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## BASE CHANGE FOR UNIT ELEMENTS OF HECKE ALGEBRAS

Robert E. Kottwitz

One of the ingredients in the comparison of trace formulas involves matching the orbital integrals of spherical functions; this is what Langlands [L] refers to as the “fundamental lemma”. There is a special case of the fundamental lemma that has a simple local proof. Let  $G$  be a connected reductive group over a  $p$ -adic field  $F$  and assume that  $G$  is unramified (that is, quasi-split over  $F$  and split over an unramified extension of  $F$ ). Let  $E$  be a finite unramified extension of  $F$ , let  $\theta$  be a generator of  $\text{Gal}(E/F)$ , and let  $l = [E : F]$ .

Consider a hyperspecial point  $x_0$  in the building of  $G$  over  $F$ . We denote by  $K$  the stabilizer of  $x_0$  in  $G(F)$  and by  $\mathcal{H} = \mathcal{H}(G(F), K)$  the corresponding Hecke algebra. Of course  $x_0$  also gives rise to  $K_E \subset G(E)$  and  $\mathcal{H}_E = \mathcal{H}(G(E), K_E)$ . There is a canonical homomorphism  $b : \mathcal{H}_E \rightarrow \mathcal{H}$ , characterized by the following property:

$$\text{tr } \pi_\varphi(b(f)) = \text{tr } \pi_\psi(f)$$

for all  $f \in \mathcal{H}_E$  and all unramified admissible homomorphisms  $\varphi : W_F \rightarrow {}^l G$ . Here  $\psi$  denotes the restriction of  $\varphi$  to  $W_E$ , and  $\pi_\varphi$  (resp.  $\pi_\psi$ ) denotes the  $K$ -spherical (resp.  $K_E$ -spherical) representation of  $G(F)$  (resp.  $G(E)$ ) corresponding to  $\varphi$  (resp.  $\psi$ ).

The fundamental lemma for the homomorphism  $b : \mathcal{H}_E \rightarrow \mathcal{H}$  relates the stable orbital integrals of  $b(f)$  to the “stable” twisted orbital integrals of  $f$  for any  $f \in \mathcal{H}_E$ . The precise statement requires definitions for stable conjugacy, stable orbital integrals, the twisted analogues, and the norm mapping  $\mathcal{N}$ . All of these are easier to define if the derived group  $G_{\text{der}}$  is simply connected. To keep the exposition simple we will now assume that  $G_{\text{der}}$  is simply connected, and then in the last section of the paper we will sketch a proof of the general case.

There are two forms of the norm mapping. The first is the mapping  $N : G(E) \rightarrow G(E)$  defined by

$$N\delta = \delta\theta(\delta)\theta^2(\delta)\dots\theta^{l-1}(\delta).$$

The second is a mapping  $\mathcal{N}$  from  $G(E)$  to the set of stable conjugacy

classes in  $G(\underline{F})$ . Since  $G_{\text{der}}$  is simply connected, stable conjugacy is the same as  $G(\underline{F})$ -conjugacy, where  $\underline{F}$  is an algebraic closure of  $F$ . The conjugacy class of  $N\delta$  in  $G(\underline{F})$  is defined over  $F$  and therefore contains an element  $\gamma \in G(F)$  (since  $G$  is quasi-split,  $G_{\text{der}}$  is simply connected, and  $\text{char}(F) = 0$  [K2]). By definition,  $\mathcal{N}\delta$  is the stable conjugacy class of  $\gamma$ . The fiber of  $\mathcal{N}$  through  $\delta$  is the stable twisted conjugacy class of  $\delta$ . The construction of  $\mathcal{N}$  when  $G_{\text{der}}$  is not simply connected is given in [K2].

Let  $dg$  (resp.  $dg_E$ ) be the Haar measure on  $G(F)$  (resp.  $G(E)$ ) that gives  $K$  (resp.  $K_E$ ) measure 1. For  $\gamma \in G(F)$  and  $f \in C_c^\infty(G(F))$  we denote by  $O_\gamma(f)$  the orbital integral

$$\int_{G_\gamma(F) \backslash G(F)} f(g^{-1}\gamma g) dg/dt.$$

This requires a choice of Haar measure  $dt$  on  $G_\gamma(F)$ , but we leave the measure out of the notation.

Let  $I = \text{Res}_{E/F} G$ . Then the automorphism  $\theta$  of  $E/F$  induces an  $F$ -automorphism of  $I$ ; this automorphism agrees with  $\theta$  on  $I(F) = G(E)$ , and we will abuse notation slightly by using  $\theta$  to denote both of them. For  $\delta \in G(E)$  and  $f \in C_c^\infty(G(E))$  we have the twisted orbital integral  $O_{\delta\theta}(f)$ , given by

$$\int_{I_{\delta\theta}(F) \backslash G(E)} f(g^{-1}\delta\theta(g)) dg_E/du,$$

where  $I_{\delta\theta}$  denotes the fixed points of  $\text{Int}(\delta) \circ \theta$  on  $I$ . Of course  $I_{\delta\theta}(F)$  is simply the twisted centralizer of  $\delta$  in  $G(E)$ .

For semisimple  $\gamma \in G(F)$  we have the stable orbital integral  $SO_\gamma$ , given as a linear form on  $C_c^\infty(G(F))$  by

$$SO_\gamma = \sum_{\gamma'} e(G_{\gamma'}) O_{\gamma'},$$

where the sum is taken over a set of representatives  $\gamma'$  for the conjugacy classes within the stable conjugacy class of  $\gamma$ , and where  $e(G_{\gamma'}) = \pm 1$  is the sign [K3] attached to the connected reductive  $F$ -group  $G_{\gamma'}$ . The distribution  $SO_\gamma$  depends on a choice of Haar measure on  $G_\gamma(F)$ . This measure is then transported to the inner twists  $G_{\gamma'}$  and used to form  $O_{\gamma'}$ . One expects that  $SO_\gamma$  is a stable distribution for all semisimple  $\gamma$ . Of course this is true by definition if  $\gamma$  is regular semisimple.

For  $\delta \in G(E)$  such that  $N\delta$  is semisimple we have the “stable” twisted orbital integral

$$SO_{\delta\theta} = \sum_{\delta'} e(I_{\delta'\theta}) O_{\delta'\theta},$$

where the sum is taken over a set of representatives  $\delta'$  for the twisted conjugacy classes within the stable twisted conjugacy class of  $\delta$ . In the same way as for  $SO_\gamma$  we use compatible measures on the groups  $I_{\delta'\theta}(F)$ .

Let  $f_E \in C_c^\infty(G(E))$  and  $f \in C_c^\infty(G(F))$ . As usual we say that  $f_E, f$  have matching orbital integrals if for every semisimple  $\gamma \in G(F)$  the stable orbital integral  $SO_\gamma(f)$  vanishes unless the stable conjugacy class of  $\gamma$  is equal to  $\mathcal{N}\delta$  for some  $\delta \in G(E)$ , in which case it is given by

$$SO_\gamma(f) = SO_{\delta\theta}(f_E).$$

Of course we are using compatible Haar measures on  $G_\gamma(F)$ ,  $I_{\delta\theta}(F)$  to form the orbital integrals; this has a meaning since  $I_{\delta\theta}$  is an inner twist of  $G_\gamma$  [K2, Lemma 5.8].

The (conjectural) fundamental lemma for  $b: \mathcal{H}_E \rightarrow \mathcal{H}$  asserts that  $f_E, b(f_E)$  have matching orbital integrals for all  $f_E \in \mathcal{H}_E$ . The main result of this paper is that  $f_E, b(f_E)$  have matching orbital integrals if  $f_E$  is the unit element of  $\mathcal{H}_E$ , namely, the characteristic function of  $K_E$  (recall that we normalized the measure on  $G(E)$  so that  $K_E$  has measure 1). In this case  $b(f_E)$  is the unit element of  $\mathcal{H}$ , namely, the characteristic function of  $K$ .

For  $G = GL_n$  this result is not new – it follows immediately from Lemma 8.8 of [K1]. In fact that lemma shows that some other pairs of functions have matching orbital integrals (characteristic functions of corresponding parahoric subgroups of  $GL_n(F)$ ,  $GL_n(E)$ , divided by the measures of the subgroups). Following a suggestion of J.-P. Labesse, this paper also proves a matching theorem for more general pairs of functions.

This more general matching theorem is the subject of §1. In §2 we make some remarks about twisted  $\kappa$ -orbital integrals of the functions considered in §1. In §3 we return to the unit elements of  $\mathcal{H}_E, \mathcal{H}$  and follow a suggestion of J. Arthur by proving a matching theorem for some weighted orbital integrals. In §4 we sketch what to do when  $G_{\text{der}}$  is not simply connected.

## 1. Main result

In this section our situation will be somewhat more general than in the introduction. Let  $F, E, \theta, l$  be as before. In particular we still insist that  $E/F$  be unramified. Let  $L$  denote the completion of the maximal unramified extension  $E^{\text{un}}$  of  $E$ . We have  $E^{\text{un}} = F^{\text{un}}$  and we denote by  $\sigma$  the Frobenius automorphism of  $L$  over  $F$ .

Let  $G$  be a connected reductive group over  $F$ , no longer assumed to be unramified. We do assume, however, that  $G_{\text{der}}$  is simply connected. As before we write  $I$  for  $\text{Res}_{E/F}G$  and  $\theta$  for the  $F$ -automorphism of  $I$

obtained from the field automorphism  $\theta$ . Let  $K_L$  be an open bounded subgroup of  $G(L)$  satisfying the following three conditions:

- (a)  $\sigma(K_L) = K_L$ .
- (b) The mapping  $k \mapsto k^{-1}\sigma(k)$  from  $K_L$  to  $K_L$  is surjective.
- (c) The mapping  $k \mapsto k^{-1}\sigma'(k)$  from  $K_L$  to  $K_L$  is surjective.

Let  $K$  (resp.  $K_E$ ) be  $G(F) \cap K_L$  (resp.  $G(E) \cap K_L$ ). Note that the situation in the introduction can be recovered by taking  $K_L$  to be the stabilizer in  $G(L)$  of the hyperspecial point  $x_0$ ; then (a) is obvious and (b), (c) follow from a result of Greenberg [G] since the special fiber of  $\mathbf{G}$  is connected, where  $\mathbf{G}$  is the extension of  $G$  to a group scheme over  $\mathfrak{o}$  determined by  $x_0$  ( $\mathfrak{o}$  denotes the valuation ring of  $F$ ).

Let  $X$ ,  $X_E$ ,  $X_L$  denote  $G(F)/K$ ,  $G(E)/K_E$ ,  $G(L)/K_L$  respectively. There are obvious inclusions  $X \subset X_E \subset X_L$  and  $\sigma$  acts on  $X_L$ . Condition (b) (resp. (c)) implies that the fixed point set of  $\sigma$  (resp.  $\sigma'$ ) on  $X_L$  is equal to  $X$  (resp.  $X_E$ ).

Choose Haar measures  $dg$ ,  $dg_E$  on  $G(F)$ ,  $G(E)$  such that  $K$ ,  $K_E$  have measure 1, and use these measures in forming orbital integrals. Let  $f$  (resp.  $f_E$ ) denote the characteristic function of  $K$  (resp.  $K_E$ ) in  $G(F)$  (resp.  $G(E)$ ).

The groups  $G(F)$ ,  $G(E)$ ,  $G(L)$  act on  $X$ ,  $X_E$ ,  $X_L$  respectively. Furthermore  $\theta$  acts on  $X_E$  (by some power of  $\sigma$ ). Let  $\delta \in G(E)$ . Then

$$O_{\delta\theta}(f_E) = \sum_g \text{meas}(I_{\delta\theta}(F) \backslash I_{\delta\theta}(F)gK_E),$$

where  $g$  runs over a set of representatives for the elements of

$$I_{\delta\theta}(F) \backslash G(E)/K_E$$

such that  $g^{-1}\delta\theta(g) \in K_E$ . Writing  $x$  for  $gK_E \in X_E$ , we have  $g^{-1}\delta\theta(g) \in K_E$  if and only if  $\delta\theta x = x$ . Let  $I_{\delta\theta}(F)_x$  denote the stabilizer of  $x$  in  $I_{\delta\theta}(F)$ . Then

$$\text{meas}(I_{\delta\theta}(F) \backslash I_{\delta\theta}(F)gK_E) = \text{meas}(I_{\delta\theta}(F)_x)^{-1}.$$

Let  $X_E^{\delta\theta}$  denote the set of fixed points of  $\delta\theta$  on  $X_E$  (the product of  $\delta$  and  $\theta$  is taken in the semidirect product of  $G(E)$  and  $\text{Gal}(E/F)$ ). Then we have shown that

$$O_{\delta\theta}(f_E) = \sum_x \text{meas}(I_{\delta\theta}(F)_x)^{-1},$$

where  $x$  runs through a set of representatives for the orbits of  $I_{\delta\theta}(F)$  on  $X_E^{\delta\theta}$ . Taking the special case  $E = F$ , we get a corollary that for  $\gamma \in G(F)$

$$O_\gamma(f) = \sum_x \text{meas}(G_\gamma(F)_x)^{-1},$$

where  $x$  runs through a set of representatives for the orbits of  $G_\gamma(F)$  on  $X^\gamma$ , the set of fixed points of  $\gamma$  on  $X$ .

Choose an integer  $j$  such that  $\theta$  is equal to the restriction of  $\sigma^j$  to  $E$ . Of course  $j$  is relatively prime to  $l$ , and hence we can choose integers  $a, b$  such that  $bl - aj = 1$ . We are going to define a correspondence between  $G(F)$  and  $G(E)$ . Let  $\gamma \in G(F)$  and  $\delta \in G(E)$ . We write  $\gamma \leftrightarrow \delta$  if there exists  $c \in G(L)$  such that the following two conditions hold:

$$(A) \quad c\gamma^a\sigma^lc^{-1} = \sigma^l,$$

$$(B) \quad c\gamma^b\sigma^jc^{-1} = \delta\sigma^j.$$

In (A) and (B) the equalities are of elements in the semidirect product of  $G(L)$  and the infinite cyclic group  $\langle \sigma \rangle$  generated by  $\sigma$ . Let  $\langle \gamma, \sigma \rangle$  be the subgroup generated by  $\gamma, \sigma$ . Then if  $\gamma, \delta, c$  satisfy (A) and (B), it follows that  $c\langle \gamma, \sigma \rangle c^{-1} = \langle \sigma^l, \delta\sigma^j \rangle$ , the point being that  $\gamma^a\sigma^l, \gamma^b\sigma^j$  generate the same subgroup as  $\gamma, \sigma$ . Let  $Y$  be any set on which the semidirect product acts. Then  $y \mapsto cy$  induces a bijection from the fixed points of  $\langle \gamma, \sigma \rangle$  on  $Y$  to the fixed points of  $\langle \sigma^l, \delta\sigma^j \rangle$  on  $Y$ . Taking  $Y = X_L$ , we see that  $x \mapsto cx$  induces a bijection from  $X^\gamma$  to  $X_E^{\delta\theta}$ . Taking  $Y = G(L)$  with  $G(L)$  acting by conjugation, we see that  $g \mapsto cgc^{-1}$  induces an isomorphism from  $G_\gamma(F)$  to  $I_{\delta\theta}(F)$ . It is then immediate from the expressions we obtained for  $O_{\delta\theta}(f_E)$  and  $O_\gamma(f)$  that

$$O_{\delta\theta}(f_E) = O_\gamma(f)$$

if the measures used on  $G_\gamma(F)$ ,  $I_{\delta\theta}(F)$  correspond under the isomorphism above.

What remains is to get a better understanding of the correspondence  $\gamma \leftrightarrow \delta$ . For which  $\gamma \in G(F)$  do there exist  $\delta \in G(E)$  such that  $\gamma \leftrightarrow \delta$ ? Conditions (A), (B) can be rewritten as

$$(A') \quad \gamma^a = c^{-1}\sigma^l(c),$$

$$(B') \quad \delta = c\gamma^b\sigma^j(c^{-1}).$$

If  $\delta$  exists, then (A') can be solved. Conversely, suppose that (A') can be solved. Then we can use (B') to define  $\delta \in G(L)$  such that  $\gamma, \delta, c$  satisfy (A), (B). But then  $c\langle \gamma, \sigma \rangle c^{-1} = \langle \sigma^l, \delta\sigma^j \rangle$ , which implies that  $\sigma^l, \delta\sigma^j$  commute, and this in turn implies that  $\delta \in G(E)$ . We conclude that  $\delta$  exists if and only if (A') can be solved. The element  $c \in G(L)$  appearing in (A') is clearly determined up to left multiplication by an element of  $G(E)$ . Making such a change in  $c$  replaces  $\delta$  by a  $\theta$ -conjugate under  $G(E)$ . Thus if  $\gamma \leftrightarrow \delta$ , then  $\gamma \leftrightarrow \delta'$  if and only if  $\delta, \delta'$  are  $\theta$ -conjugate under  $G(E)$ .

Next we consider  $\delta \in G(E)$  and ask whether there exists  $\gamma \in G(F)$  such that  $\gamma \leftrightarrow \delta$ . Inverting the matrix

$$\begin{bmatrix} a & l \\ b & j \end{bmatrix},$$

we see that (A), (B) are equivalent to

$$(C) \quad (\delta\sigma^j)^l \sigma^{-jl} = c\gamma c^{-1},$$

$$(D) \quad (\delta\sigma^j)^{-a} \sigma^{bj} = c\sigma c^{-1}$$

(of course we are using that  $\gamma, \sigma$  commute and that  $\sigma^l, \delta\sigma^j$  commute). We can rewrite (C), (D) as

$$(C') \quad N\delta = c\gamma c^{-1},$$

$$(D') \quad (\delta\sigma^j)^{-a} \sigma^{aj} = c\sigma(c^{-1}).$$

If  $\gamma$  exists, then (D') can be solved. Conversely, suppose that (D') can be solved. Then we can use (C') to define  $\gamma \in G(L)$  such that  $\gamma, \delta, c$  satisfy (C), (D). But (C) and (D) imply that  $\gamma, \sigma$  commute and hence that  $\gamma \in G(F)$ . We conclude that  $\gamma$  exists if and only if (D') can be solved. Furthermore (D') determines  $c$  up to right multiplication by  $G(F)$ , and changing  $c$  by an element of  $G(F)$  replaces  $\gamma$  by a conjugate under  $G(F)$ . Thus if  $\gamma \leftrightarrow \delta$ , then  $\gamma' \leftrightarrow \delta$  if and only if  $\gamma, \gamma'$  are conjugate in  $G(F)$ .

What we now know about the correspondence  $\gamma \leftrightarrow \delta$  can be summarized as follows. The correspondence sets up a bijection from the set of conjugacy classes in  $G(F)$  of elements  $\gamma \in G(F)$  such that (A') can be solved to the set of  $\theta$ -conjugacy classes in  $G(E)$  of elements  $\delta \in G(E)$  such that (D') can be solved. Furthermore (C') tells us that if  $\gamma \leftrightarrow \delta$ , then  $\mathcal{N}\delta = \gamma$ .

To complete the picture we need to know that there are enough corresponding elements of  $G(F), G(E)$ . First we show that if  $\gamma \in G(F)$  and  $X^\gamma$  is non-empty, then there exists  $\delta \in G(E)$  such that  $\gamma \leftrightarrow \delta$ . Indeed, replacing  $\gamma$  by a conjugate, we may assume that  $\gamma \in K$ . Then our assumption (c) on  $K_L$  implies that (A') can be solved.

Next we show that if  $\delta \in G(E)$  and  $X_E^{\delta\theta}$  is non-empty, then there exists  $\gamma \in G(F)$  such that  $\gamma \leftrightarrow \delta$ . Indeed, replacing  $\delta$  by a  $\theta$ -conjugate in  $G(E)$ , we may assume that  $\delta\theta$  fixes the base point of  $X_E = G(E)/K_E$ . Then  $(\delta\sigma^j)^{-a}$  and  $\sigma^{aj}$  both fix the base point of  $X_L$ , as does their product  $(\delta\sigma^j)^{-a} \sigma^{aj} \in G(L)$ . Therefore  $(\delta\sigma^j)^{-a} \sigma^{aj} \in K_L$  and assumption (b) on  $K_L$  implies that (D') can be solved.

There is one further remark that we need to make before stating the main result of the paper. Suppose that  $\gamma \leftrightarrow \delta$ . Choose  $c \in G(L)$  such

that  $\gamma, \delta, c$  satisfy (C), (D). We have already seen that  $g \mapsto cgc^{-1}$  induces an isomorphism from  $G_\gamma(F)$  to  $I_{\delta\theta}(F)$ . Since (C), (D) determine  $c$  up to right multiplication by an element of  $G_\gamma(F)$ , the isomorphism is canonical up to inner automorphisms of  $G_\gamma(F)$ .

**THEOREM:** *The correspondence  $\gamma \leftrightarrow \delta$  induces a bijection from the set of conjugacy classes of  $\gamma \in G(F)$  such that  $O_\gamma(f) \neq 0$  to the set of  $\theta$ -conjugacy classes of  $\delta \in G(E)$  such that  $O_{\delta\theta}(f_E) \neq 0$ . Moreover if  $\gamma \leftrightarrow \delta$ , then  $\gamma = \mathcal{N}\delta$ ,  $G_\gamma(F)$  is isomorphic to  $I_{\delta\theta}(F)$ , and  $O_\gamma(f) = O_{\delta\theta}(f_E)$ .*

Since  $O_\gamma(f) \neq 0$  (resp.  $O_{\delta\theta}(f_E) \neq 0$ ) if and only if  $X^\gamma$  (resp.  $X_E^{\delta\theta}$ ) is non-empty, the theorem follows from the remarks made above.

In order to use the theorem to prove that  $f, f_E$  have matching orbital integrals, there is a technical point to check. Suppose that  $\gamma, \delta, c$  satisfy (A), (B). Then Lemma 5.8 of [K2] gives us an inner twisting  $\beta: I_{\delta\theta} \rightarrow G_\gamma$ , canonical up to inner automorphisms of  $G_\gamma(\bar{F})$ . Assume now that  $\gamma$  is semisimple. We want to check that there exists an  $F$ -isomorphism  $\alpha: I_{\delta\theta} \xrightarrow{\sim} G_\gamma$  whose restriction to  $I_{\delta\theta}(F)$  is given by  $g \mapsto c^{-1}gc$  and which differs from  $\beta$  by an inner automorphism of  $G_\gamma(\bar{L})$ . This will show that if we use  $g \mapsto c^{-1}gc$  to transport a Haar measure on  $I_{\delta\theta}(F)$  over to  $G_\gamma(F)$ , the two measures will be compatible in the sense that arises in the definition of matching orbital integrals. It will also show that the signs  $e(G_\gamma)$  and  $e(I_{\delta\theta})$  are equal. We see from [K2] that if  $d \in G(\bar{F})$  and  $N\delta = d\gamma d^{-1}$ , then we can take  $\beta$  to be  $\text{Int}(d)^{-1} \circ p$ , where  $p: I_{\delta\theta} \rightarrow G_{N\delta}$  (over  $E$ ) is the restriction to  $I_{\delta\theta}$  of the projection of  $I_E = G_E \times \cdots \times G_E$  onto the factor indexed by the identity element of  $\text{Gal}(E/F)$  (the  $l$  factors are indexed by the elements of  $\text{Gal}(E/F)$ ). Let  $\alpha = \text{Int}(c)^{-1} \circ p$ . Then  $\alpha, \beta$  differ by an inner automorphism of  $G_\gamma(\bar{L})$  (use (C') to see this), and what remains is to show that  $\alpha$  is defined over  $F$ . It is obvious that  $\alpha$  is defined over  $L$ . Since the functor  $A \mapsto \text{Isom}_A(I_{\delta\theta}, G_\gamma)$  from ( $F$ -algebras) to (sets) is representable by a scheme over  $F$  (here we use that  $\gamma$  is semisimple and that  $G_{\text{der}}$  is simply connected in order to conclude that the groups  $G_\gamma, I_{\delta\theta}$  are connected and reductive), it is enough to show that  $\alpha$  commutes with  $\sigma$ . We will do this by showing that  $\alpha$  commutes with  $\sigma^j$  and  $\sigma^l$ ; this is enough since  $j, l$  are relatively prime. Direct calculation shows that

$$\sigma^j(\alpha) = \text{Int}(\sigma^j(c^{-1})\delta^{-1}c) \circ \alpha,$$

$$\sigma^l(\alpha) = \text{Int}(\sigma^l(c^{-1})c) \circ \alpha,$$

and (A'), (B') imply that

$$\sigma^j(c^{-1})\delta^{-1}c = \gamma^{-b},$$

$$\sigma^l(c^{-1})c = \gamma^{-a}.$$

Since  $\gamma$  is central in  $G_\gamma$ , this proves that  $\sigma^j(\alpha) = \sigma^l(\alpha) = \alpha$ .

**COROLLARY:** *The functions  $f, f_E$  have matching orbital integrals.*

This follows immediately from the theorem and the technical point that we just checked. However, we need to say a few more words about the corollary. If  $G$  is not quasi-split, the most natural notion of matching orbital integrals would involve “stable” twisted orbital integrals on  $G(E)$  and stable orbital integrals on a quasi-split inner form of  $G$ . In fact, if  $G$  is not quasi-split, the stable conjugacy class of  $N\delta$  need not contain any  $F$ -rational elements, hence the stable norm  $\mathcal{N}\delta$  does not always exist in  $G(F)$ . Nevertheless the corollary is true and even has the following supplement: if  $\mathcal{N}\delta$  does not exist in  $G(F)$ , then  $SO_{\delta\theta}(f_E) = 0$ . To prove the supplement, note that if  $SO_{\delta\theta}(f_E) \neq 0$ , then there exists a stable  $\theta$ -conjugate  $\delta'$  of  $\delta$  such that  $O_{\delta'\theta}(f_E) \neq 0$ ; therefore there exists  $\gamma \in G(F)$  such that  $\gamma \leftrightarrow \delta'$ , and then it follows that  $\gamma = \mathcal{N}\delta' = \mathcal{N}\delta$ .

## 2. $\kappa$ -orbital integrals and the dependence of $\gamma \leftrightarrow \delta$ on $j, a, b$

We keep the notation and assumptions of §1. We have not yet used the full strength of the theorem in §1, which proved a matching result for orbital integrals, not just stable orbital integrals. Consider an element  $\delta \in G(E)$  such that  $N\delta$  is regular and semisimple. Then  $I_{\delta\theta}$  is a torus. For any stable  $\theta$ -conjugate  $\delta' \in G(E)$  of  $\delta$  there is an invariant

$$\text{inv}(\delta, \delta') \in H^1(F, I_{\delta\theta})$$

measuring the difference between  $\delta, \delta'$ . This invariant sets up a bijection from the set of  $\theta$ -conjugacy classes in the stable  $\theta$ -conjugacy class of  $\delta$  to the set

$$\ker[H^1(F, I_{\delta\theta}) \rightarrow H^1(F, I)].$$

As usual we can define twisted  $\kappa$ -orbital integrals  $O_{\delta\theta}^\kappa$  for any character  $\kappa$  on the group  $H^1(F, I_{\delta\theta})$  by putting

$$O_{\delta\theta}^\kappa = \sum_{\delta'} \langle \text{inv}(\delta, \delta'), \kappa \rangle O_{\delta'\theta},$$

where  $\delta'$  runs over a set of representatives for the  $\theta$ -conjugacy classes in the stable  $\theta$ -conjugacy class of  $\delta$ . Suppose that  $O_{\delta'\theta}(f_E) \neq 0$  for some stable  $\theta$ -conjugate  $\delta'$  of  $\delta$ . It does no harm to replace  $\delta$  by  $\delta'$ , and so we may as well assume that  $O_{\delta\theta}(f_E) \neq 0$ . Then there exists  $\gamma \in G(F)$  such that  $\gamma \leftrightarrow \delta$ . Of course  $\gamma$  is regular and semisimple, and  $G_\gamma$  is a torus  $T$ .

Lemma 5.8 of [K2] gives us a canonical isomorphism  $T \xrightarrow{\sim} I_{\delta\theta}$ , allowing us to view  $\kappa$  as a character on  $H^1(F, T)$  and to form  $\kappa$ -orbital integrals

$$O_{\gamma}^{\kappa} = \sum_{\gamma'} \langle \text{inv}(\gamma, \gamma'), \kappa \rangle O_{\gamma'},$$

where  $\gamma'$  runs over a set of representatives for the conjugacy classes in the stable conjugacy class of  $\gamma$ .

PROPOSITION 1:  $O_{\delta\theta}^{\kappa}(f_E) = O_{\gamma}^{\kappa}(f)$ .

Of course the significance of the proposition is that whenever one is able to express the  $\kappa$ -orbital integrals of  $f$  in terms of stable orbital integrals of a function on an endoscopic group  $H$  of  $G$ , the proposition will then express  $O_{\delta\theta}^{\kappa}(f_E)$  in terms of stable orbital integrals on  $H$ , which may also be regarded as an endoscopic group for the pair  $(I, \theta)$  [S].

To prove the proposition it is enough to show that if  $\gamma'$  is stably conjugate to  $\gamma$ , if  $\delta'$  is stably  $\theta$ -conjugate to  $\delta$ , and if  $\gamma' \leftrightarrow \delta'$ , then  $\text{inv}(\gamma, \gamma') = \text{inv}(\delta, \delta')$ . This is sufficient since the elements  $\gamma', \delta'$  that do not take part in the correspondence contribute zero to  $O_{\gamma}^{\kappa}(f)$ ,  $O_{\delta\theta}^{\kappa}(f_E)$ . In order to prove that  $\text{inv}(\gamma, \gamma') = \text{inv}(\delta, \delta')$  it is convenient to use the injection

$$H^1(F, T) \rightarrow B(T)$$

defined in [K4, §1], where  $B(T)$  denotes  $H^1(\langle \sigma \rangle, T(L))$ . Choose  $c, c' \in G(L)$  such that  $\gamma, \delta, c$  and  $\gamma', \delta', c'$  satisfy (A), (B). Since  $H^1(L, T)$  is trivial, we can also choose  $g \in G(L)$  such that  $\gamma' = g\gamma g^{-1}$ . The image of  $\text{inv}(\gamma, \gamma')$  in  $B(T)$  is represented by the 1-cocycle

$$\sigma^k \mapsto g^{-1} \sigma^k(g)$$

of  $\langle \sigma \rangle$  in  $T(L)$ .

As in §1 we write  $p: I_{\delta\theta} \rightarrow G_{N\delta}$  (over  $E$ ) for the restriction to  $I_{\delta\theta}$  of the projection of  $I_E = G_E \times \cdots \times G_E$  on the factor indexed by the identity element of  $\text{Gal}(E/F)$ . The canonical isomorphism from  $I_{\delta\theta}$  to  $T$  is given by  $\text{Int}(c)^{-1} \circ p$ . It is easy to see that there exists a unique element  $h \in I(L)$  such that

- (a) the image of  $h$  under the projection of  $I(L) = G(L) \times \cdots \times G(L)$  onto the factor indexed by the identity element of  $\text{Gal}(E/F)$  is equal to  $dgc^{-1}$  (note that  $dgc^{-1}$  conjugates  $N\delta$  into  $N\delta'$ ),
- (b)  $\delta' = h\delta\theta(h)^{-1}$ .

The image of  $\text{inv}(\delta, \delta')$  in  $B(T)$  is represented by the 1-cocycle

$$\sigma^k \mapsto (\text{Int}(c)^{-1} \circ p)(h^{-1} \sigma^k(h))$$

of  $\langle \sigma \rangle$  in  $T(L)$ .

We will now show that with the choices we have made the two 1-cocycles of  $\langle \sigma \rangle$  in  $T(L)$  are equal (not just cohomologous). Since  $j, l$  are relatively prime, it is enough to show that

$$(\text{Int}(c)^{-1} \circ p)(h^{-1}\sigma^k(h)) = g^{-1}\sigma^k(g)$$

for  $k = j, l$ . First we take  $k = j$ . The equality  $\delta' = h\delta\theta(h)^{-1}$  implies that  $p(h^{-1}\sigma^j(h))$  is equal to

$$(dgc^{-1})^{-1} \cdot \delta' \cdot \sigma^j(dgc^{-1}) \cdot \delta^{-1}.$$

Therefore  $(\text{Int}(c)^{-1} \circ p)(h^{-1}\sigma^j(h))$  is equal to

$$g^{-1}d^{-1}\delta'\sigma^jdgc^{-1}\sigma^{-j}\delta^{-1}c.$$

Using (B) for  $\delta$  and  $\delta'$ , we can simplify this expression, obtaining

$$g^{-1}(\gamma')^b\sigma^j(g)\gamma^{-b}.$$

Using  $\gamma' = g\gamma g^{-1}$ , we can simplify it further, obtaining  $g^{-1}\sigma^j(g)$ .

Next we take  $k = l$ . Then  $(\text{Int}(c)^{-1} \circ p)(h^{-1}\sigma^l(h))$  is equal to

$$c^{-1} \cdot (dgc^{-1})^{-1} \cdot \sigma^l(dgc^{-1}) \cdot c.$$

Using (A') for  $c$  and  $d$  we can simplify this expression, obtaining

$$g^{-1}(\gamma')^a\sigma^l(g)\gamma^{-a}.$$

Using  $\gamma' = g\gamma g^{-1}$ , we can simplify it further, obtaining  $g^{-1}\sigma^l(g)$ . This completes the proof of the proposition.

In order to define the correspondence  $\gamma \leftrightarrow \delta$  we had to choose integers  $j, a, b$  such that the restriction of  $\sigma^j$  to  $E$  was  $\theta$  and such that  $bl - aj = 1$ . This raises an obvious question: How does the correspondence depend on the choice of  $j, a, b$ ? It turns out that the correspondence is independent of  $j, b$ , but is dependent on  $a$ . To see how the correspondence changes when  $j, a, b$  are replaced by  $j', a', b'$ , we suppose that we have  $\gamma, \gamma' \in G(F)$ ,  $\delta \in G(E)$ ,  $c, c' \in G(L)$  such that  $\gamma, \delta, c$  satisfy (A), (B) for  $j, a, b$  and  $\gamma', \delta, c'$  satisfy (A), (B) for  $j', a', b'$ . Then  $\gamma, \gamma'$  are stably conjugate and we can measure the difference between the two correspondences by calculating  $\text{inv}(\gamma, \gamma') \in H^1(F, G_\gamma)$ . At this point we assume that  $\gamma$  is semisimple, so that  $G_\gamma$  is connected and we can embed  $H^1(F, G_\gamma)$  in  $B(G_\gamma)$ . The set  $B(G_\gamma)$  can be identified with the set of  $\sigma$ -conjugacy classes in  $G_\gamma(L)$ .

**PROPOSITION 2:** *The image of  $\text{inv}(\gamma, \gamma')$  in  $B(G_\gamma)$  is equal to the*

$\sigma$ -conjugacy class of  $\gamma^{-n}$  in  $G_\gamma(L)$ , where  $n$  is defined by the equality  $a' = a + nl$ .

We also write  $j' = j + ml$ ; then  $b' = b + nj + ma + mnl$ . We have  $c\gamma c^{-1} = N\delta = c'\gamma'(c')^{-1}$ , and hence  $\gamma' = g\gamma g^{-1}$ , where  $g = (c')^{-1}c$ . Therefore the image of  $\text{inv}(\gamma, \gamma')$  in  $B(G_\gamma)$  is equal to the  $\sigma$ -conjugacy class of  $x$ , where  $x = g^{-1}\sigma(g)$ .

We will now show that  $x = \gamma^{-n}$ . We have

$$x = c^{-1}c' \cdot \sigma \cdot (c')^{-1}c \cdot \sigma^{-1},$$

and using (D) for  $c'$  and then replacing  $j'$  by  $j + ml$ , we find that

$$x = c^{-1}(\delta\sigma^j)^{-a'}\sigma^{l(b+nj)}c\sigma^{-1}.$$

Finally, replacing  $a'$  by  $a + nl$  and then using (C) and (D) for  $c$ , we find that  $x = \gamma^{-n}$ . In carrying out these steps we must remember that  $\sigma'$  commutes with  $\delta$ . This finishes the proof of the proposition.

### 3. Weighted orbital integrals

We return to the situation in the introduction, so that  $G$  is again unramified. The hyperspecial point  $x_0$  determines an extension of  $G$  to a connected reductive group over the valuation ring  $\mathfrak{o}$  of  $F$ , and we have  $K_L = G(\mathfrak{o}_L)$ . Let  $M$  be a Levi subgroup of  $G$  over  $\mathfrak{o}$ . We write  $\mathfrak{a}_M$  for the real vector space

$$\text{Hom}_{\mathbf{Z}}(\text{Hom}_F(M, \mathbf{G}_m), \mathbb{R})$$

and define a homomorphism

$$H_M: M(L) \rightarrow \mathfrak{a}_M$$

by requiring that for  $x \in M(L)$

$$\exp\langle H_M(x), \lambda \rangle = |\lambda(x)|$$

for all  $\lambda \in \text{Hom}_F(M, \mathbf{G}_m)$ . Here we have extended the normalized absolute value on  $F^\times$  to an absolute value on  $L^\times$ . Let  $P$  be a parabolic subgroup of  $G$  having  $M$  as Levi component and write  $N$  for the unipotent radical of  $P$ . We define a function

$$H_P: G(L) \rightarrow \mathfrak{a}_M$$

by putting  $H_P(g) = H_M(m)$ , where  $g$  has been written as  $mnk$  for

$m \in M(L)$ ,  $n \in N(L)$ ,  $k \in K_L$ . For  $g \in G(L)$  and  $\lambda \in \text{Hom}_{\mathbf{R}}(\mathfrak{a}_M, \mathbb{C})$  we set

$$v_p(\lambda, g) = e^{-\lambda(HP(g))}.$$

For fixed  $g$  the functions  $\lambda \mapsto v_p(\lambda, g)$  form a  $(G, M)$  family [A], and this  $(G, M)$  family determines a number  $v_M(g)$ . In this way we have constructed a weight function  $v_M$  on  $G(L)$ ; it is left invariant under  $M(L)$  and right invariant under  $K_L$ .

It is obvious that the restriction of  $v_M$  to  $G(F)$  is the weight function on  $G(F)$  that Arthur uses to define weighted orbital integrals. Let  $\gamma$  be a regular semisimple element of  $M(F)$ . The weighted orbital integral that we are referring to is

$$WO_{\gamma}(\phi) = \int_{G_{\gamma}(F) \backslash G(F)} \phi(g^{-1}\gamma g) v_M(g) dg/dt$$

for  $\phi \in C_c^{\infty}(G(F))$ .

After working through Arthur's definition of twisted weighted orbital integrals, one finds that the necessary weight function on  $G(E)$  is none other than the restriction of  $v_M$  to  $G(E)$  (up to a scalar which will be 1 in a suitable normalization). Let  $\delta \in M(E)$  and assume that  $N\delta$  is regular and semisimple. Then the twisted weighted orbital integral that we are referring to is

$$WO_{\delta\theta}(\phi) = \int_{I_{\delta\theta}(F) \backslash G(E)} \phi(g^{-1}\delta\theta(g)) v_M(g) dg_E/du$$

for  $\phi \in C_c^{\infty}(G(E))$ .

As before we let  $f, f_E$  denote the characteristic functions of  $K, K_E$ . Suppose that our elements  $\gamma \in M(F)$  and  $\delta \in M(E)$  are related by the correspondence  $\gamma \leftrightarrow \delta$  for the group  $M$ , so that there exists  $c \in M(L)$  such that  $\gamma, \delta, c$  satisfy (A) and (B).

**PROPOSITION:**  $WO_{\delta\theta}(f_E) = WO_{\gamma}(f)$ .

The proof is a slight variant of the proof that  $O_{\delta\theta}(f_E) = O_{\gamma}(f)$ . Since  $v_M$  is right invariant under  $K_L$  it descends to a function  $w_M$  on  $X_L = G(L)/K_L$ . We have

$$WO_{\gamma}(f) = \sum_x \text{meas}(G_{\gamma}(F)_x)^{-1} w_M(x),$$

where  $x$  runs through a set of representatives for the orbits of  $G_{\gamma}(F)$  on  $X^{\gamma}$ . There is a similar formula for  $WO_{\delta\theta}(f_E)$ . The bijection  $x \mapsto cx$  from

$X^\gamma$  to  $X_E^{\delta\theta}$  matches up the terms in the two formulas, and to finish the proof of the proposition we have only to note that the left invariance of  $v_M$  under  $M(L)$  implies that  $w_M(x) = w_M(cx)$ .

Before finishing this section we should observe that enough  $\gamma, \delta$  are related by the correspondence  $\gamma \leftrightarrow \delta$  for  $M$ . Suppose that  $\gamma$  is a regular semisimple element of  $M(F)$  such that  $WO_\gamma(f) \neq 0$ . Then there exists  $g \in G(F)$  such that  $g^{-1}\gamma g \in K$ . Choose a parabolic subgroup  $P$  of  $G$  with Levi component  $M$  and unipotent radical  $N$ . Writing  $g = mnk$  with  $m \in M(F)$ ,  $n \in N(F)$ ,  $k \in K$  and using that  $P(o) = M(o)N(o)$ , we see that  $m^{-1}\gamma m \in M(o)$ . The discussion in §1 then shows that there exists  $\delta \in M(E)$  such that  $\gamma \leftrightarrow \delta$  in the group  $M$ . Similarly, if  $WO_{\delta\theta}(f_E) \neq 0$ , then there exists  $\gamma \in M(F)$  such that  $\gamma \leftrightarrow \delta$  in the group  $M$ .

#### 4. Groups $G$ for which $G_{\text{der}}$ is not simply connected

In proving our special case of the fundamental lemma we assumed that  $G_{\text{der}}$  was simply connected. We will now show that this assumption can be dropped. Choose a finite unramified extension  $F'$  of  $F$  that splits  $G$  and contains  $E$ ; then there exists an extension  $H$  of  $G$  by a central torus  $Z$  such that

- (a)  $H_{\text{der}}$  is simply connected,
- (b)  $Z$  is a product of copies of  $\text{Res}_{F'/F}\mathbb{G}_m$ .

In the terminology of [K2, §5]  $H$  is an unramified  $z$ -extension of  $G$  adapted to  $E$ . Note that  $H(F)$  maps onto  $G(F)$ .

It is not hard to see that the fundamental lemma for  $G, E, \theta$  follows from the fundamental lemma for  $H, E, \theta$ . The point is that there is a surjective homomorphism from the Hecke algebra of  $H$  (for the hyperspecial maximal compact subgroup of  $H(F)$  corresponding to  $K$ ) to the Hecke algebra of  $G$ , obtained by mapping  $f_H$  to  $f_G$ , where

$$f_G(x) = \int_{Z(F)} f_H(x_0 z) \, dz.$$

Here  $x_0$  is an element of  $H(F)$  that maps to  $x$  and  $dz$  is the Haar measure on  $Z(F)$  that gives measure 1 to the maximal compact subgroup of  $Z(F)$ . The mapping  $f_H \mapsto f_G$  gives us (by means of the Satake isomorphism) a mapping

$$\mathbb{C}[X_*(S_H)]^{\Omega(F)} \rightarrow \mathbb{C}[X_*(S_G)]^{\Omega(F)},$$

where  $S_G$  is a maximal  $F$ -split torus of  $G$ ,  $S_H$  is the corresponding maximal  $F$ -split torus of  $H$ , and  $\Omega(F)$  is the relative Weyl group of  $S_G$  in  $G$ . The mapping is simply the homomorphism induced by  $X_*(S_H) \rightarrow X_*(S_G)$ , which is surjective since  $H^1(F, X_*(Z))$  is trivial. From this it is

also clear that  $f_H \mapsto f_G$  is compatible with the base change homomorphisms  $b$  for  $H$  and  $G$ . Furthermore, the orbital integrals of  $f_G$  can be obtained from the orbital integrals of  $f_H$  by integrating over  $Z(F)$ . There is an analogous statement for twisted orbital integrals, in which the integration is over  $Z(E)/(\theta - id)Z(E)$ . Finally, the assumption that  $F'$  contains  $E$  implies that the norm map induces an isomorphism

$$Z(E)/(\theta - id)Z(E) \xrightarrow{\sim} Z(F).$$

Putting all this together, one can now check that the fundamental lemma for  $H$  implies the fundamental lemma for  $G$ .

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