we have $c_1(\mathcal{R} \otimes \mathcal{O}_{E_i}) = -c_1(\mathcal{F} \otimes \mathcal{O}_{E_i}) = c_1(\mathcal{G} \otimes \mathcal{O}_{E_i}) - c_1(\mathcal{T} \otimes \mathcal{O}_1^{\otimes \bar{x}}(\mathcal{G}, \mathcal{O}_{E_i}))$. The claim is that this number is equal to $E_i \cdot D$, where D is the divisor of Lemma 3.5.

Let τ be the torsion subsheaf of $\mathcal{G}$. By Lemma 3.5. we have an exact sequence

$$0 \rightarrow \tau \rightarrow \mathcal{G} \rightarrow \mathcal{O}_D \rightarrow 0.$$

Tensoring this sequence with $\mathcal{O}_{E_i}$ we obtain

$$c_1(\mathcal{G} \otimes \mathcal{O}_{E_i}) - c_1(\mathcal{T} \otimes \mathcal{O}_1^{\otimes \bar{x}}(\mathcal{G}, \mathcal{O}_{E_i})) = c_1(\mathcal{O}_D \otimes \mathcal{O}_{E_i}) - c_1(\mathcal{T} \otimes \mathcal{O}_1^{\otimes \bar{x}}(\mathcal{O}_D, \mathcal{O}_{E_i})).$$

But $\mathcal{O}_D \otimes \mathcal{O}_{E_i} \cong \mathcal{O}_{E_i}$, $\mathcal{T} \otimes \mathcal{O}_1^{\otimes \bar{x}}(\mathcal{O}_D, \mathcal{O}_{E_i}) \cong \mathcal{O}_{E_i}(-D)$.

The following theorem contains the main result of this paper:

THEOREM 4.8.

Let $\pi: \tilde{X} \rightarrow X$ be the minimal resolution of a rational surface singularity $(X, 0)$, let $E = \bigcup_{i=1}^r E_i$ be the decomposition of the exceptional set $E = \pi^{-1}(0)$ into irreducible components, and let $-b_i$ be the self-intersection number of E_i . Denote by t_i the number of components of E different from E_i which meet E_i , and by s_i the number of chains of curves of self-intersection number -2 and trivial intersection with E that meet E_i . Assume that

- a) $b_i \geq t_i + 1$ for $b_i > 2$, $b_i \geq t_i$ for $b_i = 2$
- b) $s_i \leq b_i - t_i - 2$ if $b_i - t_i \geq 2$
- c) $s_i = 0$ if $b_i = t_i + 1$, $b_i \neq 2$.

Furthermore assume that inequality b) is strict for at least one E_i . Then

$$\dim T_X^1 = \dim T_{\tilde{X}}^1 + \text{emb}(X) - 4.$$

Proof

From Corollary 2.3. we get

$$\begin{aligned} \dim T_X^1 &= \dim T_{\tilde{X}}^1 + \sum_{i=1}^r (b_i - 3) + \dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) \\ &\quad + \dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}) \end{aligned}$$

By our assumptions the formula for the embedding dimension in [Artin, 1966] gives $\text{emb}(X) = 1 - E \cdot E = 1 + \sum_{i=1}^r (b_i - t_i)$. Hence

$$\begin{aligned} \dim T_X^1 - (\dim T_{\tilde{X}}^1 + \text{emb } X - 4) &= \dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) \\ &\quad + \dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}) - (r - 1) \end{aligned} \tag{4.9}$$

Let $L_1, \dots, L_p$ be the maximal chains of (-2) -curves $L_j = E_1^{(j)} \cup \dots \cup E_{t_j}^{(j)}$ such that $E \cdot E_\tau^{(j)} = 0$ for $1 \leq \tau \leq t_j$. To each L_j we associate the divisor

$$D_j = \sum_{\tau=1}^{t_j} \max(\tau, t_j - \tau + 1) \cdot E_\tau^{(j)}$$

as in (3.5). Then we have the exact sequence

$$\begin{aligned} \dots &\rightarrow H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}(-D_1 - \dots - D_p)) \rightarrow H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}) \\ \dots &\rightarrow H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1 + \dots + D_p}) \rightarrow 0 \end{aligned} \tag{4.10}$$

Then Theorem 4.8. follows from (4.9), (4.10) and

LEMMA 4.11.

Under the assumptions of Theorem 4.8. we have

- i) $\dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) + \dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1 + \dots + D_p}) = r - 1$
- ii) $\dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}(-D_1 - \dots - D_p)) = 0.$

Proof

i) Every point, where two curves E_i, E_j with $b_i > t_i, b_j > t_j$ meet, gives a one-dimensional contribution to $H^0(\tilde{X}, \mathcal{E} \otimes \omega_{\tilde{X}})$, and all other contributions to the sum above come from the chains of (-2) -curves.

By Serre-duality and the adjunction formula $H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D_1 + \dots + D_p})$ has the same dimension as $\bigoplus_{j=1}^p H^0(|D_j|, \mathcal{R}^* \otimes \mathcal{O}_{D_j}(D_j))$. Recall from (4.7) that on D_j the bundle $\mathcal{R}^*$ decomposes into a trivial bundle and a line bundle, say $\mathcal{L}_j$, with $\mathcal{L}_j \otimes \mathcal{O}_{E^{(j)}} \cong \mathcal{O}_{E^{(j)}}(-D_j)$. By the negativity of the intersection matrix $\mathcal{O}_{D_j}(D_j)$ has no sections, hence

$$H^0(|D_j|, \mathcal{R}^* \otimes \mathcal{O}_{D_j}(D_j)) \cong H^0(|D_j|, \mathcal{L}_j \otimes \mathcal{O}_{D_j}(D_j)) \cong H^0(|D_j|, \mathcal{O}_{D_j})$$

has dimension $[(t_j + 1)/2]$ by Lemma 3.6. On the other hand $\dim H^0(|D_j|, \mathcal{E} \otimes \omega_{\tilde{X}}) = [(t_j + 2)/2]$ by (3.5) and (3.6). So each chain L_j contributes $t_j + 1$ to the sum on the right hand side of (4.11).

Using the fact that the resolution graph of X is a tree, one easily sees that the number of intersection points of curves not contained in $\bigcup_{j=1}^p L_j$ and the numbers $t_j + 1$ for every chain L_j sum up to r .

ii) We first check that $H^1(|E|, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \dots - D_p)) = 0$. By Serre duality this means that $H^0(|E|, \mathcal{R}^* \otimes \mathcal{O}_E(E + D_1 + \dots + D_p)) = 0$. Our hypotheses and the Propositions 4.1. and 4.7. imply that the restriction $\mathcal{R}^* \otimes \mathcal{O}_{E_i}(E + D_1 + \dots + D_p)$ to E_i is a direct sum of line bundles of degree at most 0, and for one index i it is a direct sum of line bundles of degree at most -1 . Hence $\mathcal{R}^* \otimes \mathcal{O}_E(E + D_1 + \dots + D_p)$ has no nontrivial global sections.

Since the fundamental cycle is reduced, the sheaves $\mathcal{O}_{\tilde{X}}(-nE)$ are generated by their global sections. This gives surjections of direct sums of copies of $\mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \dots - D_p)$ to $\mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \dots - D_p)(-nE)$. Hence $H^1(|E|, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-D_1 - \dots - D_p)(-nE)) = 0$ for all positive integers n too, and from the exact sequences

$$\begin{aligned} & H^1(|E|, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_E(-nE - D_1 - \dots - D_p)) \\ & \rightarrow H^1(|E|, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{(n+1)E}(-D_1 - \dots - D_p)) \\ & \rightarrow H^1(|E|, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{nE}(-D_1 - \dots - D_p)) \rightarrow 0 \end{aligned}$$

we get our vanishing result (see also [Wahl, 1975], Lemma 5.15.1). ■

EXAMPLE 4.12.

Consider the weighted dual graph



If $b_0 \geq r - 1$ this is the dual resolution graph of a rational surface singularity. Its embedding dimension is $\text{emb}(X) = 3 + \sum_{i=0}^r (b_i - 2)$ for $r \geq b_0$, and $b_1 + \dots + b_r - 3$ for $r = b_0 + 1$ (cf. [Artin, 1966]).

Theorem 4.8. gives $\dim T_X^1 = \dim T_{\tilde{X}}^1 + \sum_{i=0}^r (b_i - 2) - 1$ provided $b_0 \geq r + 3$, or $b_0 = r + 2$, and at least one of $b_1, \dots, b_r$ is greater than 3.

For the dimension of $T_{\tilde{X}}^1$ one computes from the exact sequence

$$0 \rightarrow \text{Der}_E(\tilde{X}) \rightarrow \Theta_{\tilde{X}} \rightarrow \bigoplus_{i=0}^r \mathcal{O}_{E_i}(E_i) \rightarrow 0$$

$$\text{that } \dim T_{\tilde{X}}^1 = \sum_{i=0}^r (b_i - 1) + \dim H^1(\tilde{X}, \text{Der}_E(\tilde{X})).$$

The cohomology group $H^1(\tilde{X}, \text{Der}_E(\tilde{X}))$ parametrizes the infinitesimal deformations of $\tilde{X}$ to which all the E_i lift. By Theorem 4.1. of [Laufer, 1973] the analytic type of $\tilde{X}$ (and of X) is determined by the location of the r intersection points on the central curve, hence $H^1(\tilde{X}, \text{Der}_E(\tilde{X}))$ has dimension $r - 3$.

Putting everything together, we get $\dim T_{\tilde{X}}^1 = \sum_{i=0}^r (b_i - 3)$ and

$$\dim T_X^1 = \sum_{i=0}^r (2b_i - 2) - 4$$

under the assumptions made above.

EXAMPLE 4.13.

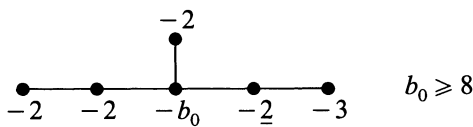
Let X be twodimensional quotient singularity of type T_m , O_m , I_m (cf. [Brieskorn, 1968] 2.9) and assume that the selfintersection number of the central curve of the exceptional set is at least $6 + p$, where p denotes the number of chains of (-2) -curves E_i with $E \cdot E_i = 0$. Then the equality

$$\dim T_X^1 = \dim T_{\tilde{X}}^1 + \text{emb}(X) - 4 = \sum_{i=0}^r (2b_i - 3) - 1$$

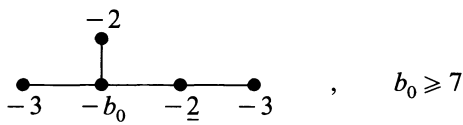
holds.

Proof

Theorem 4.8. applies to all cases of quotient singularities as listed in [Brieskorn, 1968] 2.11, apart from the following two types: I_m , $m = 30(b_0 - 2) + 7$ with resolution graph



I_m , $m = 30(b_0 - 2) + 17$ with resolution graph



In both cases there is a chain (of length one) of (-2) -curves which meets a (-3) -curve. Let $L_1 = E_1$ be the (-2) -curve and E_2 the (-3) -curve in question. We replace the divisor D_1 in the proof of Theorem 4.8. by $D'_1 := E_1 + E_2$. Put $D' := D'_1 + D_2$ in the first case, and $D' := D_1$ in the second case. In analogy to Lemma 4.11. we have

CLAIM (4.14)

$$H^1(\tilde{X}, \mathcal{R}^* \otimes \omega_{\tilde{X}}(-D')) = 0.$$

Proof

As in 4.13. we have to show that $H^0(E, \mathcal{R}^* \otimes \mathcal{O}_E(E + D')) = 0$. The restriction of $\mathcal{R}^* \otimes \mathcal{O}_E(E + D')$ to the central curve E_0 and to E_2 is a direct sum of line bundles of negative degree (cf. 4.1), and it has degree ≤ 0 on all components but E_1 . On E_1 it is a direct sum of a line bundle of degree one and of line bundles of degree -1. This shows that the vectorbundle $\mathcal{R}^* \otimes \mathcal{O}_E(E + D')$ cannot have any global sections on E .

CLAIM (4.15)

$$\dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{D'_1}) = 1.$$

Proof

As in the proof of Lemma 4.11. i) it suffices to show that $\dim H^0(E_1 \cup E_2, \mathcal{R}^* \otimes \mathcal{O}_{E_1+E_2}(E_1 + E_2)) = 1$.

Let g_1, g_2, g_3 be the global functions on $\tilde{X}$ of Remark 3.4., whose differentials generate $\mathcal{F}$ in a neighbourhood of E_1 . We may assume that g_1 vanishes with multiplicity 1 along E_1 and E_2 , g_2 vanishes with multiplicity 3 along E_1 and multiplicity 1 along E_2 , and g_3 vanishes with multiplicity 3 both along E_1 and E_2 .

Call $\mathcal{F}' \subset \Omega_{\tilde{X}}^1(\log E)(-E)$ the subsheaf generated by dg_1, dg_2, dg_3 and let $\mathcal{L}$ be the sheaf of relations between them:

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{O}_{\tilde{X}}^{\oplus 3} \rightarrow \mathcal{F}' \rightarrow 0. \quad (4.16)$$

One easily sees that $(\Omega_{\tilde{X}}^1(\log E)(-E)/\mathcal{F}') \otimes \mathcal{O}_{E_2}$ is a torsion sheaf of length at least one, so $c_1(\mathcal{F}' \otimes \mathcal{O}_{E_2}) \leq 1$.

Hence by (4.16)

$$\deg \mathcal{L}|_{E_2} \geq -1, \quad \text{while by Proposition 4.7.}$$

$$\deg \mathcal{L}|_{E_1} = -2.$$

Now by Proposition 4.7. the restriction of $\mathcal{R}^* \otimes \mathcal{O}_{E_1+E_2}(E_1 + E_2)$ to E_1 is a sum of line bundles of negative degrees and one line bundle of degree one, namely $\mathcal{L}^* \otimes \mathcal{O}_{E_1}(E_1 + E_2)$. By Proposition 4.1. and (4.6) the vectorbundle $\mathcal{R}^* \otimes \mathcal{O}_{E_2}(E_1 + E_2)$ has at most one line bundle summand of non-negative degree, which then is trivial. This summand does not agree with $\mathcal{L}^* \otimes \mathcal{O}_{E_2}(E_1 + E_2)$ (which has degree ≤ -1), so a holomorphic section of $\mathcal{R}^* \otimes \mathcal{O}_{E_1+E_2}(E_1 + E_2)$ has to vanish on E_2 . This proves claim (4.15).

The rest of the proof for the equality $\dim T_X^1 = \dim T_X^{\frac{1}{2}} + \text{emb}(X) - 4$ for the singularities under consideration is analogous to the proof of (4.8).

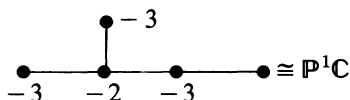
REMARK 4.17.

There are 63 individual quotient singularities of type T, O, I that are not covered by Example 4.15.

EXAMPLE 4.18.

Finally we want to give a partial analysis of the example of J. Wahl mentioned in the introduction.

Let X be the rational surface singularity with dual resolution graph



The fundamental cycle is $Z = 2E_0 + E_1 + E_2 + E_3$, where E_0 denotes the central curve. We have $\text{emb}(X) = 6$, $\dim T_X^1 = 7$, so formula (0.1) would give 9 for $\dim T_X^1$. We want to show that $\dim T_X^1 \geq 10$. We apply Corollary 2.3.:

$$\dim T_X^1 = \dim H^0(\tilde{X}, \mathcal{C} \otimes \omega_{\tilde{X}}) + \dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}}) + 7.$$

By lemma 3.3. $\mathcal{C}$ is a skyscraper sheaf supported at the points of intersection of E_0 with the other components and with stalks of length 1 there. Hence, $\dim T_X^1 = 10 + \dim H^1(\tilde{X}, \mathcal{R} \otimes \omega_{\tilde{X}})$.

REMARK 4.19.

In this example one can compute the map $(\mu' \otimes 1)$ from Section 2 quite explicitly using the canonical Gorenstein cover of X . One actually gets $\dim T_X^1 = 10$. For details see [Behnke et al., in prep.], Section 8.

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Appendix

In this section, which is entirely due to Jonathan Wahl, lower estimates for the dimension of T_X^1 of rational and minimally elliptic surface singularities are given. Let X be a normal surface singularity with minimal good resolution $\pi: \tilde{X} \rightarrow X$, let E be the exceptional divisor for π , and let Z be an effective cycle supported on E . In [Wahl, 1979], §2, Wahl defines a deformation functor R_Z by

$$R_Z(T) := \{\text{isomorphism classes of triples } (\tilde{\mathcal{X}}, \rho, \mathcal{Z}), \rho: \tilde{\mathcal{X}} \rightarrow T \text{ a deformation of } \tilde{X}, \text{ and } \mathcal{Z} \text{ a relative effective Cartier divisor which lifts } Z\}$$

Since the normal bundle of Z has no nontrivial sections, $\mathcal{X}$ is uniquely determined by $\tilde{\mathcal{X}}$, so that R_Z is a subfunctor of the deformation functor of the resolution. In 2.5.ii of [Wahl, 1979] it is checked that R_Z has a formally semiuniversal deformation space.

If X is rational or minimally elliptic, and Z is the fundamental cycle, then there is a natural blowing down map which maps R_Z finite to one to the deformation space of X ([Wahl, 1979], 2.7). Let $\sigma: \mathcal{X} \rightarrow V$ be the semiuniversal deformation of X , and let $\Phi(R_Z)$ be the image of R_Z in V . By Theorem 1 of [Karras, 1983] the fibre $\mathcal{X}_t$ over a general point of $\Phi(R_Z)$ has exactly one singular point x_t , and $(\mathcal{X}_t, x_t)$ is isomorphic to a cone over a rational (resp. elliptic) curve of degree $-Z^2$.

THEOREM A.1.

Let X be rational or minimally elliptic, $\pi: \tilde{X} \rightarrow X$ the minimal good resolution, and Z the fundamental cycle.

- i) If X is rational, then $\dim T_X^1 \geq \dim H^1(\tilde{X}, \Theta_{\tilde{X}}) - Z^2 - 3$.
- ii) If X is minimally elliptic, of degree $d = -Z^2 \geq 5$, then $\dim T_X^1 \geq \dim R_Z + d$.

Proof

By construction T_X^1 is the tangent space of the base space V of the semiuniversal deformation of X at the special point 0. If t is a general point of $\Phi(R_Z) \subset V$, by standard semicontinuity $\dim \Theta_{V,t} \otimes \mathbb{C} \leq \dim \Theta_{V,0} \otimes \mathbb{C}$. Openness of versality (cf. [Pourcin, 1974]) shows that locally around t the base space V is the product of the base space of the semiuniversal deformation of $(\mathcal{X}_t, x_t)$ and of a smooth factor, over which the deformation of $\mathcal{X}_t$ induced by $\mathcal{X} \rightarrow V$ is trivial.

In the rational case $\Phi(R_Z)$ induces trivial deformations of $\mathcal{X}_t$, since the exceptional curve of a cone over a rational curve only lifts to trivial deformations. By Theorem 2.12. of [Wahl, 1979], $\Phi(R_Z)$ has dimension $\dim H^1(\tilde{X}, \Theta_{\tilde{X}}) - \dim H^1(|Z|, \mathcal{O}_Z(Z))$, which is $\dim H^1(\tilde{X}, \Theta_{\tilde{X}}) + Z^2 - 1$, as a little calculation shows.

It is well known, that the base space of the semiuniversal deformation of the cone over a rational curve of degree d has embedding dimension $2d - 4$, $d \geq 3$, and $2d - 3$, $d = 2$. Hence

$$\dim T_X^1 \geq \dim(\Theta_{V,t} \otimes \mathbb{C}) \geq \dim H^1(\tilde{X}, \Theta_{\tilde{X}}) - Z^2 - 3$$

For X minimally elliptic, $d \geq 5$, we have a $(d + 1)$ dimensional space of infinitesimal deformations for the simply elliptic singularity of the same

degree. The space $\Phi(R_Z)$ on the other hand induces a nontrivial one parameter family in the versal deformation of the fibre singularity $(\mathcal{X}_t, x_t)$, Hence

$$\dim T_X^1 \geq (\dim \Phi(R_Z) - 1) + (d + 1).$$

In the minimally elliptic case R_Z contains exactly those deformations of the resolution $\tilde{X}$, which blow down to deformations of the singularity. The dimension of this deformation space is computed in [Wahl, 1979], Corollary 5.7., 5.8..

PROPOSITION A.2.

Let X be a minimally elliptic singularity, let $\pi: \tilde{X} \rightarrow X$ be the minimal resolution, $E = \bigcup_{i=1}^r E_i$ the exceptional divisor, and let $\text{Der}_E(\tilde{X})$ be the sheaf of vector fields, logarithmic along each E_i .

- i) If all irreducible components E_i are rational, and have normal crossings, then $\dim \Phi(R_Z) = \dim H^1(\tilde{X}, \text{Der}_E(\tilde{X})) + r + Z(Z - E)$.
- ii) If X is simply elliptic, $\dim \Phi(R_Z) = 1$.
- iii) Otherwise $\Phi(R_Z)$ has dimension $\dim H^1(\tilde{X}, \text{Der}_E \tilde{X}) + \dim H^0(E, T_E^1)$.

REMARK

In case iii) of A.2. the curve E has exactly one singular point, and the second summand measures the space of infinitesimal deformations of that plane curve singularity.

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