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CLASSIFICATION OF $\mathscr{D}$-MODULES WITH REGULAR SINGULARITIES ALONG NORMAL CROSSINGS

M.G.M. van Doorn

To classify regular holonomic $\mathscr{D}_1$-modules Boutet de Monvel [1] uses pairs of finite dimensional $\mathbb{C}$-vector spaces related by certain $\mathbb{C}$-linear maps.

Galligo, Granger and Maisonobe [2] obtain, using the Riemann-Hilbert correspondene, a classification of holonomic $\mathscr{D}_n$-modules with regular singularities along $x_1 \dots x_n$ by means of 2^n-tuples of $\mathbb{C}$-vector spaces provided with a set of linear maps. We mention that also Deligne (not published) gets a classification of regular holonomic $\mathscr{D}_1$-modules.

The aim of this paper is to get such a classification in a direct way. The idea is roughly as follows. Denote by $\tilde{\mathscr{C}}_1$ the category whose objects are diagrams $E \underset{v}{\overset{u}{\rightleftarrows}} F$ of finite dimensional $\mathbb{C}$-vector spaces such that $\{\lambda \mid \lambda \text{ eigenvalue of } vu\} \subseteq \{\alpha \in \mathbb{C} \mid 0 \leqslant \operatorname{Re} \alpha < 1\}$. We construct $\mathscr{D}_1$-modules $\mathscr{F}'$ ("Nilsson class functions"), $\mathscr{F}''$ ("micro Nilsson class functions") and $\mathscr{D}_1$-linear maps $\mathscr{U}$: $\mathscr{F}' \to \mathscr{F}''$ ("canonical map"), $\mathscr{V}$: $\mathscr{F}'' \to \mathscr{F}'$ ("variation"). For $M \in \operatorname{Mod}_l(\mathscr{D}_1)_{hr}$, i.e. M is a regular holonomic left $\mathscr{D}_1$-module, we consider the solutions of M with values in $\mathscr{F}'$ (resp. $\mathscr{F}''$), i.e. $\operatorname{Hom}_{\mathscr{D}_1}(M, \mathscr{F}')$ (resp. $\operatorname{Hom}_{\mathscr{D}_1}(M, \mathscr{F}'')$). In this way we get an object in $\tilde{\mathscr{C}}_1$, i.e. a functor S: $\operatorname{Mod}_\ell(\mathscr{D}_1)_{hr} \to \tilde{\mathscr{C}}_1$. In order to prove that S defines an equivalence of categories we exhibit an inverse functor T of S. As a matter of fact $T(E \rightleftarrows F) = \operatorname{Hom}(E \rightleftarrows F, \mathscr{F}' \rightleftarrows \mathscr{F}'')$. The proof that S and T are inverse to each other reduces to a study of what happens to simple objects of both categories.

The generalization to several variables is more or less straightforward, but the proofs get more involved. In proving statements we use induction on n to step down to the case $n = 1$ (or $n = 0$ if you wish). This causes some technical problems (cf. Lemma 4). At the end the proof of the equivalence (Proposition 3) becomes a formal exercise.

NOTATIONS: Let $n \in \mathbb{N}$. Write $\partial_i = \dfrac{\partial}{\partial x_i}$, $i \in \{1, \dots, n\}$. $\mathscr{O} = \mathscr{O}_n = \mathbb{C}[[x_1, \dots, x_n]]$ (resp. $\mathbb{C}\{x_1, \dots, x_n\}$); $\mathscr{D}_n = \mathscr{O}_n[\partial_1, \dots, \partial_n]$. $\mathscr{O}_{(n)} = \mathbb{C}[[x_n]]$ (resp. $\mathbb{C}\{x_n\}$); $\mathscr{D}_{(n)} = \mathscr{O}_{(n)}[\partial_n]$.
Let $\mathscr{D}$ be $\mathscr{D}_n$ or $\mathscr{D}_{(n)}$. $\operatorname{Mod}_\ell(\mathscr{D})$ denotes the category of left $\mathscr{D}$-modules.

If $P \in \mathscr{D}$ the left $\mathscr{D}$-module $\mathscr{D}/\mathscr{D}P$ is denoted by $\mathscr{D}/(P)$. If $M \in \mathrm{Mod}_l(\mathscr{D})$ and $P \in \mathscr{D}$, left multiplication with P on M is denoted by $M \xrightarrow{P.} M$.

$J = \{\alpha \in \mathbb{C} \mid 0 \leqslant \mathrm{Re}\ \alpha < 1\}$.

Throughout the paper we assume that the reader has some familiarity with the language of $\mathscr{D}$-modules. He may consult for example [6], [7].

Let M, $N \in \mathrm{Mod}_\ell(\mathscr{D}_n)$. Then the tensorproduct $M \underset{\mathscr{O}}{\otimes} N$ has in natural way a left $\mathscr{D}_n$-module structure, namely given by $\partial_i(m \otimes n) = \partial_i(m) \otimes n + m \otimes \partial_i(n)$, all i. Let $M \in \mathrm{Mod}_\ell(\mathscr{D}_{n-1}) \cdot \mathscr{O} \underset{\mathscr{O}_{n-1}}{\otimes} M$ has a left $\mathscr{D}_n$-module structure given by $\partial_i(a \otimes m) = \partial_i(a) \otimes m + a \otimes \partial_i(m)$, all $i \in \{1, \ldots, n-1\}$, $\partial_n(a \otimes m) = \partial_n(a) \otimes m$ (cf [6], Ch. 2, 12.2).
In a similar way $\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} N$ has a left $\mathscr{D}_n$-module structure if $N \in \mathrm{Mod}_\ell(\mathscr{D}_{(n)})$. If $Q_i \in \mathscr{D}_{(i)}$, the following is easily verified

$$\left(\mathscr{O} \underset{\mathscr{O}_{n-1}}{\otimes} \mathscr{D}_{n-1}/(Q_1, \ldots, Q_{n-1})\right) \underset{\mathscr{O}}{\otimes} \left(\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{D}_{(n)}/(Q_n)\right)$$
$$\cong \mathscr{D}/(Q_1, \ldots, Q_n).$$

§1. The operation $\mathscr{C}$

In order to state the results in a neat way we introduce some general notions. Let $\mathscr{A}$ be a category. $\mathscr{C}(\mathscr{A})$ is the category whose objects are quadruples (E, F, u, v), where E, F are objects of $\mathscr{A}$, $u \in \mathrm{Hom}_{\mathscr{A}}(E, F)$ and $v \in \mathrm{Hom}_{\mathscr{A}}(F, E)$. If (E, F, u, v) and (E', F', u', v') belong to $\mathscr{C}(\mathscr{A})$, then

$$\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}((E, F, u, v), (E', F', u', v'))$$
$$= \{(f, g) \in \mathrm{Hom}_{\mathscr{A}}(E, E')$$
$$\times \mathrm{Hom}_{\mathscr{A}}(F, F') \mid u'f = gu,\ fv = v'g\}.$$

Hence $\mathscr{C}(\mathscr{A})$ is the category of diagrams in $\mathscr{A}$ over the scheme "$\cdot \rightleftarrows \cdot$". Cf. Grothendieck [3] and Mitchell [4], Ch. II §1. $\mathscr{C}(\mathscr{A})$ may be seen as a functor category and as such it inherits the properties of $\mathscr{A}$. In particular $\mathscr{C}(\mathscr{A})$ is an abelian category if $\mathscr{A}$ is abelian. We have two evaluation functors e_0 and e_1 from $\mathscr{C}(\mathscr{A})$ to $\mathscr{A}$. If $X = (E, F, u, v) \in \mathscr{C}(\mathscr{A})$ then $e_0(X) = E$, $e_1(X) = F$. If $\mathscr{A}$ is an abelian category these functors are exact and collectively faithful. Hence in particular: $X' \to X \to X''$ is exact in $\mathscr{C}(\mathscr{A})$ if and only if $e_i(X') \to e_i(X) \to e_i(X'')$ is exact in $\mathscr{A}$, all $i \in \{0, 1\}$. Notice that we have natural transformations u: $e_0 \to e_1$, v: $e_1 \to e_0$. If F: $\mathscr{A} \to \mathscr{B}$ is a functor between categories $\mathscr{A}$ and $\mathscr{B}$, there is obviously an induced functor $\mathscr{C}(F)$: $\mathscr{C}(\mathscr{A}) \to \mathscr{C}(\mathscr{B})$. Clearly if $\mathscr{A}$ and $\mathscr{B}$ are additive and F is an additive functor, then $\mathscr{C}(F)$ is additive. Exactness properties of F are transferred to $\mathscr{C}(F)$. Furthermore, if G: $\mathscr{A} \to \mathscr{B}$ is another functor and η: $F \to G$ a natural transformation (resp. equivalence), there is a natural transformation (resp. equivalence) $\mathscr{C}(\eta)$: $\mathscr{C}(F) \to \mathscr{C}(G)$.

Let $\mathscr{A}$ be a category. For all $n \in \mathbb{N}$ we define inductively

$$\mathscr{C}_0(\mathscr{A}) := \mathscr{A}$$

$$\mathscr{C}_{n+1}(\mathscr{A}) := \mathscr{C}(\mathscr{C}_n(\mathscr{A})).$$

For each $n \in \mathbb{N}$ we have 2^n evaluation functors defined inductively as follows: for all $i_1, \dots, i_{n+1} \in \{0, 1\}$

$$e_{i_1 \dots i_{n+1}} = e_{i_1 \dots i_n} \circ e_{i_{n+1}}.$$

If $E \in \mathscr{C}_n(\mathscr{A})$ and $i_1, \dots, i_n \in \{0, 1\}$ we mostly write $E(i_1 \dots i_n)$ or $E_{i_1 \dots i_n}$ instead of $e_{i_1 \dots i_n}(E)$.
For every $j \in \{1, \dots, n\}$ and all $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_n \in \{0, 1\}$ we get $\mathscr{A}$-morphisms

$$E(i_1 \dots i_{j-1} 0 i_{j+1} \dots i_n) \to E(i_1 \dots i_{j-1} 1 i_{j+1} \dots i_n)$$

$$E(i_1 \dots i_{j-1} 1 i_{j+1} \dots i_n) \to E(i_1 \dots i_{j-1} 0 i_{j+1} \dots i_n)$$

It is easily seen that the category $\mathscr{C}_n(\mathscr{A})$ can be identified with the category whose objects are 2^n-tuples $(E(i_1 \dots i_n);\ i_1, \dots, i_n \in \{0, 1\})$ of objects of $\mathscr{A}$, connected by $\mathscr{A}$-morphisms, for all $j \in \{1, \dots, n\}$, all $i_1, \dots i_n \in \{0, 1\}$,

$$u\colon E(-0-) \to E(-1-), \quad v\colon E(-1-) \to E(-0-),$$

where $E(-r-)$ stands for $E(i_1 \dots i_{j-1} r i_{j+1} \dots i_n)$. The following diagrams have to commute

$$\begin{array}{ccc} E_{00} & \xrightarrow{u} & E_{01} \\ \downarrow u & & u \downarrow \\ E_{10} & \xrightarrow{u} & E_{11} \end{array} \qquad \begin{array}{ccc} E_{00} & \xleftarrow{v} & E_{01} \\ \uparrow v & & v \uparrow \\ E_{10} & \xleftarrow{v} & E_{11} \end{array} \qquad \begin{array}{ccc} E_{00} & \xrightarrow{u} & E_{01} \\ \uparrow v & & v \uparrow \\ E_{10} & \xrightarrow{u} & E_{11} \end{array} \qquad \begin{array}{ccc} E_{00} & \xleftarrow{v} & E_{01} \\ \downarrow u & & u \downarrow \\ E_{10} & \xleftarrow{v} & E_{11} \end{array}$$

where for simplicity we have written E_{rs} instead of

$$E\left(i_1 \dots i_{j-1} r i_{j+1} \dots i_{k-1} s i_{k+1} \dots i_n\right), \quad \text{all } r, s \in \{0, 1\}.$$

REMARK: Let A be a ring and let $\mathrm{Mod}_\ell(A)$ be the category of left A-modules. We write $\mathscr{C}_n(A)$ instead of $\mathscr{C}_n(\mathrm{Mod}_\ell(A))$. Furthermore we set $\mathscr{C}_n = \mathscr{C}_n(\mathbb{C})$.

§2. Definition and properties of $\mathscr{F}_n$

Our next goal is to construct a particular object $\mathscr{F}_n$ of $\mathscr{C}_n(\mathscr{D}_n)$. Let therefore $n \in \mathbb{N}$, $n \neq 0$. For $\alpha \in J$, $i \in \mathbb{N} - \{0\}$ define

$$\mathscr{F}'_{(n),\alpha,i} := \mathscr{D}_{(n)} / \left((\partial_n x_n - \alpha)^i \right), \quad F''_{(n),\alpha,i} := \mathscr{D}_{(n)} / \left((x_n \, \partial_n - \alpha)^i \right).$$

For each $\alpha \in J$, the $\mathscr{D}_{(n)}$-linear maps

$$\mathscr{F}'_{(n),\alpha,i} \to \mathscr{F}'_{(n),\alpha,i+1}, \quad \text{induced by } P \mapsto P(\partial_n x_n - \alpha)$$

and

$$\mathscr{F}''_{(n),\alpha,i} \to \mathscr{F}''_{(n),\alpha,i+1}, \quad \text{induced by } P \mapsto P(x_n \, \partial_n - \alpha)$$

yield inductive systems $(\mathscr{F}'_{(n),\alpha,i})_i$ and $(\mathscr{F}''_{(n),\alpha,i})_i$.
Define

$$\mathscr{F}'_{(n)} := \bigoplus_{\alpha \in J} \varinjlim_{i} \mathscr{F}'_{(n),\alpha,i}, \quad \mathscr{F}''_{(n)} := \bigoplus_{\alpha \in J} \varinjlim_{i} \mathscr{F}''_{(n),\alpha,i}.$$

Furthermore, the $\mathscr{D}_{(n)}$-linear maps

$$\mathscr{F}'_{(n),\alpha,i} \to \mathscr{F}''_{(n),\alpha,i}, \quad \text{induced by } P \mapsto P \, \partial_n,$$

$$\mathscr{F}''_{(n),\alpha,i} \to \mathscr{F}'_{(n),\alpha,i}, \quad \text{induced by } P \mapsto P x_n,$$

give rise to $\mathscr{D}_{(n)}$-linear maps

$$\mathscr{U}_{(n)} \colon \mathscr{F}'_{(n)} \to \mathscr{F}''_{(n)}, \quad \mathscr{V}_{(n)} \colon \mathscr{F}''_{(n)} \to \mathscr{F}'_{(n)}.$$

Hence we have constructed an object

$$\mathscr{F}_{(n)} := \left(\mathscr{F}'_{(n)}, \mathscr{F}''_{(n)}, \mathscr{U}_{(n)}, \mathscr{V}_{(n)} \right) \in \mathscr{C}_1\left(\mathscr{D}_{(n)}\right).$$

By extending coefficients we get $\mathscr{O} \otimes_{\mathscr{O}_{(n)}} \mathscr{F}_{(n)} \in \mathscr{C}_1(\mathscr{D}_n)$.

REMARK: Instead of the clumsy notation $\mathscr{C}_1(\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} .)(\mathscr{F}_{(n)})$ we prefer to write $\mathcal{O} \underset{\mathcal{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}$.

The preceding constructions lead immediately to

LEMMA 1: *There exists short exact sequences of $\mathscr{D}_{(n)}$-modules*

$$\mathcal{O}_{(n)} = \mathscr{D}_{(n)}/(\partial_n) \hookrightarrow \mathscr{F}'_{(n)} \overset{\mathscr{U}_{(n)}}{\twoheadrightarrow} \mathscr{F}''_{(n)}$$

$$\mathscr{D}_{(n)}/(x_n) \hookrightarrow \mathscr{F}''_{(n)} \overset{\mathscr{V}_{(n)}}{\twoheadrightarrow} \mathscr{F}'_{(n)}$$

$$\mathscr{D}_{(n)}/(\partial_n x_n - \alpha) \hookrightarrow \mathscr{F}'_{(n)} \overset{\mathscr{V}_{(n)}\mathscr{U}_{(n)} - \alpha 1}{\twoheadrightarrow} \mathscr{F}'_{(n)}, \ \alpha \in J - \{0\}.$$

PROOF: Let $\alpha \in J - \{0\}$. The $\mathscr{D}_{(n)}$-linear map $\mathscr{D}_{(n)}/(\partial_n x_n - \alpha) \to \mathscr{D}_{(n)}/(x_n \partial_n - \alpha)$, induced by $P \mapsto P\,\partial_n$ is an isomorphism (left to the reader).
We have the commutative diagram with exact rows

$$\begin{array}{ccccc}
\mathscr{F}'_{(n),\alpha,i} & \underset{1 \mapsto \partial_n x_n - \alpha}{\hookrightarrow} & \mathscr{F}'_{(n),\alpha,i+1} & \underset{1 \mapsto 1}{\twoheadrightarrow} & \mathscr{D}_{(n)}/(\partial_n x_n - \alpha) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{F}''_{(n),\alpha,i} & \overset{1 \mapsto x_n \partial_n - \alpha}{\hookrightarrow} & \mathscr{F}''_{(n),\alpha,i+1} & \overset{1 \mapsto 1}{\twoheadrightarrow} & \mathscr{D}_{(n)}/(x_n \partial_n - \alpha)
\end{array}$$

where the vertical maps are induced by $P \mapsto P\,\partial_n$.
Hence, by induction on i, it follows that $\mathscr{F}'_{(n),\alpha,i} \to \mathscr{F}''_{(n),\alpha,i}$, $1 \mapsto \partial_n$, is an isomorphism for all $i \in \mathbb{N} - \{0\}$.

It is easily verified that we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathscr{D}_{(n)}/(\partial_n) & \overset{1 \mapsto x_n(\partial_n x_n)^{i-1}}{\hookrightarrow} & \mathscr{F}'_{(n),0,i} & \overset{1 \mapsto \partial_n}{\to} & \mathscr{F}''_{(n),0,i} & \overset{1 \mapsto 1}{\twoheadrightarrow} & \mathscr{D}_{(n)}/(\partial_n) \\
\downarrow & & \downarrow 1 \mapsto \partial_n x_n & & \downarrow 1 \mapsto x_n \partial_n & & \downarrow 0 \\
\mathscr{D}_{(n)}/(\partial_n) & \overset{1 \mapsto x_n(\partial_n x_n)^{i}}{\hookrightarrow} & \mathscr{F}'_{(n),0,i+1} & \overset{1 \mapsto \partial_n}{\to} & \mathscr{F}''_{(n),0,i+1} & \overset{1 \mapsto 1}{\twoheadrightarrow} & \mathscr{D}_{(n)}/(\partial_n)
\end{array}$$

Taking the direct limit and summing over $\alpha \in J$ we obtain the exact sequence of $\mathscr{D}_{(n)}$-modules

$$\mathscr{D}_{(n)}/(\partial_n) \hookrightarrow \mathscr{F}'_{(n)} \overset{\mathscr{U}_{(n)}}{\twoheadrightarrow} \mathscr{F}''_{(n)}.$$

The other two sequences are obtained in a similar way.

Consider the bifunctor $\underset{\mathscr{O}}{\otimes}$: $\mathrm{Mod}_\ell(\mathscr{D}_n) \times \mathrm{Mod}_\ell(\mathscr{D}_n) \to \mathrm{Mod}_\ell(\mathscr{D}_n)$, $(M, N) \mapsto M \underset{\mathscr{O}}{\otimes} N$. It induces a bifunctor from $\mathscr{C}_{n-1}(\mathscr{D}_n) \times \mathscr{C}_1(\mathscr{D}_n)$ to $\mathscr{C}_n(\mathscr{D}_n)$, also denoted by $\underset{\mathscr{O}}{\otimes}$. Keeping this in mind we define inductively on $n \in \mathbb{N}$

$$\mathscr{F}_0 := \mathbb{C}$$

$$\mathscr{F}_n := \left(\mathscr{O} \underset{\mathscr{O}_{n-1}}{\otimes} \mathscr{F}_{n-1}\right) \underset{\mathscr{O}}{\otimes} \left(\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}\right) \in \mathscr{C}_n(\mathscr{D}_n).$$

Hence $\mathscr{F}_n(i_1 \dots i_n) = (\mathscr{O} \underset{\mathscr{O}_{(1)}}{\otimes} \mathscr{F}_{(1)}(i_1)) \underset{\mathscr{O}}{\otimes} \dots \underset{\mathscr{O}}{\otimes} (\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}(i_n))$, all $i_1, \dots, i_n \in \{0, 1\}$. The $\mathscr{D}_n$-linear maps are identified as

$$\mathscr{F}_n(i_1 \dots i_{j-1} 0 i_{j+1} \dots i_n) \underset{1 \otimes \mathscr{V}_{(j)} \otimes 1}{\overset{1 \otimes \mathscr{U}_{(j)} \otimes 1}{\rightleftarrows}} \mathscr{F}_n(i_1 \dots i_{j-1} 1 i_{j+1} \dots i_n).$$

We are ready now to define the functor S_n. Therefore consider the bifunctor H_n: $\mathrm{Mod}_\ell(\mathscr{D}_n) \times \mathrm{Mod}_\ell(\mathscr{D}_n) \to \mathscr{C}_0$, $(M, N) \mapsto \mathrm{Hom}_{\mathscr{D}_n}(M, N)$. It induces a bifunctor $\mathscr{C}_n(H_n)$: $\mathrm{Mod}_\ell(\mathscr{D}_n) \times \mathscr{C}_n(\mathscr{D}_n) \to \mathscr{C}_n$. So there arises a contravariant functor

$$S_n\colon \mathrm{Mod}_\ell(\mathscr{D}_n) \to \mathscr{C}_n, \quad S_n(M) := \mathscr{C}_n(H_n)(M, \mathscr{F}_n).$$

Notice that S_n is characterized by

$$S_n(M)(i_1 \dots i_n) = \mathrm{Hom}_{\mathscr{D}_n}(M, \mathscr{F}_n(i_1 \dots i_n)),$$

$$\text{all } i_1, \dots, i_n \in \{0, 1\}.$$

§3. Study of the functor S_n

We restrict our attention to the category $\mathrm{Mod}_\ell(\mathscr{D}_n)_{hr}^{x_1 \dots x_n}$ the full subcategory of $\mathrm{Mod}_\ell(\mathscr{D}_n)$ consisting of holonomic $\mathscr{D}_n$-modules with regular singularities along $x_1 \dots x_n$. For a definition we refer to van den

Essen [5], Ch. I, Def. 1.16. He gives also a description of the simple objects in $\mathrm{Mod}_\ell(\mathscr{D}_n)_{hr}^{x_1 \cdots x_n}$ (Ch. I, Th. 2.7). They are of the form $\mathscr{D}/(q_1, \dots, q_n)$ with $q_i \in \{x_i, \partial_i\} \cup \{\partial_i x_i - \alpha_i \mid \alpha_i \in \mathbb{C},\ 0 < \operatorname{Re} \alpha_i < 1\}$, all $i \in \{1, \dots, n\}$.
It is suitable for us to write this as

$$\left(\mathscr{O} \underset{\mathscr{O}_{n-1}}{\otimes} N\right) \underset{\mathscr{O}}{\otimes} \left(\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{D}_{(n)}/(q_n)\right)$$

where $N = \mathscr{D}_{n-1}/(q_1, \dots, q_{n-1})$ is a simple object from $\mathrm{Mod}_\ell(\mathscr{D}_{n-1})_{hr}^{x_1 \cdots x_{n-1}}$. To simplify notations we introduce:
For $\alpha \in J \cup \{1\}$ define $q_n(\alpha) \in \mathscr{D}_{(n)}$ as:

$$q_n(0) \blacksquare\ \partial_n;\ q_n(1) \blacksquare\ x_n;\ q_n(\alpha) \blacksquare\ \partial_n x_n - \alpha, \quad \alpha \in J - \{0\}.$$

For $N \in \mathrm{Mod}_\ell(\mathscr{D}_{n-1})$ define $P_\alpha(N) \blacksquare\ (\mathscr{O} \underset{\mathscr{O}_{n-1}}{\otimes} N) \underset{\mathscr{O}}{\otimes} (\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{D}_{(n)}/(q_n(\alpha)))$.
For $M \in \mathrm{Mod}_\ell(\mathscr{D}_n)$ define $Q_\alpha(M) \blacksquare\ \operatorname{Ker}(M \overset{q_n(\alpha)\cdot}{\to} M)$.

So for each $\alpha \in J \cup \{1\}$ we have a pair of functors (P_α, Q_α) P_α: $\mathrm{Mod}_\ell(\mathscr{D}_{n-1}) \to \mathrm{Mod}_\ell(\mathscr{D}_n)$, Q_α: $\mathrm{Mod}_\ell(\mathscr{D}_n) \to \mathrm{Mod}_\ell(\mathscr{D}_{n-1})$. Obviously:

- Q_α is left exact.
- P_α is exact, because $\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{D}_{(n)}/(q_n(\alpha))$ is a flat $\mathscr{O}_{n-1}$-module.
- P_α is a left adjoint of Q_α.

By a direct calculation, using the definitions of $\mathscr{F}_n'$ and $\mathscr{F}_n''$, one establishes

LEMMA 2: There exist short exact sequences of $\mathscr{D}_{n-1}$-modules

$$\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}'' \overset{\partial_n \cdot}{\hookrightarrow\!\!\!\!\rightarrow} \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}'' \qquad \mathscr{O}_{n-1} \hookrightarrow \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}' \overset{\partial_n \cdot}{\twoheadrightarrow} \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}'$$

$$\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}' \overset{x_n \cdot}{\hookrightarrow\!\!\!\!\rightarrow} \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}' \qquad \mathscr{O}_{n-1} \hookrightarrow \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}'' \overset{x_n \cdot}{\twoheadrightarrow} \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}''$$

$$\mathscr{O}_{n-1} \hookrightarrow \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}' \overset{q_n(\alpha)\cdot}{\twoheadrightarrow} \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}' \qquad \mathscr{O}_{n-1} \hookrightarrow \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}'' \overset{q_n(\alpha)\cdot}{\twoheadrightarrow} \mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}'',$$

for all $\alpha \in J - \{0\}$.

PROOF: During the proof we write $\otimes$ in stead of $\underset{\mathcal{O}_{(n)}}{\otimes}$. Let $\alpha \in J$. It is straightforward to verify that $\mathcal{O}_{n-1} \cong Q_\alpha(\mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i})$. One may use e.g.

$$\mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i} = \mathcal{O}\left[\frac{1}{x_n}\right] x_n^{\alpha-1}(\log x_n)^{i-1} + \ldots + \mathcal{O}\left[\frac{1}{x_n}\right] x_n^{\alpha-1}$$

or the lemma on page 39 in [6].

Furthermore

$$\mathcal{O}_{n-1} \cong \operatorname{Coker}\left(\mathcal{O} \otimes \mathcal{D}_{(n)}/(\partial_n x_n - \alpha) \overset{(\partial_n x_n - \alpha)\cdot}{\rightarrow} \mathcal{O} \otimes \mathcal{D}_{(n)}/(\partial_n x_n - \alpha)\right).$$

Consider the short exact sequence of $\mathcal{D}$-modules

$$\mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i} \hookrightarrow \mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i+1} \twoheadrightarrow \mathcal{O} \otimes \mathcal{D}_{(n)}/(\partial_n x_n - \alpha).$$

Writing ϕ_j for the map: left multiplication with $\partial_n x_n - \alpha$ on $\mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,j}$, all $j \in \mathbb{N} - \{0\}$, we obtain a long exact sequence

$$\begin{aligned} \mathcal{O}_{n-1} &= \operatorname{Ker} \phi_i \hookrightarrow \operatorname{Ker} \phi_{i+1} = \mathcal{O}_{n-1} \rightarrow \operatorname{Ker} \phi_0 \\ &= \mathcal{O}_{n-1} \overset{\delta}{\rightarrow} \operatorname{Coker} \phi_i \overset{\epsilon}{\rightarrow} \operatorname{Coker} \phi_{i+1} \twoheadrightarrow \operatorname{Coker} \phi_0 = \mathcal{O}_{n-1} \end{aligned}$$

where the maps are $\mathcal{D}_{n-1}$-linear.

By induction on i we have Coker $\phi_i = \mathcal{O}_{n-1}$. Now $\mathcal{O}_{n-1}$ is a simple $\mathcal{D}_{n-1}$-module, hence δ is an isomorphism. Moreover $\epsilon = 0$ and Coker $\phi_{i+1} = \mathcal{O}_{n-1}$. So we have for all $i \in \mathbb{N}$-$\{0\}$, a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathcal{O}_{n-1} & \hookrightarrow & \mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i} & \overset{(\partial_n x_n - \alpha)\cdot}{\rightarrow} & \mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i} & \twoheadrightarrow & \mathcal{O}_{n-1} \\ 1\downarrow & & \downarrow & & \downarrow & & \downarrow 0 \\ \mathcal{O}_{n-1} & \hookrightarrow & \mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i+1} & \overset{(\partial_n x_n - \alpha)\cdot}{\rightarrow} & \mathcal{O} \otimes \mathcal{F}'_{(n),\alpha,i+1} & \twoheadrightarrow & \mathcal{O}_{n-1} \end{array}$$

Another calculation learns that left multiplication with $\partial_n x_n - \alpha$ on $\mathcal{O} \otimes \mathcal{F}'_{(n),\beta,i}$ is a bijection, for all $i \in \mathbb{N} - \{0\}$, all $\beta \in J$, $\beta \neq \alpha$ (use induction on i). After taking the direct limit and summing over $\beta \in J$ we arrive at the short exact sequence

$$\mathcal{O}_{n-1} \hookrightarrow \mathcal{O} \otimes \mathcal{F}'_{(n)} \overset{(\partial_n x_n - \alpha)\cdot}{\twoheadrightarrow} \mathcal{O} \otimes \mathcal{F}'_{(n)}.$$

Using that left multiplication with $\partial_n x_n - \alpha$ on $\mathcal{O}$ is bijective and the commutativity of the next diagram with exact rows (Lemma 1)

$$\begin{array}{ccccc} \mathcal{O} & \hookrightarrow & \mathcal{O}\otimes\mathscr{F}'_{(n)} & \twoheadrightarrow & \mathcal{O}\otimes\mathscr{F}''_{(n)} \\ \downarrow(\partial_n x_n-\alpha)\cdot & & \downarrow(\partial_n x_n-\alpha)\cdot & & \downarrow(\partial_n x_n-\alpha)\cdot \\ \mathcal{O} & \hookrightarrow & \mathcal{O}\otimes\mathscr{F}'_{(n)} & \twoheadrightarrow & \mathcal{O}\otimes\mathscr{F}''_{(n)} \end{array}$$

one estblishes the exactness of

$$\mathcal{O}_{n-1} \hookrightarrow \mathcal{O}\otimes\mathscr{F}''_{(n)} \overset{(\partial_n x_n-\alpha)\cdot}{\twoheadrightarrow} \mathcal{O}\otimes\mathscr{F}''_{(n)}.$$

It is immediately verified that left multiplication with x_n on $\mathcal{O}\otimes\mathscr{F}'_{(n)}$ is bijective. Furthermore left multiplication with x_n on $\mathcal{O}\otimes\mathscr{D}_{(n)}/(x_n)$ is surjective and has Ker $\cong \mathcal{O}_{n-1}$. Consider the second sequence in Lemma 1, argue as above and obtain the exactness of $\mathcal{O}_{n-1} \hookrightarrow \mathcal{O}\otimes\mathscr{F}''_{(n)} \overset{x_n\cdot}{\twoheadrightarrow} \mathcal{O}\otimes\mathscr{F}''_{(n)}$. Combining results on left multiplication with $\partial_n x_n$ and left multiplication with x_n yields the exactness of the upper sequences in the lemma.

At this point we introduce a category $\tilde{\mathscr{C}}$ as follows:
$\tilde{\mathscr{C}}_0$ is the category of finite dimensional $\mathbb{C}$-vector spaces,
$\tilde{\mathscr{C}}_{n+1}$ is the full subcategory of $\mathscr{C}_{n+1}$ consisting of the objects $(E, F, u, v) \in \mathscr{C}_{n+1}$ such that
(i) $E, F \in \tilde{\mathscr{C}}_n$
(ii) $\{\lambda \mid \lambda$ eigenvalue of $e_{i_1 \ldots i_n}(vu)\} \subset J$ for all $i_1, \ldots, i_n \in \{0, 1\}$.
Notice that $\tilde{\mathscr{C}}_n$ is a thick abelian subcategory of $\mathscr{C}_n$. For each $\alpha \in J \cup \{1\}$ we introduce a functor L_α: $\mathscr{C}_{n-1} \to \mathscr{C}_n$ by setting for all $E \in \mathscr{C}_{n-1}$:

$$L_0(E) := (E, 0, 0, 0),$$

$$L_1(E) := (0, E, 0, 0),$$

$$L_\alpha(E) := (E, E, 1, \alpha 1),\ \alpha \in J - \{0\}.$$

These are all exact functors. Clearly for each $\alpha \in J \cup \{1\}$ L_α restricts to a functor from $\tilde{\mathscr{C}}_{n-1}$ to $\tilde{\mathscr{C}}_n$, denoted also L_α.
Putting $n = 1$ in Lemma 2 we may reformulate it as

$$S_1(\mathscr{D}_1/(q_1(\alpha))) = L_\alpha(\mathbb{C}) \in \tilde{\mathscr{C}}_1,$$

$$\mathrm{Ext}^1_{\mathscr{D}1}(\mathscr{D}_1/(q_1(\alpha)), \mathscr{F}_1(i)) = 0, \quad \text{all } \alpha \in J \cup \{1\}, \quad \text{all } i \in \{0, 1\}.$$

However elements of $\mathrm{Mod}_\ell(\mathscr{D}_1)_{hr}$ have finite length. Hence S_1 induces a contravariant exact functor, denoted S_1, from $\mathrm{Mod}_\ell(\mathscr{D}_1)_{hr}$ to $\tilde{\mathscr{C}}_1$. This result generalizes to

Proposition 1: *S_n induces a contravariant exact functor S_n: $\mathrm{Mod}_\ell(\mathscr{D}_n)_{hr}^{x_1\dots x_n} \to \tilde{\mathscr{C}}_n$.*

Proof: By induction on n. We need only to consider a simple module $M \in \mathrm{Mod}_\ell(\mathscr{D}_n)_{hr}^{x_1\dots x_n}$. Hence let $\alpha \in J \cup \{1\}$, $N \in \mathrm{Mod}_\ell(\mathscr{D}_{n-1})_{hr}^{x_1\dots x_{n-1}}$ such that $M = P_\alpha N$. Let $i_1, \dots i_n \in \{0, 1\}$. Write $q = q_n(\alpha)$, $P = P_\alpha$, $Q = Q_\alpha$, $L = L_\alpha$. Lemma 2 says that left multiplication with q is surjective on $\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}(i_n)$. Furthermore $\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}(i_n)$ is a flat $\mathscr{O}_{n-1}$-module and $q \in \mathscr{D}_{(n)}$, hence

$$Q(\mathscr{F}_n(i_1 \dots i_n)) = \mathscr{F}_{n-1}(i_1 \dots i_{n-1}) \underset{\mathscr{O}_{n-1}}{\otimes} Q\left(\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}(i_n)\right).$$

Again using Lemma 2 we get $\mathscr{C}_1(Q)(\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}) = L(\mathscr{O}_{n-1})$. It follows that

$$\begin{aligned} S_n(PN) &= \mathscr{C}_n(H_n)(PN, \mathscr{F}_n) = \mathscr{C}_n(H_{n-1})(N, \mathscr{C}_n(Q)(\mathscr{F}_n)) \\ &= \mathscr{C}_n(H_{n-1})\left(N, \mathscr{F}_{n-1} \underset{\mathscr{O}_{n-1}}{\otimes} \mathscr{C}_1(Q)\left(\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}\right)\right) \\ &= \mathscr{C}_n(H_{n-1})\left(N, \mathscr{F}_{n-1} \underset{\mathbb{C}}{\otimes} L(\mathbb{C})\right) \\ &= (\mathscr{C}_{n-1}(H_{n-1})(N, \mathscr{F}_{n-1})) \underset{\mathbb{C}}{\otimes} L(\mathbb{C}) = LS_{n-1}N \end{aligned}$$

The exactness of S_n follows, by induction, from the next general result.

Lemma 3: *Let $\mathscr{A}$, $\mathscr{B}$ be abelian categories with enough injectives. Let G: $\mathscr{B} \to \mathscr{A}$ be a left adjoint of F: $\mathscr{A} \to \mathscr{B}$ and assume that G is exact. Furthermore, let $A \in \mathscr{A}$ be such that $R^1F(A) = 0$.*
Then $\mathrm{Ext}^1_{\mathscr{A}}(G(B), A) \cong \mathrm{Ext}^1_{\mathscr{B}}(B, F(A))$, all $B \in \mathscr{B}$.

Remark: $R^1Q(\mathscr{F}_n(i_1 \dots i_n)) = 0$ because left multiplication with q is surjective.

Proof: Notice that for an injective object $I \in \mathscr{A}$ $F(I)$ is injective in $\mathscr{B}$, because one has $\mathrm{Hom}_{\mathscr{B}}(\cdot, F(I)) \cong \mathrm{Hom}_{\mathscr{A}}(G(\cdot), I)$ and this last functor is exact. Consider a short exact sequence $A \hookrightarrow I \twoheadrightarrow R$ in $\mathscr{A}$ with I

injective object in $\mathscr{A}$. Because $R^1F(A)=0$ we get an exact sequence in $\mathscr{B}$

$$F(A) \hookrightarrow F(I) \twoheadrightarrow F(R).$$

(Obvious F is left exact.) Let $B \in \mathscr{B}$. There results a commutative diagram of abelian groups with exact rows

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{A}}(G(B), A) & \hookrightarrow & \mathrm{Hom}_{\mathscr{A}}(G(B), I) \\ \| \wr & & \| \wr \\ \mathrm{Hom}_{\mathscr{B}}(B, F(A)) & \hookrightarrow & \mathrm{Hom}_{\mathscr{B}}(B, F(I)) \end{array}$$

$$\begin{array}{ccc} \to \mathrm{Hom}_{\mathscr{A}}(G(B), R) & \twoheadrightarrow & \mathrm{Ext}^1_{\mathscr{A}}(G(B), A) \\ \| \wr & & \\ \to \mathrm{Hom}_{\mathscr{B}}(B, F(R)) & \twoheadrightarrow & \mathrm{Ext}^1_{\mathscr{B}}(B, F(A)). \end{array}$$

Hence the lemma follows.

§4. The inverse functor

In order to prove that S_n defines an equivalence of categories we come up with an inverse functor. First some generalities. Let $\mathscr{A}$ be an additive category and let R be a ring. A left R-object in $\mathscr{A}$ is an object $A \in \mathscr{A}$ together with a homomorphism of rings $\rho\colon R \to \mathrm{Hom}_{\mathscr{A}}(A, A)$. (Cf. Mitchell [4], Ch. II, §13). For example the objects of $\mathscr{C}_n(R)$ are R-objects. Further if $A \in \mathscr{A}$ is any left R-object, then the abelian group $\mathrm{Hom}_{\mathscr{A}}(B, A)$ gets in a canocial way a left R-module structure. If $\alpha \in \mathrm{Hom}_{\mathscr{A}}(B, B')$ then $\mathrm{Hom}_{\mathscr{A}}(\alpha, A)$ is a morphism of left R-modules. In particular we have a left exact contravariant functor

$$T_n\colon \mathscr{C}_n \to \mathrm{Mod}_\ell(\mathscr{D}_n),\ E \mapsto \mathrm{Hom}_{\mathscr{C}_n}(E, \mathscr{F}_n).$$

In order to study this functor T_n we first consider the operation $\mathscr{C}$. We recall that for any additive category $\mathscr{A}$, we defined $\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(E, F)$ for all E, $F \in \mathscr{A}$ in such a way that the following sequence of abelian groups is exact

$$\mathrm{Hom}_{\mathscr{C}(\mathscr{A})}(E, F) \hookrightarrow \mathrm{Hom}_{\mathscr{A}}(E_0, F_0) \times \mathrm{Hom}_{\mathscr{A}}(E_1, F_1)$$

$$\to \mathrm{Hom}_{\mathscr{A}}(E_0, F_1) \times \mathrm{Hom}_{\mathscr{A}}(E_1, F_0)$$

$$(f, g) \mapsto (u_F f - g u_E, f v_E - v_F g).$$

This observation enables us to prove the following.

LEMMA 4: *Let A be a $\mathbb{C}$-algebra, B an A-algebra. Let $\mathscr{A}$ be an abelian subcategory of $\mathscr{C}_0$. Suppose $\mathscr{G}$:* $\mathrm{Mod}_\ell(B) \to \mathrm{Mod}_\ell(A)$ *is a left exact functor. Let θ_0: $G(\mathrm{Hom}_{\mathbb{C}}(\cdot,\cdot)) \to \mathrm{Hom}_{\mathbb{C}}(\cdot, G(\cdot))$ be a natural transformation (resp. equivalence) of bifunctors from $\mathscr{A} \times \mathrm{Mod}_l(B)$ to $\mathrm{Mod}_\ell(A)$. Then there is a natural transformation (resp. equivalence)*

$$\theta_n: G\big(\mathrm{Hom}_{\mathscr{C}_n}(\cdot,\cdot)\big) \to \mathrm{Hom}_{\mathscr{C}_n}(\cdot, \mathscr{C}_n(G)(\cdot))$$

of bifunctors from $\mathscr{C}_n(\mathscr{A}) \times \mathscr{C}_n(B)$ to $\mathrm{Mod}_\ell(A)$.

Finally let us define for each $\alpha \in J \cup \{1\}$ a functor K_α: $\mathscr{C}_n \to \mathscr{C}_{n-1}$ as follows:

$$K_0(E, F, u, v) := \mathrm{Ker}\, u,$$

$$K_1(E, F, u, v) := \mathrm{Ker}\, v,$$

$$K_\alpha(E, F, u, v) := \mathrm{Ker}(vu - \alpha 1),\ \alpha \in J - \{0\}.$$

Clearly K_α is left exact for all $\alpha \in J \cup \{1\}$. Furthermore, as one easily verifies, L_α is a left adjoint of K_α.

Before we return to the functor T_n we need a description of the simple objects of $\tilde{\mathscr{C}}_n$. We leave it to the reader to verify:

LEMMA 5: (*i*) *Every $F \in \tilde{\mathscr{C}}_n$, $F \neq 0$, has a subobject of the form $L_\alpha E$, for some $\alpha \in J \cup \{1\}$ and some simple object $E \in \tilde{\mathscr{C}}_{n-1}$.*
(*ii*) *The simple objects in $\tilde{\mathscr{C}}_n$ are those of the form $L_\alpha E$ for some $\alpha \in J \cup \{1\}$ and some simple object $E \in \tilde{\mathscr{C}}_{n-1}$.*
(*iii*) *Every object in $\tilde{\mathscr{C}}_n$ has a finite length.*

Now we are ready to prove.

PROPOSITION 2: *T_n restricts to a contravariant exact functor, $\tilde{\mathscr{C}}_n \to$ $\mathrm{Mod}_\ell(\mathscr{D}_n)_{hr}^{x_1 \cdots x_n}$, which takes simple objects to simple objects (and is still denoted T_n).*

PROOF: By induction on n. We may assume $F \in \tilde{\mathscr{C}}_n$ to be simple. Let us say $F = L_\alpha E$, $\alpha \in J \cup \{1\}$, $E \in \tilde{\mathscr{C}}_{n-1}$ simple. Write $L = L_\alpha$, $K = K_\alpha$, $P = P_\alpha$. For each $i \in \{0, 1\}$ $\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{F}_{(n)}(i)$ is a flat $\mathscr{O}_{n-1}$-module. Hence in virtue of Lemma 1 we get $K(\mathscr{F}_n) = \mathscr{F}_{n-1} \underset{\mathscr{O}_{n-1}}{\otimes} (\mathscr{O} \underset{\mathscr{O}_{(n)}}{\otimes} \mathscr{D}_{(n)}/(q_n(\alpha))) =$ $\mathscr{C}_{n-1}(P)(\mathscr{F}_{n-1})$.

Lemma 4 applied to the equivalence $\mathrm{Hom}_{\mathbb{C}}(F, M) \otimes_{\mathscr{O}} N \cong \mathrm{Hom}_{\mathbb{C}}(F, M$

$\otimes_{\mathcal{O}} N)$, where F is a finite dimensional $\mathbb{C}$-vector space, $M \in \mathrm{Mod}(\mathcal{O})$, N a flat $\mathcal{O}$-module, gives

$$T_n(LE) \cong \mathrm{Hom}_{\mathscr{C}_{n-1}}(E, K(\mathscr{F}_n)) \cong \mathrm{Hom}_{\mathscr{C}_{n-1}}(E, \mathscr{C}_{n-1}(P)(\mathscr{F}_{n-1}))$$
$$\cong PT_{n-1}E$$

In fact these isomorphisms are $\mathscr{D}_n$-linear.
To exhibit the exactness of T_n we use

LEMMA 6: *Let* R: $\mathrm{Mod}_{\ell}(\mathscr{D}_n) \to \mathscr{C}_0$ *be an exact functor. Then* $\mathrm{Ext}^1_{\mathscr{C}_n}(E, \mathscr{C}_n(R)(\mathscr{F}_n)) = 0$, *all* $E \in \tilde{\mathscr{C}}_n$.

PROOF: According to lemma 1 $\mathscr{U}_{(n)}$, $\mathscr{V}_{n)}$ and $\mathscr{V}_{(n)}\mathscr{U}_{(n)} - \alpha 1$ are surjective, hence $R^1K(\mathscr{F}_n) = 0$. Because R is exact it commutes with K and $R^1K(\mathscr{C}_n(R)(\mathscr{F}_n)) = 0$. Hence according to Lemma 3 it follows that

$$\mathrm{Ext}^1_{\mathscr{C}_n}(LE, \mathscr{C}_n(R)(\mathscr{F}_n)) \cong \mathrm{Ext}^1_{\mathscr{C}_{n-1}}(E, \mathscr{C}_{n-1}(RP)(\mathscr{F}_{n-1})) = 0.$$

REMARK: According to Mitchell [4], Ch. VI, Corollary 4.2, (with $R = \mathbb{C}$) $\mathscr{C}_n$ is equivalent to a category of right modules over a certain ring of endomorphisms. (Recall, cf. §1, that $\mathscr{C}_n$ is a functor category of the kind mentioned in this Corollary.) Hence $\mathscr{C}_n$ has enough injectives.

§5. The equivalence of categories

In the preceding pages we have shown the existence of two contravariant exact functors

$$S_n: \mathrm{Mod}_{\ell}(\mathscr{D}_n)_{hr}^{x_1 \dots x_n} \to \tilde{\mathscr{C}}_n, \; T_n: \tilde{\mathscr{C}}_n \to \mathrm{Mod}_{\ell}(\mathscr{D}_n)_{hr}^{x_1 \dots x_n}$$

By some formal considerations it follows now that S_n defines an equivalence of categories with inverse T_n.

PROPOSITION 3: S_n *and* T_n *are inverse to each other.*

PROOF: First we mention the natural equivalence of $\mathbb{C}$-vector spaces $\mathrm{Hom}_{\mathbb{C}}(E, \mathrm{Hom}_{\mathscr{D}_n}(M, N)) \cong \mathrm{Hom}_{\mathscr{D}_n}(M, \mathrm{Hom}_{\mathbb{C}}(E, N))$, where $E \in \mathscr{C}_0$, M, $N \in \mathrm{Mod}_{\ell}(\mathscr{D}_n)$. By Lemma 4 there results a natural equivalence

$$\mathrm{Hom}_{\mathscr{C}_n}(E, \mathscr{C}_n(H_n)(M, F)) \cong \mathrm{Hom}_{\mathscr{D}_n}(M, \mathrm{Hom}_{\mathscr{C}_n}(E, F)),$$

where $E \in \mathscr{C}_n$, $M \in \mathrm{Mod}_l(\mathscr{D}_n)$, $F \in \mathscr{C}_n(\mathscr{D}_n)$.
So in particular we get a natural equivalence

$$\mathrm{Hom}_{\mathscr{C}_n}(E, S_n(M)) \cong \mathrm{Hom}_{\mathscr{D}_n}(M, T_n(E)),$$

where $E \in \tilde{\mathscr{C}}_n$, $M \in \mathrm{Mod}_{\ell}(\mathscr{D}_n)_{hr}^{x_1 \dots x_n}$.

Or, working in the dual category $\mathscr{C}_n^0$,

$$\mathrm{Hom}_{\mathscr{C}_n^0}(S_n^0(M),\ E) \cong \mathrm{Hom}_{\mathscr{D}_n}(M,\ T_n^0(E)).$$

Hence S_n^0 is a left adjoint of T_n^0. This gives rise to natural transformations ψ: $1 \to T_n^0 S_n^0 = T_n S_n$, ϕ: $S_n^0 T_n^0 \to 1$ and dual ϕ^0: $1 \to S_n T_n$.

Both S_n^0 and T_n^0 are exact and take simple objects to simple objects. Hence in particular both functors are faithful. Hence $\psi(M)$ and $\phi^0(E)$ are monomorphisms if $M \in \mathrm{Mod}_l(\mathscr{D}_n)_{hr}^{x_1 \cdots x_n}$, $E \in \tilde{\mathscr{C}}_n$. Hence both are isomorphisms in case the object is simple. So, by induction on the length, ψ and ϕ^0 are equivalences.

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