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ON THE NÉRON MODEL OF JACOBIANS OF SHIMURA CURVES

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Let \( \mathcal{B} \) be an indefinite rational quaternion algebra of discriminant \( \text{Disc} \mathcal{B} > 1 \) and denote by \( V_{\mathcal{B}} = V_{\mathcal{B}}/\mathbb{Q} \) the corresponding Shimura curve. \( V_{\mathcal{B}} \) has bad reduction exactly at the primes \( p \) dividing \( \text{Disc} \mathcal{B} \); fix such a prime \( p \). Let \( \mathcal{J}/\mathbb{Z}_p \) be the Néron model of the jacobian of \( V_{\mathcal{B}} \times _{\mathbb{Q}} \mathbb{Q}_p \). Denote by \( \mathcal{J}_p^0 \) the connected component of the special fiber \( \mathcal{J}_p = \mathcal{J} \times _{\mathbb{Z}_p} \mathbb{F}_p \) and by \( \Phi = \mathcal{J}_p/\mathcal{J}_p^0 \) its group of connected components.

The following problems are relevant to many arithmetic questions concerning \( V_{\mathcal{B}} \):

1. Determine the structure of \( \mathcal{J}_p^0/\mathbb{F}_p \).
2. Determine the group of connected components \( \Phi \).

It is the purpose of this paper to solve these problems.

To describe the answer we obtain, let \( \mathcal{B} \) be the rational definite quaternion algebra of discriminant \( \frac{1}{\text{Disc} \mathcal{B}} \). Denote by \( m(\mathcal{B}) \) the mass \( \frac{1}{2} \sum _{q \mid \text{Disc} \mathcal{B}} (q - 1) \) of \( \mathcal{B} \). Let \( B = B(p) \) be the Brandt matrix of degree \( p \) for \( \mathcal{B} \) relative to a fixed ordering of the ideal classes of \( \mathcal{B} \). \( B \) is an integral \( h \times h \) matrix for which \( p + 1 \) is an eigenvalue, where \( h \) is the class number of \( \mathcal{B} \). Hence we can write the characteristic polynomial \( P_B(x) \) of \( B \) as

\[
P_B(x) = (x - p - 1) \prod _{i=2}^{h} (x - \lambda_i).
\]

In response to Problem 2 we establish the

**Theorem (2.3):**

Let

\[
e_2 = \prod _{q \mid \text{Disc} \mathcal{B}} \left(1 - \left(\frac{-4}{q}\right)\right), \quad e_3 = \prod _{q \mid \text{Disc} \mathcal{B}} \left(1 - \left(\frac{-3}{q}\right)\right).
\]

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Then

\[ |\Phi| = \frac{p + 1}{m(\hat{\mathcal{B}}) c(\hat{\mathcal{B}}) 2^2 3^3} \left| \prod_{i=2}^{h} (\lambda_i - (p + 1))(\lambda_i + (p + 1)) \right|, \]

where \( c(\hat{\mathcal{B}}) = 8 \) if \( \text{Disc} \hat{\mathcal{B}} = 2 \), \( c(\hat{\mathcal{B}}) = 3 \) if \( \text{Disc} \hat{\mathcal{B}} = 3 \), and \( c(\hat{\mathcal{B}}) = 1 \) otherwise.

In fact, we explain how to describe \( \Phi \) in terms of the Brandt matrix \( B \). In Theorem 3.1 we describe the connected component \( \mathcal{J}_p^0 \).

By the results of Raynaud [8] and Deligne-Rapoport [1], questions 1 and 2 are reduced to computations in linear algebra if one has a description of a regular model of \( V_{\mathfrak{p}} \). In our case, Drinfeld [2] has constructed a scheme \( M_{\mathfrak{p}}/\mathbb{Z} \) whose fiber over \( \mathbb{Q} \) is the Shimura curve \( V_{\mathfrak{p}} \). Moreover he has given a description of \( M_{\mathfrak{p}} \times \mathbb{Z}_p \) in terms of Mumford uniformization. By resolving singularities one obtains a regular scheme \( M_{\mathfrak{p}} \times \mathbb{Z}_p \) over \( \mathbb{Z}_p \). In Section 1 we give the intersection matrix of the special fiber \( (M_{\mathfrak{p}} \times \mathbb{Z}_p)_0 \) in terms of the Brandt matrix \( B \). Then in Sections 2 and 3 we carry out the computations necessary to answer our questions. The case where the interchanged algebra \( \hat{\mathcal{B}} \) has discriminant 2 was treated by Ogg in [7].

The theorems we obtain are analogs of the results of Mazur and Rapoport [6] on elliptic modular Jacobians. The arithmetic significance of Theorem 2.3, however, seems more involved. Suppose for simplicity that \( \text{Disc} \mathfrak{p} = pq \) with \( q \) prime. Then \( P_B(x) \) is the characteristic polynomial of the Hecke operator \( T(p) \) acting on the space \( M_2(\Gamma_0(q)) \) of modular forms of weight 2 for \( \Gamma_0(q) \). What is remarkable is that the primes dividing \( |\Phi| \) are essentially the primes of congruence between modular forms in \( M_2(\Gamma_0(q)) \) and newforms of weight 2 for \( \Gamma_0(pq) \), cf. Ribet [9]. Hence \( \Phi \) apparently detects fusion between newforms and old forms.

§1. The intersection matrix

We first recall the description of the special fiber \( M_{\mathfrak{p}} \times \mathbb{F}_p \) provided by Drinfeld [2]. For details see [4] and Kurihara [5]. Fix a maximal order \( \hat{\mathcal{N}} \subset \hat{\mathcal{B}} \) and set

\[
\Gamma_0 = \left( \hat{\mathcal{N}} \otimes \mathbb{Z} \left[ \frac{1}{p} \right] \right)^\times / \mathbb{Z} \left[ \frac{1}{p} \right]^\times
\]

\[
\Gamma_+ = \left\{ x \in \left( \hat{\mathcal{N}} \otimes \mathbb{Z} \left[ \frac{1}{p} \right] \right)^\times : |\text{Norm}(x) \in p^{2\mathbb{Z}}| / \mathbb{Z} \left[ \frac{1}{p} \right]^\times \right\},
\]
where \( \text{Norm} : \mathcal{O} \rightarrow \mathbb{Q} \) is the reduced norm. Identify \( \mathcal{O} \otimes \mathbb{Q}_p \) with the algebra of \( 2 \times 2 \) matrices over \( \mathbb{Q}_p \). Then \( \Gamma_0 \) and \( \Gamma_+ \) are discrete cocompact subgroups of \( \text{PGL}_2(\mathbb{Q}_p) \). Let \( \Delta \) be the Bruhat-Tits building of \( \text{SL}_2(\mathbb{Q}_p) \) with vertices \( \text{Ver} \Delta \) and edges \( \text{Ed} \Delta \). The groups \( \Gamma_0 \) and \( \Gamma_+ \) act on \( \Delta \) and the quotients are finite oriented graphs with lengths in the sense of Kurihara [5]. The vertices \( \text{Ver}(\Gamma_0 \setminus \Delta) \) correspond to the ideal classes of \( \mathcal{O} \) and we denote them by \( v_1, \ldots, v_h \) with the same ordering used to write \( B \). The weight \( f(v) \) of a vertex \( v \in \text{Ver}(\Gamma_0 \setminus \Delta) \) and the length \( \ell(y) \) of an edge \( y \in \text{Ed}(\Gamma_0 \setminus \Delta) \) are defined as the orders of their stabilizers in \( \Gamma_0 \). The integer \( \ell(y) \) is always 1, 2, or 3. Define \( h \times h \) matrices \( N^k = (N^k_{ij})_{1 \leq i, j \leq h} \) for \( 1 \leq k \leq 3 \) by

\[
N^k_{ij} = \text{number of } y \in \text{Ed}(\Gamma_0 \setminus \Delta) \text{ with } v_i = o(y), v_j = t(y)
\]

where \( o(y) \) is the initial vertex of \( y \) and \( t(y) \) the terminal vertex. Set \( F \) equal to the \( h \times h \) diagonal matrix with \( F_{ii} = f(v_i), 1 \leq i \leq h \). Then

\[
B = \left( N^1 + \frac{1}{2} N^2 + \frac{1}{3} N^3 \right) F; \tag{1.1}
\]

see Kurihara [5], (4-4). Let \( St v_i \) denote \( \{ y \in \text{Ed}(\Gamma_0 \setminus \Delta) \mid o(y) = v_i \} \). As

\[
\# \{ y \in \text{Ed} \Delta \mid o(y) = \bar{v} \} = p + 1 \text{ for any } \bar{v} \in \text{Ver} \Delta
\]

we have

\[
p + 1 = \sum_{y \in St v_i} \frac{f(v_i)}{f(y)} = f(v_i) \sum_{j=1}^{h} \left( N^1_{ij} + \frac{1}{2} N^2_{ij} + \frac{1}{3} N^3_{ij} \right). \tag{1.2}
\]

We can write \( \Gamma_0 = \Gamma_+ \sqcup \Gamma_+ \gamma_p \) where \( \gamma_p \in \mathcal{A} \) has norm \( p \). \( \gamma_p \) induces an involution \( w_p \) of \( \Gamma_+ \setminus \Delta \) which fixes no vertex and no (oriented) edge. In fact we may write \( \text{Ver}(\Gamma_+ \setminus \Delta) = \{ v_1, \ldots, v_h \} \) with \( 1 \leq i \leq h, 1 \leq \ell \leq 2 \), where \( v_{i1} \) and \( v_{i2} \) lie above \( v_i \in \text{Ver}(\Gamma_0 \setminus \Delta) \) and \( w_p v_{i\ell} = v_{i,3-\ell} \). Moreover, we may suppose that liftings \( \bar{v}_{i\ell}, \bar{v}_{jm} \in \text{Ver} \Delta \) of \( v_{i\ell}, v_{jm} \in \text{Ver}(\Gamma_+ \setminus \Delta) \) are at a distance congruent to \( \ell - m \) modulo 2. Hence no edge connects \( v_{i\ell} \) and \( v_{j\ell'} \) \( (\ell = 1, 2; 1 \leq i, j \leq h) \). By Drinfeld [2] \( \Gamma_+ \setminus \Delta \) is canonically identified with the dual graph \( G = G(M_{\mathfrak{a}} \times \mathbb{Z}_p^* / \mathbb{Z}_p^*) \) of the special fiber \( M_{\mathfrak{a}} \times \mathbb{F}_p \), and Frobenius acts on \( G \) as \( w_p \) (for this "Geometric Eichler-Shimura Relation" see also [4]). Let \( \tilde{G} \) be the dual graph of the special fiber of the resolution of singularities \( M_{\mathfrak{a}} \times \mathbb{Z}_p^* / \mathbb{Z}_p^* \) of \( M_{\mathfrak{a}} \times \mathbb{Z}_p / \mathbb{Z}_p \). For an edge \( y \in \text{Ed}(\Gamma_0 \setminus \Delta) \) let \( \hat{y} \) be the edge above it in \( G = \Gamma_+ \setminus \Delta \) such that \( o(\hat{y}) \in \{ v_1, \ldots, v_h \} \). Then \( \tilde{G} \) is obtained from \( G \) by replacing \( \hat{y} \) together with its opposite edge by a chain

\[
o(\hat{y}) - w_{y_1} - \cdots - w_{y,\ell(\hat{y})-1} - t(\hat{y})
\]
whenever \( \ell(y) \geq 2 \). Identify

\[
\{ v_{i\ell}, w_{ym} | 1 \leq i \leq h; \ell = 1, 2; y \in \text{Ed}(\Gamma_0 \setminus \Delta) \text{ satisfying } \ell(y) \geq 2 \}
\]

and \( 1 \leq m < \ell(y) \}

with \( \text{Ver} \tilde{G} \) by letting an element \( \alpha \) in the former set correspond to a component \([\alpha]\) of \((M_\text{gr} \times \mathbb{Z}_p)_{\ell_0}\) in \( \text{Ver} \tilde{G} \). The intersection matrix for \((M_\text{gr} \times \mathbb{Z}_p)_{\ell_0}, A = ([\alpha] \cdot [\beta])_{\alpha, \beta \in \text{Ver} \tilde{G}}, \) is readily obtained from \( G \):

(i) \([v_{11}] \cdot [v_{12}] = N_{11}^1 \text{ for } i \neq j.
\]

\([w_{y1}] \cdot [o(y)] = [w_{y2}] \cdot [t(y)] \text{ if } \ell(y) = 2.
\]

\([w_{y1}] \cdot [o(y)] = [w_{y1}] \cdot [w_{y2}] = [w_{y2}] \cdot [t(y)] = 1
\]

if \( \ell(y) = 3 \).

(ii) \( A \) is symmetric.

(iii) All off-diagonal entries of \( A \) not already determined by i) and ii) are 0.

(iv) The diagonal entries of \( A \) are determined so that any row (or column) sum is 0. Thus \([w_{ym}]^2 = -2 \) and

\[
[w_{ym}]^2 = - \sum_{k=1}^{3} \sum_{j=1}^{h} N_{ij}^k.
\]

\section{2. The group of connected components}

Let \( L \) be the free abelian group on the set \( \text{Ver} \tilde{G} \). Let \( L_0 = \left\{ \sum_{v \in \text{Ver} \tilde{G}} n_v v \in L | \sum n_v = 0 \right\} \). The intersection matrix \( A \) represents a transformation \( \mathcal{A} : L \to L \) relative to the standard basis. We have \( \mathcal{A} L \subset L_0 \) by [1.3 iv]. According to Raynaud [8], \( \Phi \approx L_0 / \mathcal{A} L \) canonically. Since \( L \cong L_0 \oplus \mathbb{Z} \) (noncanonically), \( L / \mathcal{A} L \cong \mathbb{Z} \oplus \Phi \). To describe \( \Phi \) we need some linear algebra preliminaries. For \( i \neq j \) let \( R_i \to R_i + aR_j \) (respectively \( C_i \to C_i + aC_j \)) denote the operation of adding a constant multiple \( a \) of the \( j \)th row (column) of a given matrix \( Z \) to the \( i \)th row (column). Let \( Z^{ij} \) denote the matrix obtained from \( Z \) by deleting the \( i \)th row and the \( j \)th column. If \( Z \) is a square matrix we denote its characteristic polynomial by \( P_Z \).
2.1. **LEMMA:** Suppose $X$ and $Y$ are $n \times n$ matrices. Then

(i) $\det \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} = \det(X - Y) \det(X + Y)$.

Suppose in addition that $X$ is symmetric with zero row sum and that $Y$ is diagonal. Then

(ii) $(-1)^{n-1} \det(X^{ij}) = \frac{(-1)^{i+j}}{n} P_X^{i}(0)$.

(iii) $(-1)^{n-1} P_{XY}^{i}(0) = \frac{1}{n} P_X^{i}(0) P_Y^{i}(0)$.

**PROOF:** Adding the first block row to the second transforms $\begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$ to $\begin{pmatrix} X + Y & Y \\ X & X + Y \end{pmatrix}$; subtracting then the second block column from the first gives $\begin{pmatrix} X - Y & Y \\ 0 & X + Y \end{pmatrix}$, proving (i). Now suppose $X$ is symmetric with zero row sum. For a fixed $i$ let $X_j$ denote the $j$th column of the $(n - 1) \times n$ matrix obtained by omitting the $i$th row of $X$. By assumption $\sum_j X_j = 0$ so that $\det(X^{ij}) = \det(X_1 \ldots \hat{X}_j \ldots X_n) = \det(-X_2 + \ldots + X_n)$, $X_2 \ldots \hat{X}_j \ldots X_n) = \det(-X_j)$, $X_2 \ldots \hat{X}_j \ldots X_n) = (-1)^{j+1} \det(X^{1j})$. Since $X$ is symmetric

$\det(X^{ij}) = (-1)^{i+j} \det(X^{11})$. However $(-1)^{n-1} P_X^{i}(0) = \sum_{\ell=1}^{n} \det(X^{\ell\ell}) = \sum_{\ell=1}^{n} \det(X^{\ell\ell}) = n \det(X^{11})$, so (ii) follows. Finally suppose in addition that $Y$ is diagonal. Note that $(XY)^{\ell\ell} = X^{\ell\ell}Y^{\ell\ell}$, so that

$$(-1)^{n-1} P_{XY}^{i}(0) = \sum_{\ell=1}^{n} \det((XY)^{\ell\ell}) = \det(X^{11}) \sum_{\ell=1}^{n} \det(Y^{\ell\ell})$$

$$= \frac{1}{n} P_X^{i}(0) P_Y^{i}(0),$$

proving (iii).

We can now calculate the order of $\Phi$. By the theory of elementary divisors $|\Phi| = \gcd_{\alpha, \beta}(\det(A^{a\beta}))$. By Lemma 2.1, $|\Phi| = |\det(A^{a\beta})|$ for any $\alpha$ and $\beta$, which we will choose equal and among the $v_\ell$. Row and column operations $R_\gamma \rightarrow R_\gamma + aR_\delta$, $C_\gamma \rightarrow C_\gamma + aC_\delta$ ($\gamma \neq \delta$) will not change $\det(A^{a\alpha})$ so long as $\delta \neq \alpha$. We will use these to simplify $A$.

**Step 1:** Suppose $\ell(y) = 2$ for $y \in \text{Ed}(\Gamma_0 \backslash \Delta)$. Set $\alpha_1 = o(y)$, $\alpha_2 = t(y)$,
\[ \alpha_3 = w_{y1}. \text{ Then } A_{\alpha_\alpha} \neq 0 \text{ only when } \alpha \in \{ \alpha_i \}_{i=1}^3. \text{ The } 3 \times 3 \text{ minor } M = (A_{\alpha_{\alpha}})_{1 \leq i,j \leq 3} \text{ has the form} \]

\[
M = \begin{pmatrix}
a & b & 1 \\
b & c & 1 \\
1 & 1 & -2
\end{pmatrix}.
\]

Applying to \( A \) the transformations \( R_{\alpha_i} \rightarrow R_{\alpha_i} + \frac{1}{2} R_{\alpha_j} \), \( R_{\alpha_2} \rightarrow R_{\alpha_2} + \frac{1}{2} R_{\alpha_3} \), and then the symmetric operations on columns transforms the minor \( M \) to

\[
M' = \begin{pmatrix}
a + \frac{1}{2} & b + \frac{1}{2} & 0 \\
b + \frac{1}{2} & c + \frac{1}{2} & 0 \\
0 & 0 & -2
\end{pmatrix},
\]

leaves \( A \) symmetric, and doesn’t change the other elements of \( A \).

Performing these operations for all \( y \in \text{Ed}(\Gamma_0 \setminus \Delta) \) with \( \varepsilon(y) = 2 \) will transform the subminor

\[
(A_{\alpha_i \alpha_j})_{1 \leq i,j \leq 2} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{where } \alpha_k = v_{i1}, \; \alpha_k = v_{j2}
\]

(or \( \alpha_k = v_{i2} \) and \( \alpha_k = v_{j1} \), \( 1 \leq i,j \leq h \), to

\[
\begin{pmatrix}
a + \frac{1}{2} \sum_{k=1}^{h} N_{ik}^2 & b + \frac{1}{2} N_{ij}^2 \\
b + \frac{1}{2} N_{ij}^2 & c + \frac{1}{2} \sum_{k=1}^{h} N_{kj}^2
\end{pmatrix}.
\]

**Step 2:** Now suppose \( \varepsilon(y) = 3 \) for \( y \in \text{Ed}(\Gamma_0 \setminus \Delta) \). Set \( \alpha_1 = o(y) \), \( \alpha_2 = t(y) \), \( \alpha_3 = w_{y1} \), \( \alpha_4 = w_{y2} \). The corresponding \( 4 \times 4 \) minor has the form

\[
M = \begin{pmatrix}
a & b & 1 & 0 \\
b & c & 0 & 1 \\
1 & 0 & -2 & 1 \\
0 & 1 & 1 & -2
\end{pmatrix}
\]

and \( A_{\alpha_{\alpha}} = A_{\alpha_{\alpha}} = 0 \) for \( \alpha \notin \{ \alpha_i \}_{i=1}^4 \). Applying \( R_{\alpha_3} \rightarrow R_{\alpha_2} + \frac{1}{2} R_{\alpha_4}, \cdot R_{\alpha_3} \rightarrow R_{\alpha_3} + \frac{1}{2} R_{\alpha_4} \) and then \( C_{\alpha_2} \rightarrow C_{\alpha_2} + \frac{1}{2} C_{\alpha_4}, \; C_{\alpha_3} \rightarrow C_{\alpha_3} + \frac{1}{2} C_{\alpha_4} \) transforms \( M \) to

\[
M' = \begin{pmatrix}
a & b & 1 & 0 \\
b & c + \frac{1}{2} & \frac{1}{2} & 0 \\
1 & \frac{1}{2} & -\frac{3}{2} & 0 \\
0 & 0 & 0 & -2
\end{pmatrix}.
\]
Applying next \( R_\alpha \rightarrow R_\alpha + \frac{3}{2} R_\alpha \), \( R_\alpha \rightarrow R_\alpha + \frac{1}{3} R_\alpha \), \( C_\alpha \rightarrow C_\alpha + \frac{2}{3} C_\alpha \), and \( C_\alpha \rightarrow C_\alpha + \frac{1}{3} C_\alpha \) gives

\[
\begin{pmatrix}
  a + \frac{2}{3} & b + \frac{1}{3} & 0 & 0 \\
  b + \frac{1}{3} & c + \frac{2}{3} & 0 & 0 \\
  0 & 0 & -3/2 & 0 \\
  0 & 0 & 0 & -2
\end{pmatrix}.
\]

Performing these operations for all \( y \in Ed(\Gamma_0 \setminus \Delta) \) with \( \ell(y) = 3 \) will transform the subminor

\[
(A_{\alpha_k, \alpha_j})_{1 \leq k, \ell \leq 2} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{where } \alpha_k = v_{im}, \quad \alpha_\ell = v_{j,3-m}
\]

for \( m = 1, 2; 1 \leq i, j \leq h \), to

\[
\begin{pmatrix}
 a + \frac{2}{3} \sum_{k=1}^{n} N_{ik}^3 & b + \frac{1}{3} N_{ij}^3 \\
 b + \frac{1}{3} N_{ij}^3 & c + \frac{2}{3} \sum_{k=1}^{h} N_{jk}^3
\end{pmatrix}.
\]

**Step 3:** Suppose that \( \text{Ver} \, \tilde{G} \) is ordered so that the first \( h \) rows (and columns) of \( A \) correspond to \( \{ v_{i1} \}_{i=1}^{h} \) (in order) and the next \( h \) rows and columns similarly correspond to \( \{ v_{i2} \}_{i=1}^{h} \). After Steps 1 and 2 \( A \) is transformed to a matrix with block form \( \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \), where \( U \) is a \( 2h \times 2h \) matrix. For \( 1 \leq \ell \leq 3 \) let \( n_\ell \) be the number of oriented edges of length \( \ell \) in \( Ed(\Gamma_0 \setminus \Delta) \). The matrix \( V \) is diagonal with \( n_2 + n_3 \) entries equal to \(-2\) and \( n_3 \) entries equal to \(-\frac{3}{2}\). \( U \) has the block form \( U = \begin{pmatrix} J & N \\ N & J \end{pmatrix} \), where \( N = N^1 + \frac{1}{2} N^2 + \frac{1}{3} N^3 \) (see Section 1). By our calculation \( J \) is the diagonal matrix given by

\[
J_{ii} = A_{ii} + \frac{1}{2} \sum_{j=1}^{h} N_{ij}^2 + \frac{2}{3} \sum_{j=1}^{h} N_{ij}^3 \quad \text{for } 1 \leq i \leq h.
\]

Hence by \([1.3, iv]\)

\[
J_{ii} = - \sum_{j=1}^{h} \left( N_{ij}^1 + \frac{1}{2} N_{ij}^2 + \frac{1}{3} N_{ij}^3 \right).
\]
It follows that $U$ is a symmetric zero row sum matrix. By [1.1] $N = BF^{-1}$ and by [1.2] $-J = (p + 1)F^{-1}$. Hence $U = \hat{U}F^{-1}$, where $\hat{U} = \begin{pmatrix} -(p + 1)I & B \\ B & -(p + 1)I \end{pmatrix}$ and $\hat{F} = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$. Using Lemma 2.1, (iii) we now obtain

$$|\Phi| = |\det(A^{11})| = |\det(U^{11})| \det(V) | = 2^{n^2}3^{n_3} \frac{1}{2h} |P_{\hat{U}}(0)|$$

$$= 2^{n^2}3^{n_3} |P_{\hat{U}}(0)/P_{\hat{F}}(0)|.$$  

Firstly, $|P_{\hat{F}}(0)| = 2 |P_F(0)P_{\hat{F}}(0)| = 2(\tr F^{-1})(\det F)^2$. Next, using Lemma 2.1 (i), $P_{\hat{U}}(x) = \det \begin{pmatrix} (x + p + 1)I & -B \\ -B & (x + p + 1)I \end{pmatrix} = (\det((x + p + 1)I + B) \det((x + p + 1)I - B)) = (-1)^hP_B(-x - p - 1)P_B(x + p + 1)$. Differentiating at $x = 0$ gives $P'_{\hat{U}}(0) = (-1)^hP_B(-p - 1)P_B(p + 1)$, since $p + 1$ is an eigenvalue for $B$, so that $P_B(p + 1) = 0$. Hence we have proven:

2.2. THEOREM:

Using the results of Eichler [3] and Kurihara [5] we can rewrite Theorem 2.2 in a more convenient form. Let

$$e_2 = \prod_{q \mid \Disc} \left(1 - \left(-\frac{4}{q}\right)\right), \quad e_3 = \prod_{q \mid \Disc} \left(1 - \left(-\frac{3}{q}\right)\right).$$

2.3. THEOREM:

$$|\Phi| = \frac{1}{2m(\hat{D})c(\hat{D})2^{n^2}3^{n_3}} |P_B(-p - 1)P_B'(p + 1)|$$

where $c(\hat{D}) = 8$ if $\Disc = 2$, $c(\hat{D}) = 3$ if $\Disc = 3$, and $c(\hat{D}) = 1$ otherwise.

PROOF: By Eichler's mass formula $\tr F^{-1} = m(\hat{D})$. Suppose $\Disc \geq 5$. Then $f(v) \in \{1, 2, 3\}$ for all $v \in \Ver(\Gamma_0 \setminus \Delta)$; set $h_\ell = \#\{v \in \Ver(\Gamma_0 \setminus \Delta) \mid f(v) = \ell\}$. By Kurihara [5], Section 4 we have

$$h_2 = \frac{1}{2} \prod_{q \mid \Disc} \left(1 - \left(-\frac{4}{q}\right)\right) \quad \text{and} \quad h_3 = \frac{1}{2} \prod_{q \mid \Disc} \left(1 - \left(-\frac{3}{q}\right)\right).$$
From Kurihara's table ([5], Proposition 4-2) we obtain

\[
\frac{(\det F)^2}{2^n 3^n} = \frac{2^{2h_2} 3^{2h_3}}{2^{h_2 (1+(-4/p))} 3^{h_3 (1+(-3/p))}} = 2^{e_2} 3^{e_3}.
\]

Suppose next Disc \( \mathcal{D} = 3 \). Then \( F \) is the \( 1 \times 1 \) matrix (6) and Kurihara's table gives

\[
\frac{(\det F)^2}{2^n 3^n} = \frac{36}{2^{(1+(-4/p))} 3^{(1/2)(1+(-3/p))}} = 3 \cdot 2^{e_2} \cdot 3^{e_3}.
\]

Finally if Disc \( \mathcal{D} = 2 \), \( F = (12) \) and

\[
\frac{(\det F)^2}{2^n 3^n} = \frac{144}{2^{(1/2)(1+(-4/p))} 3^{(1+(-3/p))}} = 8 \cdot 2^{e_2} \cdot 3^{e_3}.
\]

The theorem follows.

2.4. REMARK: In the course of the proof of Theorem 2.2 we inverted only 2 and 3. Likewise the proof of Lemma 2.1, i) shows that one can transform \( \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \) to \( \begin{pmatrix} X - Y & 0 \\ 0 & X + Y \end{pmatrix} \) by elementary row and column transformations \( R_i \rightarrow R_i + aR_j, C_i \rightarrow C_i + aC_j \) with \( a \in \mathbb{Z}^{[\frac{1}{2}]} \). Hence setting

\[
M = \mathbb{Z}^{[\frac{1}{2}]}, \quad M_0 = \left\{ (a_1, \ldots, a_h) \in M \mid \sum \frac{a_i}{f(v_i)} = 0 \right\},
\]

we have

\[
\Phi \otimes \mathbb{Z}^{[\frac{1}{2}]} \simeq M_0/(B - (p + 1) I) M \oplus M/(B + (p + 1) I) M.
\]

§3. The connected component

Since all components of the special fiber \( (\overline{M_{\mathcal{D}}} \times \mathbb{Z}_p)_0 \) are rational the connected component \( J_p^0 \) is a torus.

3.1. THEOREM: \( J_p^0 \simeq H^1((\Gamma_+ \setminus \Delta), \mathbb{Z}) \otimes G_m \). The action of Frobenius is \( w_p \otimes \text{Frob}_{G_m} \).

PROOF: We need only remark that \( \Gamma_+ \setminus \Delta, \tilde{G} \), and the graph of the special fiber as defined in Deligne and Rapoport [1], p. 164, are all naturally homotopic, so that [1], 3.7b applies.
3.2. Corollary: Let $\ell \not= p$ be a prime. Then the Tate module

$$Ta_{\ell} (\mathfrak{g}_p^0) \cong H^1 (\Gamma_+ \backslash \Delta, \mathbb{Z}_\ell)$$

with Frobenius acting as $p^w p$.

References


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