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**L_2 -COHOMOLOGY AND INTERSECTION HOMOLOGY
OF LOCALLY SYMMETRIC VARIETIES, II**

Steven Zucker

Introduction

In this paper, we continue our study of the L_2 -cohomology (with the customary coefficients $\mathbb{E}$, and natural metrics) of quotients $\Gamma \backslash X$ of symmetric spaces by arithmetic groups. Though it bears the title of [16], it is actually the sequel to [15]. (In fact, [16] is but a condensed account of [15].) The latter was conceived in an attempt to better understand the L_2 -cohomology of itself, and to obtain better injectivity results than were then available for the mapping of it into ordinary cohomology. However, it became almost immediately apparent that it was the conjecture [15, (6.20)] (see (3.2) here) – which indeed would imply sharp isomorphism ranges for the mapping – that was attracting most of the attention. The conjecture asserts, for X Hermitian, the natural isomorphism between the L_2 -cohomology of $\Gamma \backslash X$ and the (middle perversity) intersection homology [10] of its Baily-Borel-Satake compactification [1], denoted here $\Gamma \backslash_{\mathbb{Q}} X^*$ (a normal projective variety). There is now strong evidence that it is true.

The main value of the conjecture seems to lie in that it allows one to extend the conjectures of Langlands – on the relation between the Hasse-Weil zeta function of a Shimura variety (more generally, L -functions corresponding to certain local systems thereon) and L -functions of automorphic representations of the algebraic group G in which Γ sits – to the non-compact case, which includes most of the interesting examples. A Shimura variety, being in essence a model of $\Gamma \backslash X$ defined over a number field, admits reductions modulo primes, as does therefore its completion $\Gamma \backslash_{\mathbb{Q}} X^*$. Also, the sheaf-theoretic formulation of intersection homology [10, (3.1)] makes sense in étale cohomology. Since the automorphic forms for Γ have a clear relation to the L_2 -cohomology, it has been suggested that one use intersection homology in defining a zeta function for the singular completion. This has already been carried out successfully by Brylinski and Labesse in the case of Hilbert(-Blumenthal) modular varieties [8].

The conjecture is to be shown true for local reasons on $\Gamma \backslash_{\mathbb{Q}} X^*$. Each of the L_2 -cohomology and intersection homology is the hypercohomology of a complex of sheaves, denoted $\mathcal{L}_{(2)}(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$ and $\mathcal{IC}'(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$ respectively. Since $\mathcal{IC}'$ is characterized, up to quasi-isomorphism, by certain axioms [10,(6.1)], it is enough to verify that $\mathcal{L}_{(2)}(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$ satisfies these axioms. Concretely, this necessitates the computation of the L_2 -cohomology of (the intersection with $\Gamma \backslash X$ of) the members of a fundamental system of neighborhoods of points of $\Gamma \backslash_{\mathbb{Q}} X^*$ as the primary task. (One must prove the requisite vanishing theorems for these groups, plus a “duality” condition, to use the criterion cited in our (3.1).) This was done for $G = \mathrm{SU}(n, 1)$ (X the complex n -ball) and $G = R_{\mathbb{F}/\mathbb{Q}} \mathrm{SL}(2)$, where $\mathbb{F}$ is a totally real number field, ($\Gamma \backslash X$ a Hilbert modular variety) in [15,§6]. These are the simplest groups of $\mathbb{Q}$ -rank one. The proof in the general $\mathbb{Q}$ -rank one case has been carried out by Borel (see [4]).

The relevant features of the $\mathbb{Q}$ -rank one case are that $\Gamma \backslash_{\mathbb{Q}} X^*$ has only one singular stratum, and there exists a rather nice description of a fundamental neighborhood system of points thereon, as basic distinguished open sets (see [5,(10.1)], and [15,(3.18)]), for which the L_2 cohomology admits a calculation by a “Künneth formula” (see [4,§5], [15,§§4.5], and our (2.4)). When the $\mathbb{Q}$ -rank of G is r , $\Gamma \backslash_{\mathbb{Q}} X^*$ has r singular strata $S^{j(i)}$ ($i = 1, \dots, r$), whose codimensions $j(1), \dots, j(r)$ are arranged to form an increasing sequence. The connected components of these strata are arithmetic quotients of the so-called rational boundary components of X ; these are Hermitian symmetric spaces of lower rank, normalized by maximal parabolic subgroups of G . Along any of them, the geometry and Riemannian structure are locally constant. For points in $S^{j(1)}$, the fundamental neighborhoods are as in the $\mathbb{Q}$ -rank one case, and the verification of the axioms can proceed analogously. On the other hand, for the higher codimension strata, the fundamental neighborhoods are more difficult to deal with.

In this paper, we give two constructions of (equivalent) fundamental systems of neighborhoods of points on the various strata. Both can be described in terms of distinguished coordinates on the manifold with corners $\Gamma \backslash \bar{X}$ constructed in [5]. With this goal in mind, we had described $\Gamma \backslash_{\mathbb{Q}} X^*$ as the natural quotient of $\Gamma \backslash \bar{X}$ (leaving $\Gamma \backslash X$ itself unaltered), and identified the fibers as submanifolds with corners [17]. Assume that X is irreducible, and let Q be a maximal parabolic subgroup. The symmetric space of the Levi quotient of Q decomposes into two factors, one of which is the associated boundary component; the other will be tentatively denoted X_Q . The fiber F over a point of the corresponding stratum is then a nilmanifold fibration over an arithmetic quotient $\Gamma' \backslash X_Q$, contained in $e'(Q) \subset \Gamma \backslash \bar{X}$. Our neighborhoods are, of course, obtained from neighborhoods of these fibers in $\Gamma \backslash \bar{X}$. We can proceed from here in two ways:

(a) ((3.6)–(3.8)) Take a finite union of basic distinguished open sets. Then the L_2 cohomology of the pieces and their intersections can be computed by the methods of [15] (see (2.4)). One can then try to use a Mayer-Vietoris argument to calculate the L_2 -cohomology of the full neighborhood. This approach works particularly well on $S^{j(2)}$, for X_Q is a symmetric space (not necessarily Hermitian) of $\mathbb{Q}$ -rank one.

(b) ((3.16)–(3.19)) Express the neighborhood as a product of F , a disc and a half-line. In order to do this, we must make an explicit change of variables from the distinguished coordinates with respect to Q . We then write down a “Künneth formula” for the L_2 -cohomology of the neighborhood (3.19), in which the change of variables has introduced a weighting of the L_2 -cohomology of $\Gamma' \backslash X_Q$ that could be suppressed in (2.4). It would be nice to be able to calculate this weighted L_2 -cohomology without breaking the neighborhood into smaller pieces; the latter would come down to the method described in (a) above. (See, however, (3.23ff).)

Using the method (a), we verify the requisite vanishing theorem along $S^{j(2)}$ for $G = \mathrm{Sp}(2n)$ ($n > 1$; (3.10)–(3.12)) and by method (b) for $G = \mathrm{SU}(p, q)$ ($p \geq q \geq 2$; (3.25)–(3.28)). Coupled with results on duality and L_2 -cohomology, both general ((3.13)–(3.15)) and special ((3.20)), *this proves the conjecture for the rank two cases of the preceding: $\mathrm{Sp}(4)$ and $\mathrm{SU}(p, 2)$.* It is likely that, with sufficient energy, one could push this method through for all of the $\mathbb{Q}$ -rank two, $\mathbb{R}$ -rank two groups. With existing methods, this seems to be a rather tedious task; barring (heaven forbid!) a counterexample, – unless, of course, the need to use a particular instance arises – we feel that it might be wiser to seek a better understanding of the argument, even in the known cases, so that a proof in general may suggest itself.¹

Method (b) also has potential for use in higher rank as well. We illustrate this in (3.29)–(3.30) by *proving the conjecture for the most basic rank-three example, viz. $G = \mathrm{Sp}(6)$.*

The paper is divided into three large chapters, and numerous sub-sections. In §1, we recall some basic facts about compactifications of symmetric spaces and their arithmetic quotients. In particular, we review in (1.2) the construction of $\bar{X}$ in terms of parabolic subgroups and geodesic action, and in (1.6) the definition of a Satake compactification and the quotient mapping from $\Gamma \backslash \bar{X}$ onto $\Gamma \backslash_{\mathbb{Q}} X^*$. In (1.3) and (1.5), we introduce the notion of an *influx*, namely an open subset of X or $\Gamma \backslash X$ defined by a geodesic action and a cross-section thereto, generalizing the basic distinguished open sets.

In (2.1)–(2.3), we recall the basic definitions and properties of L_2 cohomology. This is followed by the calculation of the L_2 -cohomology of

¹ A proof for the general $\mathbb{Q}$ -rank two case is announced in [21] (see also [22]).

a basic distinguished open set (for a relatively compact subset of a face of $\Gamma \setminus \bar{X}$) in (2.4), where we assert that it is given by a “Künneth formula” (cf. [15,(4.20),(5.6)]). We have continued to use our “purely geometric” approach of [15,§4], rather than appealing to the representation-theoretic techniques introduced by Borel (see [4,§5], [8,§1]), for the possibility of its finding applications to other, non-homogeneous principal bundles has been suggested by Cheeger. As part of the exposition, we give in (2.5) a complete “abstract” account of the argument concerning complexes comprised of sums of Hilbert space tensor products that leads to the Künneth formula, in view of its importance in the proof, for it is used (correctly!) without being explicitly formulated in [15,p.200]. In (2.6) and (2.7), we obtain a rather easy generalization of the calculation: that it carries over to other influxes.

In §3, we begin with a summary of the basic properties of intersection homology in (3.1), and restate our conjecture (3.2)¹. We presume Borel’s calculation [4,(5.4)] that shows that L_2 -cohomology satisfies the axioms for intersection homology along $S^{j(1)}$, and pass to the other strata. The Mayer-Vietoris sequence mentioned earlier, which we use for $S^{j(2)}$, is discussed in (3.9). We give, in (3.21), a description of the link of the general stratum in the case $rk_{\mathbb{Q}}G = rk_{\mathbb{R}}G$, and use it to interpret the weighted L_2 -cohomology in (3.19) in the rank two cases mentioned above. The contents of the remaining sub-sections of §3 have already been summarized in sufficient detail.

We operate under two conventions:

1. Some of our results are easier to state when the $\mathbb{Q}$ -rank and $\mathbb{R}$ -rank of G are equal. We have chosen to assert them under this hypothesis, and then to comment, as unpedantically as possible, on their generalization.
2. The irreducible restricted root systems that occur for groups of Hermitian type are of classification type BC_r or C_r . In making calculations, we find it convenient to pretend that they are of type B_r : (using classical notation) we choose as simple “roots” $\beta_1 = \epsilon_1 - \epsilon_2, \dots, \beta_{r-1} = \epsilon_{r-1} - \epsilon_r$, $\beta_r = \epsilon_r$, even if this means that β_r has multiplicity zero!

With convention 2, we can state an interesting fact, which is bound to be relevant to the general conjecture, that we have proved in Appendix (A.1). *Let δ denote the half-sum of the positive roots. Then its restriction to a maximal $\mathbb{Q}$ -split torus is*

$$\delta = \sum_{i=1}^r j(i)\beta_i.$$

¹ Borel has extended the conjecture to a larger class of Satake compactifications of arithmetic quotients of symmetric spaces. See Appendix (A.2).

I want to thank Avner Ash, Armand Borel, Jiri Dadok and Vinay Deodhar for the helpful conversations.

Not so long ago, I came across some long-lost notes I had taken during an extended conversation with Robert Langlands one afternoon in January, 1979. The discussion included L_2 -cohomology on Γ/X . Having forgotten by now exactly how things then stood with my research plans, I find it impossible to assess the extent of my indebtedness to him. I want to thank him, retrospectively to [15], for, at the very least, encouraging my work in this direction, and also for general encouragement.

The first version of this paper, which contained in some form all but the contents of (3.21)–(3.23), (3.29), (3.30) and (A.2), was written in the congenial environment at Indiana University during academic year 1982/83.

§1. Compactifications of symmetric spaces and their quotients

(1.1). Let H be an algebraic group defined over the subfield $\mathbb{F}$ of the complex numbers. We write $H_{\mathbb{F}}$ for the $\mathbb{F}$ -valued points of H . We will be interested primarily in the cases $\mathbb{F} = \mathbb{Q}$ and $\mathbb{F} = \mathbb{R}$.

Let U_H denote the unipotent radical of H . Then

$$\hat{L}_H = H/U_H. \quad (1)$$

is the canonical (reductive) *Levi quotient* of H , also defined over $\mathbb{F}$ (see [13,p.413]).

We denote by ${}_{\mathbb{F}}\mathfrak{P}(H)$ the set of parabolic subgroups of H that are defined over $\mathbb{F}$ (*$\mathbb{F}$ -parabolic subgroups*, for short). By convention, we do not exclude $H \in {}_{\mathbb{F}}\mathfrak{P}(H)$, the improper parabolic subgroup. Every member of ${}_{\mathbb{F}}\mathfrak{P}(H)$ contains U_H , so there is then a natural one-to-one correspondence between ${}_{\mathbb{F}}\mathfrak{P}(H)$ and ${}_{\mathbb{F}}\mathfrak{P}(\hat{L}_H)$; and likewise with ${}_{\mathbb{F}}\mathfrak{P}(\hat{L}_H/\hat{C}_H)$, where $\hat{C}_H$ is the center of $\hat{L}_H$.

Let $\hat{T}_H$ denote a maximal $\mathbb{F}$ -split torus of $\hat{L}_H/\hat{C}_H$, and let ${}_{\mathbb{F}}\hat{\Delta}_H$ be a system of positive simple roots with respect to $\hat{T}_{\mathbb{F}}$. One has the notion of the *standard $\mathbb{F}$ -parabolic subgroups* of $\hat{L}_H/\hat{C}_H$ (hence also of H) relative to ${}_{\mathbb{F}}\hat{T}$ and ${}_{\mathbb{F}}\hat{\Delta}_H$ (see [6,(4.2,ii)]). These are denoted ${}_{\mathbb{F}}\hat{P}_{\Theta}$ (resp. ${}_{\mathbb{F}}P_{\Theta}$) for $\Theta \subseteq {}_{\mathbb{F}}\hat{\Delta}_H$. We write U_{Θ} instead of $U_{{}_{\mathbb{F}}P_{\Theta}}$. The standard $\mathbb{F}$ -parabolic subgroups are partially ordered by inclusion, and the correspondence of Θ to ${}_{\mathbb{F}}P_{\Theta}$ defines a lattice isomorphism between the set of subsets of ${}_{\mathbb{F}}\hat{\Delta}_H$ and the set of standard $\mathbb{F}$ -parabolic subgroups of H . Moreover, every element of ${}_{\mathbb{F}}P(H)$ can be written in the form ${}^h({}_{\mathbb{F}}P_{\Theta})$ for some $\Theta \subseteq {}_{\mathbb{F}}\hat{\Delta}_H$ and $h \in H_{\mathbb{F}}$, where Θ is uniquely determined (see [6,(4.6),(4.13c)]); here and elsewhere, hS denotes hSh^{-1} whenever $S \subseteq H$.

(1.2). Let G be a semi-simple algebraic group defined over $\mathbb{Q}$, and let X be the associated symmetric space of maximal compact subgroups of

$G_{\mathbb{R}}$. According to [5], X can be realized as the interior of a manifold with corners $\bar{X}$. We recall the construction of the latter.

Let $P \in {}_{\mathbb{F}}\mathfrak{P}(G)$, and let ${}_{\mathbb{F}}\hat{S}_P$ denote the maximal $\mathbb{F}$ -split torus of $\hat{C}_P$. We write

$${}_{\mathbb{F}}\hat{A}_P = ({}_{\mathbb{F}}\hat{S}_P)_{\mathbb{R}}^0, \quad (1)$$

the identity component with respect to the classical topology. There is a *geodesic action* of ${}_{\mathbb{F}}\hat{A}_P$ on X [5, §3]¹, whose definition is as follows. To each $x \in X$ is associated the Cartan involution θ_x of G that acts trivially on the corresponding maximal compact subgroup. There is a unique θ_x -stable lifting ${}_{\mathbb{F}}A_{P,x}$ of ${}_{\mathbb{F}}\hat{A}_P$ (and likewise $L_{P,x}$ of $(\hat{L}_P)_{\mathbb{R}}$) to $P_{\mathbb{R}}$. One then defines the geodesic action by the formula

$$a \circ x = a_x \cdot x, \quad (2)$$

where $a_x \in {}_{\mathbb{F}}A_{P,x}$ is the lifting of $a \in {}_{\mathbb{F}}\hat{A}_P$.² The geodesic action of ${}_{\mathbb{F}}\hat{A}_P$ commutes with the usual (transitive) action of $P_{\mathbb{R}}$ on X . One puts

$${}_{\mathbb{F}}e(P) = {}_{\mathbb{F}}\hat{A}_P \backslash X. \quad (3)$$

We also let ${}_{\mathbb{F}}\hat{e}(P)$ denote the quotient of ${}_{\mathbb{F}}e(P)$ by the action of $(U_P)_{\mathbb{R}}$, with projection

$${}_{\mathbb{F}}P_P : {}_{\mathbb{F}}e(P) \rightarrow {}_{\mathbb{F}}\hat{e}(P). \quad (4)$$

We consider now the case $\mathbb{F} = \mathbb{Q}$, for which we will drop the left-subscript $\mathbb{Q}$. One adjoins $e(P)$ to X as the set of limits, in the positive direction defined by P , of the $\hat{A}_P$ orbits in X under the geodesic action. (For a more precise statement of this, see [5, (5.1)].) Then

$$\bar{X} = \coprod_{P \in {}_{\mathbb{Q}}\mathfrak{P}(G)} e(P) \quad (5)$$

has a natural structure of a C^∞ manifold with corners, such that the usual action of $G_{\mathbb{Q}}$ on X extends to define diffeomorphisms of $\bar{X}$ ([5, (7.6)]). We remark that the codimension of the subspace $e(P)$ in $\bar{X}$ equals the dimension of $\hat{A}_P$, sometimes called the *parabolic $\mathbb{Q}$ -rank* of P .

For $Q \in {}_{\mathbb{Q}}\mathfrak{P}(G)$, the space $e(Q)$ is of type $S - \mathbb{Q}$ (in the sense of [5, (2.3)]), and as such allows for a parallel construction of a manifold with corners. One has that

$$\overline{e(Q)} = \coprod_{P \in {}_{\mathbb{Q}}\mathfrak{P}(Q)} e(P) = \coprod_{Q \supseteq P \in {}_{\mathbb{Q}}\mathfrak{P}(G)} e(P) \quad (6)$$

¹ The convention in [5] is that G acts on X on the *right*. With G acting on the left, we must replace a by a^{-1} in all formulas from [3] and [5] that involve $\hat{A}_P$.

² By transport of structure, one could say that ${}_{\mathbb{F}}A_{P,x}$ also acts geodesically on X .

is diffeomorphic to the closure of $e(Q)$ in $\bar{X}$ [5,(7.3,i)]. In fact, whenever $P \subset Q$, one can identify $\hat{A}_Q$ canonically as a subgroup of $\hat{A}_P$, such that the geodesic actions are compatible [5,(3.11)]. Then, through $\hat{A}_P/\hat{A}_Q$, $\hat{A}_P$ acts geodesically on $e(Q)$, with quotient $e(P)$ (see (3)); and $e(P)$ sits inside $\overline{e(Q)}$ in (6) as the set of limit points of this geodesic action.

The principal $(U_P)_{\mathbb{R}}$ -fibration (4) extends to give a principal fibration (see [5,(7.2)])

$$\bar{p}_P: \overline{e(P)} \rightarrow \overline{\hat{e}(P)}. \quad (7)$$

(1.3). Let

$$q_P: X \rightarrow e(P) \quad (1)$$

denote the quotient mapping. For any open subset V of $e(P)$, a cross-section σ of (1) over V determines, in view of (1.2(5)), a translation of V from the boundary of $\bar{X}$ into the interior X .

For any $t \in \hat{A}_P$, we put

$$\hat{A}_P(t) = \{a \in \hat{A}_P: a^\alpha > t^\alpha \text{ for all } \alpha \in \Delta_P\}. \quad (2)$$

Here, Δ_P is the set of those simple roots, with respect to a lifting of $\hat{T}_P$, that occur in U_P (transported back to $\hat{A}_P$); it is complementary to $\hat{\Delta}_P$ (defined in (1.1)).

For any cross-section $\sigma(V)$, a set of the form

$$W(V, \sigma, t) = \hat{A}_P(t) \circ \sigma(V) \quad (3)$$

will be called an *open set defined by geodesic influx from V into X* . We remark that the set (3) is a domain with corners in X , and that σ can be recovered (up to $\hat{A}_P$ -translation) from the lowest-dimensional face of the boundary. For any $t_0 \in \hat{A}_P$, the collection $\{W(V, \sigma, t): t \geq t_0\}$ of such sets will be called also a *system of collars of V determined by σ* .

Let 0P denote the subgroup of P defined as the intersection of the kernels of the absolute values of the rational characters of P , as in [5]. Then

$$({}^0P)_{\mathbb{R}} \cdot x \quad (4)$$

defines the *canonical cross-section* $\sigma_{P,x}$ over all of $e(P)$ to the geodesic action of $\hat{A}_P$, given in [5,(5.4)], and the resulting sets defined by geodesic influx are precisely the basic *distinguished neighborhoods* of [5,(10.1)]. In this case, we write $W_x(V, t)$ for $W(V, \sigma_{P,x}, t)$. The following relation is easy to verify:

$$g \cdot W_x(V, t) = W_{g \cdot x}(gV, {}^g t) \quad \text{if } g \in G_{\mathbb{Q}}. \quad (5)$$

For any cross-section σ , the geodesic action of $\hat{A}_p(t)$ upon $\sigma(V)$ defines what we call *distinguished coordinates*; there is a natural isomorphism (cf. (3)):

$$\mu_\sigma: \hat{A}_p(t) \times V \xrightarrow{\sim} W(V, \sigma, t). \quad (6)$$

In the construction of $\bar{X}$, one adjoins points to $\hat{A}_p$ where the elements of Δ_p are permitted to take on the value ∞ , giving a space $\bar{A}_p$ that is diffeomorphic to $(0, \infty]^{\Delta_p}$. The justification for the terminology for (3) is that μ_σ extends to a diffeomorphism

$$\bar{\mu}_\sigma: \bar{A}_p(t) \times V \xrightarrow{\sim} \bar{W}(V, \sigma, t), \quad (7)$$

where $\bar{A}_p$ is defined as in (3) with $\bar{A}_p$ replacing $\hat{A}_p$. Then $\bar{W}(V, \sigma, t)$ is a neighborhood of V in $\bar{X}$, with $\bar{\mu}_\sigma(\{(\infty, \dots, \infty)\} \times V) = V$.

We have, more generally, quotient mappings

$$q_{Q,P}: e(Q) \rightarrow e(P) \quad (8)$$

whenever $P \subset Q$, and corresponding notions of geodesic influx from subsets of $e(P)$ into $e(Q)$, distinguished neighborhoods, etc. The following simple observation will be very useful:

LEMMA: *Let V be an open subset of $e(P)$, and let $t \in \hat{A}_p$. Then $W_x(V, t)$ is an open set defined by geodesic influx from its image in $e(Q)$. The latter can be written as*

$$q_Q W_x(V, t) = W_{x_Q}(V, t), \quad (9)$$

where $x_Q = p_Q(x)$.

REMARK: It is definitely *not* the case in general that $W_x(V, t)$ is the restriction of the geodesic influx defined by the canonical cross-section $\sigma_{Q,x}$ (cf. [15,(3.19)]), for the following reason. One can identify Δ_Q as a subset of Δ_p . As in [5,(4.3)], we let

$$\hat{A}_{p,Q} = \{a \in \hat{A}_p: a^\alpha = 1 \text{ for all } \alpha \in \Delta_Q\}. \quad (10)$$

With the identification

$$\hat{A}_Q = \{a \in \hat{A}_p: a^\alpha = 1 \text{ for all } \alpha \notin \Delta_Q\}, \quad (11)$$

we have clearly

$$\hat{A}_p \simeq \hat{A}_{p,Q} \times \hat{A}_Q. \quad (12)$$

Decomposing t accordingly as (t_1, t_2) , we get

$$\hat{A}_P(t) \simeq \hat{A}_{P,Q}(t_1) \times \hat{A}_Q(t_2) \quad (13)$$

(which gives the lemma above). For any $x \in X$, (13) can be lifted to P to give

$$A_{P,x} \simeq A_{P,Q,x} \times A_{Q,x},$$

thus defining $A_{P,Q,x}$. We can also write

$$\hat{A}_P \circ ({}^0P)_{\mathbb{R}} \cdot x = \hat{A}_Q \circ B_{P,Q,x} \cdot ({}^0P)_{\mathbb{R}} \cdot x, \quad (14)$$

where

$$B_{P,Q,x} = A_{P,x} \cap ({}^0Q)_{\mathbb{R}}.$$

Let $\hat{B}_{P,Q}$ denote the common projection of $B_{P,Q,x}$ in $\hat{A}_P$ for any $x \in X$. One has also

$$\hat{A}_P \simeq \hat{B}_{P,Q} \times \hat{A}_Q. \quad (15)$$

However, unless Δ_Q is a component of Δ_P , $\hat{A}_{P,Q}$ and $\hat{B}_{P,Q}$ are different, and one cannot replace $\hat{A}_{P,Q}$ by $\hat{B}_{P,Q}$ in (13).

(1.4). Let Γ be an arithmetic subgroup of $G_{\mathbb{Q}}$. If Γ is torsion-free, as we will assume, the quotient $\Gamma \backslash X$ is a manifold. We remark that any arithmetic group Γ contains normal torsion-free subgroups of finite index.

The natural action of Γ on $\bar{X}$ is proper and discontinuous [5,(9.3)]. One has that if $\gamma \in \Gamma$, and $P \in {}_{\mathbb{Q}}\mathfrak{P}(G)$, then, in terms of (1.2(5)),

$$\gamma \cdot e(P) = e({}^{\gamma}P). \quad (1)$$

In particular, $\Gamma_P = \Gamma \cap P$ is the normalizer in Γ of $e(P)$ (and also of $e({}^{\gamma}P)$), and the action of Γ_P on $e(P)$ is the one coming from the natural P -homogeneity of $e(P)$. Put, as in [5,(9.4)],

$$e'(P) = \Gamma_P \backslash e(P). \quad (2)$$

Then one has

$$\Gamma \backslash \bar{X} = \coprod_{P \in \mathfrak{R}(\Gamma)} e'(P), \quad (3)$$

where $\mathfrak{R}(\Gamma)$ is a (necessarily finite [2,(15.6)]) set of Γ -conjugacy class

representatives in ${}_Q\mathfrak{P}(G)$. The space $\Gamma \backslash \bar{X}$ is a compact manifold with corners [5,(9.3)].

Put $\Gamma_{U_P} = \Gamma \cap U_P$, and let $\Gamma_{\hat{L}_P}$ denote the projection Γ_P/Γ_{U_P} of Γ_P in $\hat{L}_P$; it is arithmetic in $\hat{L}_P$ (see [2,(7.13)]). If $\Gamma_{\hat{L}_P}$ is also torsion-free, then the fibration p_P (1.2(4)) gives rise to the fibration

$$p'_P: e'(P) \rightarrow \hat{e}'(P), \quad (4)$$

where $\hat{e}'(P) = \Gamma_{\hat{L}_P} \backslash \hat{e}(P)$, whose fiber is the compact manifold $\Gamma_{U_P} \backslash U_P$. Thus, if Γ is *neat* in the sense of [2,(17.1)], each face $e'(P)$ of the manifold with corner admits such a fibration. Again, any arithmetic group contains neat normal subgroups of finite index [2,(17.4)].

(1.5). Let V' be an open subset of $e'(P)$, and let V denote its inverse image in $e(P)$ via the quotient mapping π_P . Suppose that σ is a Γ_P -invariant cross-section of (1.3(1)), e.g., $\sigma_{P,x}$ for any $x \in X$. Then $W(V, \sigma, t)$ (1.3(4)) is Γ_P -invariant, and we put

$$W'(V', \sigma, t) = \Gamma_P \backslash W(V, \sigma, t). \quad (1)$$

PROPOSITION: *Let V' be a relatively compact open subset of $e'(P)$. Then if $t \in \hat{A}_P$ is sufficiently large, the equivalence relations defined on $W(V, \sigma, t)$ by Γ and Γ_P are the same. For such t , $W'(V', \sigma, t) \simeq \pi W(V, \sigma, t)$, and the isomorphism μ_σ (1.3(6)) induces an isomorphism*

$$\mu'_\sigma: \hat{A}_P(t) \times V' \xrightarrow{\sim} W'(V', \sigma, t). \quad (2)$$

The geodesic action of $\hat{A}_P(1)$ on $W(V, \sigma, t)$ descends to an action on $W'(V', \sigma, t)$, and q_P determines a mapping

$$q'_P: W'(V', \sigma, t) \rightarrow e'(P) \quad (3)$$

PROOF: (See and rework [5,(10.3)].)

In case $\sigma = \sigma_{P,x}$, we write $W'_x(V', t)$ for $W'(V', \sigma_{P,x}, t)$, and μ'_x instead of $\mu'_{\sigma_{P,x}}$. In view of the last assertion of the proposition, we may speak of geodesic influx and corresponding collars of relatively compact subsets of the faces of $\Gamma \backslash \bar{X}$. As in (1.3(7)), there are extensions of (2) to the boundary:

$$\bar{\mu}'_\sigma: \bar{A}_P(t) \times V' \xrightarrow{\sim} \bar{W}'(V', \sigma, t). \quad (4)$$

REMARK: Since $\overline{e'(P)}$ is compact, it follows that there is some neighborhood of $e'(P)$ in $\Gamma \backslash \bar{X}$, whose closure is a neighborhood of $\overline{e'(P)}$, to

which the geodesic action of $\hat{A}_p(1)$, and hence the projection q_p , descend. For a description of such a neighborhood, see (3.18).

(1.6) We recall now the Satake compactifications of $\Gamma \backslash X$ and their relation to the manifold with corners $\Gamma \backslash \bar{X}$, following [17].

Let ${}_{\mathbb{R}}\Delta$ denote a set of simple real roots for G . For each non-empty subset Ξ of ${}_{\mathbb{R}}\Delta$, Satake constructed a compactification X_{Ξ}^* of X , whose structure is recalled in [17, §2]. Under two hypotheses concerning the interaction of the $\mathbb{Q}$ -structure of G , one can retopologize a certain subset ${}_{\mathbb{Q}}X_{\Xi}^*$ of X_{Ξ}^* , so that it becomes a quotient space of $\bar{X}$, which we have called (for good reason) a manifold with crumpled corners [17, (3.7)]. Then Γ acts properly on ${}_{\mathbb{Q}}X_{\Xi}^*$, and the quotient, which we call a Satake compactification of $\Gamma \backslash X$, is compact. One has a natural continuous surjection

$$p^{*'}: \Gamma \backslash \bar{X} \rightarrow \Gamma \backslash {}_{\mathbb{Q}}X_{\Xi}^*. \quad (1)$$

While it is possible to describe the structure of $p^{*'}$ in the above generality, we specialize now to the case in which we are interested.

Assume henceforth that X is Hermitian, i.e., has a $G_{\mathbb{R}}$ -invariant complex structure. Suppose first that X , or equivalently ${}_{\mathbb{R}}\Delta$, is irreducible. Then the root system is of classification type BC_r or C_r (see [1, (1.2)]), for which the diagram of ${}_{\mathbb{R}}\Delta$ is

$$\bigcirc_{\beta_1} - \bigcirc_{\beta_2} - \dots - \bigcirc_{\beta_{r-1}} = \bigcirc_{\beta_r}; \quad (2)$$

and β_r is shorter (resp. longer) than the other simple roots. The choice $\Xi = \{\beta_r\}$ gives as X^* (we drop the subscript Ξ) the closure of the realization of X as a bounded domain. If X is reducible, one merely takes the product of the compactifications of the irreducible factors. The compactification $\Gamma \backslash {}_{\mathbb{Q}}X^*$ is, in this case, homeomorphic to the closure of the image of $\Gamma \backslash X$ under the embedding in complex projective space defined by automorphic forms of sufficiently high weight [1, (10.11)], and thereby inherits the structure of a normal algebraic variety. We refer to it as the *Baily-Borel-Satake compactification* of $\Gamma \backslash X$.

We will usually assume that a maximal $\mathbb{Q}$ -split torus of G is also maximal $\mathbb{R}$ -split (and we choose one for reference), so that ${}_{\mathbb{Q}}\Delta = {}_{\mathbb{R}}\Delta$. There is then no loss in generality if we take X to be irreducible (see [1, 3.6]), as the general case is almost a direct product of irreducible factors.

For any integer s , with $0 \leq s \leq r$, we put

$$\Theta_s = \{\beta_j : s < j \leq r\}, \quad (3)$$

$$\Lambda_s = \{\beta_j : j \neq s\}. \quad (4)$$

The proper boundary components of ${}_{\mathbb{Q}}X^*$ are parametrized by the maximal $\mathbb{Q}$ -parabolic subgroups of G , i.e. those of type Λ_s for some $s > 0$, though they are more naturally described in terms of the $\mathbb{Q}$ -parabolic subgroups of type Θ_s (for the same s). Specifically, we can write

$${}_{\mathbb{Q}}X^* = \bigcup \hat{e}({}^g P_{\Theta_s}), \quad (5)$$

where the union is taken over all s , and all $g \in G_{\mathbb{Q}}/(P_{\Theta_s})_{\mathbb{Q}}$; however, if g and h are in the same $(P_{\Lambda_s})_{\mathbb{Q}}/(P_{\Theta_s})_{\mathbb{Q}}$ coset, then $\hat{e}({}^g P_{\Theta_s})$ and $\hat{e}({}^h P_{\Theta_s})$ are identified. Putting

$$\Psi_s = \{ \beta_j : s < j \}, \quad (6)$$

we have the decomposition

$$\Lambda_s = \Psi_s \amalg \Theta_s \quad (7)$$

into connected components.

In discussing the mapping (1), we first look at the mapping (to be described)

$$p^* : \bar{X} \rightarrow {}_{\mathbb{Q}}X^*, \quad (8)$$

and then just take the natural quotients by Γ . By homogeneity, it is enough to say what happens to the face $e(P_{\Theta})$ in (1.2(5)) corresponding to the standard parabolic subgroup P_{Θ} . Let Θ_s be the largest set of the form (3) contained in Θ . It follows that $\Theta \subseteq \Lambda_s$. We can now state: the restriction of p^* to $e(P_{\Theta})$ is given by the composite of natural mappings

$$e(P_{\Theta}) \rightarrow \hat{e}(P_{\Theta}) \simeq \hat{e}(P_{\Theta_s}) \times \hat{e}(P_{\Theta \cap \Psi_s}) \rightarrow \hat{e}(P_{\Theta_s}); \quad (9)$$

the topology of ${}_{\mathbb{Q}}X^*$ is the quotient topology induced by p^* . Over the subspace $\hat{e}(P_{\Theta_s})$ of ${}_{\mathbb{Q}}X^*$, the mapping p^* is given by the projection

$$\begin{aligned} (\bar{p}_{P_{\Lambda_s}})^{-1}(\hat{e}(P_{\Theta_s}) \times \overline{\hat{e}(P_{\Psi_s})}) &\subseteq \overline{\hat{e}(P_{\Lambda_s})} \\ &\downarrow \\ \hat{e}(P_{\Theta_s}) \end{aligned} \quad (10)$$

The group P_{Λ_s} is the normalizer in G of $\hat{e}(P_{\Theta_s}) \subset {}_{\mathbb{Q}}X^*$, and P_{Ψ_s} acts trivially thereon. Let $\hat{e}''(P_{\Theta_s})$ denote the quotient $\Gamma_{P_{\Lambda_s}} \backslash \hat{e}(P_{\Theta_s})$ (it has $\hat{e}'(P_{\Theta_s})$ as a finite covering space). We can finally assert:

PROPOSITION: *Let $y \in \hat{e}''(P_{\Theta_s}) \subset \Gamma \backslash {}_{\mathbb{Q}}X^*$. Then $(p^{*'})^{-1}y$ is differentially a $\Gamma_{U_{\Lambda_s}} \backslash U_{\Lambda_s}$ fibration over $\hat{e}'(P_{\Psi_s})$.*

In case $s = r$, where Θ_s is empty, $\hat{e}(P_{\Theta_s})$ is just a point, though a very nasty singularity on $\Gamma \backslash_{\mathbb{Q}} X^*$. We state for emphasis:

COROLLARY: *Let $\{y\} = \hat{e}'(P_{\emptyset}) \subset \Gamma \backslash_{\mathbb{Q}} X^*$. Then there is a natural identification*

$$(p^{*'})^{-1}y = \overline{e'(P_{\Lambda_r})}. \quad (11)$$

REMARK: We stress that these formulas are valid as given only under the assumption that the $\mathbb{Q}$ - and $\mathbb{R}$ -ranks of G are equal. From [2,(3.7)–(3.8)] and [17,(3.8)], we can see how to revise them in general. If (2) now describes the $\mathbb{Q}$ -root system, we replace $\hat{e}(P_{\Theta_s})$ by ${}_{\mathbb{R}}\hat{e}(\tilde{P}_s)$ in (10), where $\tilde{P}_s$ is a certain $\mathbb{R}$ -parabolic subgroup of P_{Θ_s} ; and if $s = r$, ${}_{\mathbb{R}}\hat{e}(\tilde{P}_r)$ need not be a point.

§2. L_2 -cohomology

(2.1). We begin by recalling the definition of L_2 -cohomology. Let M be a C^∞ Riemannian manifold. Let $\mathbb{E}$ be the locally constant sheaf of germs of horizontal sections of a flat complex vector bundle on M ; we assume the latter is equipped with an Hermitian metric (not necessarily flat), and call $\mathbb{E}$ a *metrized local system*. We denote by $A'(M, \mathbb{E})$ the complex of C^∞ $\mathbb{E}$ -valued differential forms on M .

The metrics on M and $\mathbb{E}$ determine L_2 semi-norms on each $A'(M, \mathbb{E})$:

$$\|\phi\|^2 = \int_M |\phi|^2 dV_M, \quad (1)$$

where dV_M is the volume density of M . We then put

$$A'_{(2)}(M, \mathbb{E}) = \{\phi \in A'(M, \mathbb{E}) : \|\phi\| \text{ and } \|d\phi\| \text{ are finite}\}. \quad (2)$$

By construction, $A'_{(2)}(M, \mathbb{E})$ is a subcomplex of $A'(M, \mathbb{E})$, and the L_2 -cohomology, $H'_{(2)}(M, \mathbb{E})$, is defined to be the cohomology of $A'_{(2)}(M, \mathbb{E})$. One notes immediately that $A'_{(2)}(M, \mathbb{E})$, and hence $H'_{(2)}(M, \mathbb{E})$, depends only on the quasi-isometry classes of the metrics.

Alternatively, let $L_2^i(M, \mathbb{E})$ denote the completion of $A'_{(2)}(M, \mathbb{E})$ with respect to the L_2 norm. Then d determines a densely-defined closed operator

$$d: L_2^i(M, \mathbb{E}) \rightarrow L_2^{i+1}(M, \mathbb{E}), \quad (3)$$

thus imparting $L_2^*(M, \mathbb{E})$ with the structure of a complex of Hilbert spaces. The inclusion

$$A'_{(2)}(M, \mathbb{E}) \rightarrow L_2^*(M, \mathbb{E}) \quad (4)$$

induces an isomorphism on cohomology [9,§8].

If g is a positive continuous function on M , one has the notion of the *weighted L_2 complex*

$$L_2^\bullet(M, \mathbb{E}; g),$$

and corresponding weighted L_2 -cohomology $H_{(2)}^\bullet(M, \mathbb{E}; g)$, in which one multiplies the volume density of M by g before computing the L_2 semi-norm of a form (see (1)). This process can be interpreted as multiplying the metric on $\mathbb{E}$ by the function g , hence weighted L_2 cohomology is covered by our earlier discussion.

(2.2). We summarize the basic properties of L_2 -cohomology (see [9], [15] or [16]). The inclusion

$$A_{(2)}^\bullet(M, \mathbb{E}) \subset A^\bullet(M, \mathbb{E})$$

induces a homomorphism

$$H_{(2)}^\bullet(M, \mathbb{E}) \rightarrow H^\bullet(M, \mathbb{E}). \quad (1)$$

If M is a compact Riemannian manifold-with-boundary (corners allowed), then (1) is an isomorphism. Let $\mathcal{H}_2^i(M, \mathbb{E})$ denote the subspace of $A_{(2)}^i(M, \mathbb{E})$ consisting of these forms which are harmonic in a strict operator sense¹ (see [15,(1.10)]; if M is complete, without boundary, then $\mathcal{H}_2^\bullet(M, \mathbb{E})$ is just the space of L_2 solutions of the associated Laplace equation, with no extra "boundary" conditions). This space injects into $H_{(2)}^i(M, \mathbb{E})$, and it maps isomorphically if and only if $dL_2^{-1}(M, \mathbb{E})$ is a closed subspace of $L_2^i(M, \mathbb{E})$; otherwise, it has infinite algebraic codimension.

Let $f \in A^0(M, \mathbb{C})$ be a bounded function. Then multiplication by f leaves $A_{(2)}^\bullet(M, \mathbb{E})$ invariant if and only if the Riemannian length $|df|$ is also a bounded function on M . A locally finite open covering $\mathfrak{B}$ of M will be called *L_2 -admissible* if there exists a partition of unity subordinate to $\mathfrak{B}$ consisting of functions of bounded differential. If $\mathfrak{B}$ is L_2 -admissible, there is a spectral sequence (see [15,(5.4,iii)])

$$E_2^{p,q} = H^p(\mathfrak{B}, H_{(2)}^q(\cdot, \mathbb{E})) \Rightarrow H_{(2)}^{p+q}(M, \mathbb{E}). \quad (2)$$

In case $\mathfrak{B}$ has only two elements, (2) gives rise to the Mayer-Vietoris sequence for L_2 -cohomology.

The following simple observation is convenient:

¹ These form a subspace of the closed and coclosed L_2 harmonic forms.

PROPOSITION: (i) If $p: M \rightarrow M'$ is a finite Riemannian covering, $\mathbb{E}'$ a metrized local system on M' , and $\mathbb{E} = p^{-1}\mathbb{E}'$, then

$$p^*: H_{(2)}^*(M', \mathbb{E}') \rightarrow H_{(2)}^*(M, \mathbb{E})$$

is defined and is injective.

(ii) If M' is the quotient of M by a finite group J , then

$$H_{(2)}^*(M', \mathbb{E}') \simeq H_{(2)}^*(M, \mathbb{E})^J.$$

The proof is standard, and is omitted.

(2.3). The association

$$W \mapsto A'_{(2)}(W, \mathbb{E})$$

for W open in M , defines a presheaf on M , whose 0-th Čech cohomology is, unfortunately, $A^i(M, \mathbb{E})$. In order to sheafify L_2 cohomology, we have used the following device.

Let $\bar{M}$ be a compactification of M (as a Hausdorff topological space). Then the association

$$W \mapsto A'_{(2)}(W \cap M, \mathbb{E}), \quad (1)$$

for W open in $\bar{M}$, defines a presheaf on $\bar{M}$ that is denoted $\mathcal{A}'_{(2)}(\bar{M}, \mathbb{E})$. Since $\bar{M}$ is compact, we have

$$H^0(\bar{M}, \mathcal{A}'_{(2)}(\bar{M}, \mathbb{E})) \simeq A'_{(2)}(M, \mathbb{E}). \quad (2)$$

In analogy with the discussion in (2.2), suppose that $\bar{M}$ has a cofinal set of L_2 -admissible (in the sense that the restriction to M is L_2 -admissible) open coverings. Then $\mathcal{A}'_{(2)}(\bar{M}, \mathbb{E})$ is a fine sheaf, and by standard arguments

$$H^q(\bar{M}, \mathcal{A}'_{(2)}(\bar{M}, \mathbb{E})) = 0 \quad \text{for } q > 0. \quad (3)$$

Thus,

$$H_{(2)}^*(M, \mathbb{E}) \simeq H^*(\bar{M}, \mathcal{A}'_{(2)}(\bar{M}, \mathbb{E})), \quad (4)$$

so we have written the L_2 -cohomology as the hypercohomology of a complex of sheaves on $\bar{M}$.

Using (2.1(3)) instead of (2.1(2)), one can define $\mathcal{L}_{(2)}^{\cdot}(\overline{M}, \mathbb{E})$, the non-smooth analogue of $\mathcal{A}_{(2)}^{\cdot}(\overline{M}, \mathbb{E})$. By (2.1(4)), one sees that the inclusion

$$\mathcal{A}_{(2)}^{\cdot}(\overline{M}, \mathbb{E}) \rightarrow \mathcal{L}_{(2)}^{\cdot}(\overline{M}, \mathbb{E})$$

is a quasi-isomorphism.

(2.4). Let X be the symmetric space of the semi-simple algebraic group G , as in (1.2), and suppose that a representation ρ of G , on the finite dimensional complex vector space E , is specified. For any torsion-free arithmetic (discrete) subgroup Γ of $G_{\mathbb{Q}}$, E determines a locally constant sheaf $\mathbb{E}$ on $\Gamma \backslash X$.

We fix some notation. Let K_x denote the maximal compact subgroup of $G_{\mathbb{R}}$ corresponding to $x \in X$. If P is a $\mathbb{Q}$ -parabolic subgroup of G , we put

$$K_{P,x} = K_x \cap P_{\mathbb{R}},$$

$$M_{P,x} = L_{P,x} \cap ({}^0P)_{\mathbb{R}}.$$

Then $K_{P,x}$ is maximal compact in $M_{P,x}$, and the symmetric space of $M_{P,x}$ is canonically isomorphic to $\hat{e}(P)$. The orthogonal complement of the Lie algebra of $K_{P,x}$ in that of $M_{P,x}$ ($\mathfrak{m}_{P,x}$) is denoted $\mathfrak{p}_{M,x}$. (We will sometimes suppress the subscript x .)

Let $\mathfrak{u}_P$ (resp. $\mathfrak{a}_{P,x}$) denote the Lie algebra of U_P (resp. $A_{P,x}$). One writes $\delta_P \in \mathfrak{a}_{P,x}^*$ for one-half the sum of the weights (counted with multiplicity) of $\mathfrak{u}_P$ with respect to $\mathfrak{a}_{P,x}$. The Lie algebra cohomology $H^*(\mathfrak{u}_P, E)$, and then the $\mathfrak{a}_{P,x}$ weight spaces $H_{\mu}^*(\mathfrak{u}_P, E)$ are representations of $M_{P,x}$. Consequently, we have locally constant sheaves, denoted $\mathbb{H}_{\mu}^k(\mathfrak{u}_P, E)$, on $\hat{e}'(P)$ whenever $\Gamma_{\hat{L}_P}$ is torsion-free.

The following is the basic calculation of L_2 -cohomology on subsets of $\Gamma \backslash X$:

THEOREM: *Let $\hat{O}$ be the interior of a relatively compact domain with corners in $\hat{e}'(P)$, $V' = (p'_P)^{-1}(\hat{O}) \subset e'(P)$, and $x \in X$. If $\Gamma_{\hat{L}_P}$ is torsion-free, then*

$$H_{(2)}^i(W'_x(V', t), \mathbb{E}) \simeq \bigoplus_{\substack{j+k+q=i \\ \mu \in \mathfrak{a}_P^*}} \left[H^q(\hat{O}, \mathbb{H}_{\mu}^k(\mathfrak{u}_P, E)) \right. \\ \left. \otimes H^j(A_P(t), \mathbb{C}; h_{\mu}) \right], \quad (1)$$

where $h_{\mu} = a^{-2(\mu + \delta_P)}$.

We remark that some cases of this theorem are contained in [15]: if $\hat{O}$ is contractible (for any P), and if $\hat{O}$ is arbitrary (as above) and P is maximal. One can adapt the techniques used in [15, §§3–5] to provide a proof of the theorem in general. An alternate approach has been given by Borel, where one uses a spectral sequence for C^∞ vectors (see [4]). Also, if we write

$$\mu + \delta_P = \sum_{\beta \in \Delta_P} n_\beta \beta, \quad (2)$$

then for $j > 0$

$$H^j(A_P(t), \mathbb{C}; h_\mu) \begin{cases} \text{is infinite dimensional if} \\ j \leq \# \{ \beta : n_\beta = 0 \} \text{ and } \{ \beta : n_\beta < 0 \} = \emptyset, \\ = 0 \text{ otherwise,} \end{cases} \quad (3)$$

and

$$H^0(A_P(t), \mathbb{C}; h_\mu) = \begin{cases} \mathbb{C} & \text{if all } n_\beta > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

This give us a means of evaluating (1). (For these formulas – and we note that they are independent of t – see [15, (4.51)].)

(2.5). We make explicit the underlying theme in our proof of the theorem of (2.4), for we will use it again in (3.19).

Let $L^\bullet$ be a complex of Hilbert spaces (with densely defined differential d_L), and suppose that it has an orthogonal decomposition, though only as a graded Hilbert space,

$$L^\bullet = \bigoplus_{\mu} L_{\mu}^{\bullet}, \quad (1)$$

indexed by a partially ordered set. Let $K_{\mu}^{\bullet}$ be complexes of Hilbert spaces such that for any μ and γ there is a densely defined inclusion

$$\iota_{\mu, \gamma} : K_{\mu}^{\bullet} \rightarrow K_{\gamma}^{\bullet}, \quad (2)$$

with $\iota_{\mu, \mu} = I$, $\iota_{\gamma, \mu} \iota_{\mu, \epsilon} = \iota_{\gamma, \epsilon}$ (where defined), and $\iota_{\mu, \gamma}$ is bounded whenever $\mu < \gamma$. Then

$$N^{\bullet} = \bigoplus_{\mu} (K_{\mu}^{\bullet} \hat{\otimes} L_{\mu}^{\bullet}) \quad (3)$$

has a natural structure of a complex of Hilbert spaces, with differential

$$d = d_K + \sigma_K d_L, \quad (4)$$

where σ_K is the scalar $(-1)'$ on K' , and the sum is taken in the strict operator sense (with passage to the closure). We are omitting here (and elsewhere) the following abuse of notation: if $\phi \in K_\mu^*$, and $\psi \in L^*$ is decomposed as

$$\psi = \sum_{\gamma} \psi_{\gamma}$$

according to (1), then

$$\phi \otimes \psi = \sum_{\gamma} (\iota_{\mu, \gamma} \phi) \otimes \psi_{\gamma}, \quad (5)$$

provided $\iota_{\mu, \gamma} \phi$ is defined for all γ such that $\psi_{\gamma} \neq 0$; by d_L in (4), we mean the closure of $(1 \otimes d_L)$ on linear combinations of elements of the form (5).

PROPOSITION: *With the above notation and conventions, let P be a degree zero bounded operator on L^* that preserves each L_μ^* , and B a bounded operator of degree -1 on L^* such that $I - P = d_L B + B d_L$. If*

$$B(L_\mu^*) \subseteq \bigoplus_{\gamma \geq \mu} L_\gamma^*, \quad (6)$$

then we have on N^*

$$I - (1 \otimes P) = d(\sigma_K \otimes B) + (\sigma_K \otimes B)d,$$

and $\sigma_K \otimes B$ is a bounded operator.

PROOF (cf. [15,(2.29)]): By the assumption (6) made on B , we have that $1 \otimes B$ (defined as in (5)) is a bounded operator. Let $\eta = \phi \otimes \psi \in N^*$. If $\eta \in \text{Dom}(d)$, then $\psi \in \text{Dom}(d_L)$, so we may write

$$\psi - P\psi = d_L(B\psi) + B(d_L\psi).$$

This shows that in N^* , $(1 \otimes B)\eta \in \text{Dom}(d_L)$, and

$$\eta - (1 \otimes P)\eta = d_L(1 \otimes B)\eta + (1 \otimes B)d_L\eta.$$

We also have $\phi \in \text{Dom}(d_K)$. Since d_K and $(1 \otimes B)$ evidently commute, we obtain

$$\eta - (1 \otimes P)\eta = d(\sigma_K \otimes B)\eta + (\sigma_K \otimes B)d\eta.$$

This relation persists on all of $\text{Dom}(d)$, by linearity and passage to the

closure (using (4)), as desired.

For the theorem of (2.4), one applies the above proposition with

$$K_\mu^\bullet = L_2^\bullet(A_P(t), \mathbb{C}; h_\mu),$$

$\iota_{\mu, \gamma}$ the “identity” mapping, the partial order is that defined by Δ_P , and

$$L_\mu^\bullet \subset L_2^0(\Gamma_P \backslash \tilde{V}, \mathbb{C}) \otimes \wedge_\mu^\bullet(E) \otimes \wedge^{\bullet} \mathfrak{p}_M^*,$$

where $\tilde{V}$ is the lift of V' to P . The key technical hypothesis (4) is a consequence of the fact that $W_x'(V', t)$ is a complete Riemannian manifold-with-corners (see [15,(2.27)]). By a calculation similar to [15,§4(c)], (6) is satisfied for the homotopy operators in question.

(2.6). We set up a mild generalization of the Theorem of (2.4), in a form that will also be useful in §3.

Let V be an open subset of $e(P)$. For any pair σ, σ' of cross-sections over V to the geodesic action of $\hat{A}_P$, there is a unique function

$$b = b_{\sigma, \sigma'}: V \rightarrow \hat{A}_P \quad (1)$$

such that for all $v \in V$

$$\sigma(v) = b(v) \circ \sigma'(v). \quad (2)$$

On the quotient by an arithmetic group, we have, by (1.5), a similar notion for open subsets V' of $e'(P)$ in some neighborhood of the latter.

We define a diffeomorphism (for t sufficiently large)

$$F: W(V, \sigma, t) \rightarrow W(V, \sigma', t) \quad (3)$$

by

$$F(\mu_\sigma(a, v)) = \mu_{\sigma'}(a, v). \quad (4)$$

PROPOSITION 1: *Let $\hat{O}$ be an open subset of $\hat{e}(P)$, $V = p_P^{-1}(\hat{O})$, and σ, σ' a pair of cross-sections over V to the geodesic action of $\hat{A}_P$. Suppose that*

- (i) *b is constant along the fibers of p'_P ,*
- (ii) *b takes its values in a compact subset of $\hat{A}_P$,*
- (iii) *$(db/b)^2 = \sum_{\alpha \in \Delta_P} (db^\alpha/b^\alpha)^2$ is dominated by a constant multiple of the metric on V .*

Then F is a quasi-isometry.

PROOF: Thinking of X as $\hat{A}_P \circ \sigma_x(\hat{e}(P) \times U_P)$, one obtains ([3,(4.3)]) for the metric on X , in distinguished coordinates $(a, z, u) \in \hat{A}_P \times \hat{e}(P) \times U_P$, the formula

$$ds^2 = (da/a)^2 + dz^2 + \sum_{\beta} a^{-2\beta} du_{\beta}^2(z), \quad (5)$$

where dz^2 is the invariant metric on $\hat{e}(P)$ and $du_{\beta}^2(z)$ is an inner product on $\mathfrak{u}_{\beta}$ that depends smoothly on z . From (4), we have

$$F(a \circ \sigma(v)) = a \circ \sigma'(v),$$

which we rewrite (cf. (2)) as

$$\begin{aligned} F(ab_{\sigma, \sigma_x}(v) \circ \sigma_x(v)) &= ab_{\sigma', \sigma_x}(v) \circ \sigma_x(v); \\ F(a \circ \sigma_x(v)) &= ab_{\sigma, \sigma_x}^{-1}(v) b_{\sigma', \sigma_x}(v) \circ \sigma_x(v) \\ F(\mu_x(a, v)) &= \mu_x(ab^{-1}(v), v). \end{aligned} \quad (6)$$

We therefore obtain from (5)

$$F^*ds^2 = (da/a - db/b)^2 + dz^2 + \sum a^{-2\beta} b^{2\beta} du_{\beta}^2(z). \quad (7)$$

The metric (7) is always quasi-isometric to

$$(da/a)^2 + (db/b)^2 + dz^2 + \sum a^{-2\beta} b^{2\beta} du_{\beta}^2(z). \quad (8)$$

With conditions (i)–(iii), we can drop the term $(db/b)^2$ and the factors $b^{2\beta}$ without changing the quasi-isometry class. Thus, F^*ds^2 and ds^2 are quasi-isometric on $W(V, \sigma, t)$, as desired.

Likewise, we have for the coefficients:

PROPOSITION 2: *Let V be a relatively compact subset of $e(P)$. Then the metrized local systems $\mathbb{E}$ and $F^*\mathbb{E}$ are quasi-isometric on $W(V, \sigma, t)$ if condition (ii) of Proposition 1 is satisfied.*

PROOF. Since no quotient of X has been taken, $\mathbb{E}$ is trivial as a local system. Let e be a constant section with values in the weight space E_{μ} for α_P . Then for $q \in ({}^0P)_{\mathbb{R}}$, at $y = qa \cdot x$, one has by definition

$$|e|_y = |\rho(a^{-1}q^{-1})e|_x = a^{-\mu} |\rho(a^{-1}q^{-1}a)e|_x, \quad (9)$$

so also

$$|e|_{F(y)} = a^{-\mu} b^{\mu} |\rho(ba^{-1}q^{-1}ab^{-1})e|_x, \quad (10)$$

where $b = b(qx_p)$. The desired conclusion follows from the fact that $\tilde{q} = ba^{-1}q^{-1}ab^{-1}$ lies in a compact subset of $({}^0P)_{\mathbb{R}}$ (cf. [2,(1.3)]), so the norm of $\rho(\tilde{q})$ lies in a compact subset of $\mathbb{R}^+$.

(2.7). The essential point in the proof of Proposition 2 of (2.6) is the compactness in the U_p -direction. (If $q = mu$, with $m \in M_{p,x}$ and $u \in U_p$, then $a^{-1}q^{-1}a = (a^{-1}u^{-1}a)m^{-1}$, etc.) From the theory of Siegel sets, one knows that there exist fundamental sets for the action of Γ_p that satisfy this condition. Thus, we may relax the hypotheses of Proposition 2 to cover this case. We then obtain:

THEOREM: Let $V' = (p'_p)^{-1}(\hat{O})$, where $\hat{O}$ is now an open subset of $\hat{e}'(P)$, and let σ and σ' be Γ_p -invariant cross-sections over V' satisfying (i)–(iii) of Proposition 1 of (2.6). Then

$$F: W'(V', \sigma, t) \rightarrow W'(V', \sigma', t) \quad (1)$$

induces an isomorphism on L_2 -cohomology. Thus, if $\hat{O}$ is relatively compact,

$$H_{(2)}^i(W'(V', \sigma, t), \mathbb{E}) \simeq \bigoplus_{\substack{j+k+q=i \\ \mu \in a_p^*}} \left[H^q(\hat{O}, \mathbb{H}_{\mu}^k(u_p, E)) \otimes H^j(A_p(t), \mathbb{C}; h_{\mu}) \right] \quad (2)$$

for any σ for which b_{σ, σ_x} is constant along the fibers of p'_p .

REMARK: We assume $\hat{O}$ is relatively compact. If $H^*(\hat{O}, \mathbb{H}_{\mu}^*(u_p, E))$ vanishes for all μ such that the L_2 -cohomology of $A_p(t)$ with weight h_{μ} is infinite dimensional (see (2.4(3))), it follows from (2.4(4)) that $H_{(2)}^*(W'(V', \sigma, t), \mathbb{E})$ is represented by those forms in $\mathcal{H}_2^*(V', \mathbb{E})$ for which the pullback to the collar is in L_2 :

$$H_{(2)}^*(W'(V', \sigma, t), \mathbb{E}) \simeq \bigoplus_{\mu > -\delta_p} H^*(\hat{O}, \mathbb{H}_{\mu}^*(u_p, \mathbb{E})). \quad (3)$$

§3. And intersection homology

(3.1). We recall the definition and properties of intersection homology with “middle perversity” for complex spaces, following [10].

Let Y be a complex analytic variety, and put $n = \dim_{\mathbb{C}} Y$. Then Y admits the structure of a *stratified space*; i.e., there exists a descending chain $\{Y^j\}$ of subvarieties (*stratification*)

$$Y = Y^0 \supset Y^1 \supset \dots \supset Y^n \supset \emptyset, \quad (1)$$

with the following properties:

- (a) $\dim_{\mathbb{C}} Y^j \leq n - j$,
- (b) $S^j = Y^j - Y^{j+1}$ is a complex manifold of dimension $n - j$ (possibly empty), the codimension j *stratum*,
- (c) if $y \in S^k$, there is a neighborhood of y in Y in which the stratification is topologically $\{U \times Z^{j-k}\}$ for some stratified space Z , where U is a neighborhood of y in S^k .

We remark that the stratification (1) is not uniquely determined. For example, if Y is a manifold, S^0 need not be taken to be all of Y ; there may be other reasons to impose a non-trivial stratification (see below).

In addition, there is a *PL*-structure on Y such that each Y^j is a subcomplex of Y . As is well-known, one can then compute the homology of Y from the complex of simplicial chains (with whatever coefficients). A subcomplex is defined as follows. One says that an i -chain ξ is *allowed* if for all $j > 0$

$$\dim_{\mathbb{R}}(|\xi| \cap Y^j) < i - j, \quad (2)$$

with the convention that a negative-dimensional space is empty. For any locally constant sheaf of vector spaces $\mathbb{E}$ on S^0 , one can define:

$$IC_i(Y, \mathbb{E}) = \{\text{finite } \mathbb{E}\text{-valued } i\text{-chains } \xi: \xi \text{ and } \partial\xi \text{ are allowed}\}, \quad (3)$$

$$IC'(Y, \mathbb{E}) = \{\text{locally-finite } \mathbb{E}\text{-valued } (2n - i)\text{-chains } \zeta: \zeta \text{ and } \partial\zeta \text{ are allowed}\}. \quad (4)$$

(Note that (2) implies that $|\xi| \cap S^0 \neq \emptyset$.) By construction, (3) and (4) define complexes $IC(Y, \mathbb{E})$ and $IC'(Y, \mathbb{E})$ respectively; these give rise to the *intersection homology* $IH_*(Y, \mathbb{E})$ and *intersection cohomology* $IH^*(Y, \mathbb{E})$ respectively.

It is clear from (4) that the association

$$W \mapsto IC'(W, \mathbb{E}),$$

for W open in Y , defines a complex of soft sheaves, denoted $\mathcal{IC}'(Y, \mathbb{E})$, such that

$$H^i(Y, \mathcal{IC}'(Y, \mathbb{E})) \simeq IH^i(Y, \mathbb{E}) \quad (5)$$

and

$$H_c^i(Y, \mathcal{IC}'(Y, \mathbb{E})) \simeq IH_{2n-i}(Y, \mathbb{E}). \quad (6)$$

It is an important feature of intersection homology that if $\mathbb{E}^*$ is the locally constant sheaf dual to $\mathbb{E}$, $\mathcal{IC}'(Y, \mathbb{E})$ and $\mathcal{IC}'(Y, \mathbb{E}^*)$ are Verdier-dual (up to a shift) [10, (5.3)]. This gives rise to Poincaré duality; in terms of (3) and (4), this becomes

$$IH^i(Y, \mathbb{E}^*) \simeq [IH_i(Y, \mathbb{E})]^*,$$

and the duality can be realized via intersection numbers of cycles.

There are several characterizations of $\mathcal{IC}'(Y, \mathbb{E})$ up to quasi-isomorphism. The following one is perhaps easiest to use:

THEOREM [10, §3]: *Let $\mathcal{L}^*$ be a bounded complex (beginning in degree zero) of sheaves on Y . Suppose that*

- i) The cohomology sheaves $\mathcal{H}^i \mathcal{L}^*$ are locally constant along the strata,*
- ii) $\mathcal{H}^0 \mathcal{L}^*|_{S^0} \simeq \mathbb{E}$,*
- iii) $\mathcal{H}^i \mathcal{L}^*|_{S^j} = 0$ if $i \geq j > 0$,*
- iv) if $y \in S^j$, $\lim_{\substack{\leftarrow \\ W \ni y \text{ open}}} H_c^i(W, \mathcal{L}^*) = 0$ if $i \leq 2n - j$.*

Then $\mathcal{L}^$ is quasi-isomorphic to $\mathcal{IC}'(Y, \mathbb{E})$.*

REMARK: The relevant feature of complex varieties for all of the preceding is that they possess stratifications with strata of even real codimension. All of the above is true in this latter generality.

(3.2). We take for Y the Baily-Borel-Satake compactification $\Gamma \backslash_{\mathbb{Q}} X^*$ of an arithmetic quotient of the Hermitian symmetric space X , associated to the group G . These spaces are naturally stratified, with $S^0 = \Gamma \backslash X$. When the $\mathbb{Q}$ -rank of G equals its $\mathbb{R}$ -rank, the lower-dimensional strata can be written as

$$S^{j(s)} = \coprod_g \hat{e}''({}^g P_{\Theta_g}) \quad (1)$$

(see (1.6) for notation), where g runs over a (finite) set of representatives for $\Gamma \backslash G_{\mathbb{Q}} / (P_{\Lambda_s})_{\mathbb{Q}}$. (Note that the codimension $j(s)$ of the stratum is an increasing function of s .) We take as coefficients the locally constant sheaves $\mathbb{E}$ on $\Gamma \backslash X$ associated to finite dimensional representations of G , as in (2.3).

In [15, (6.20)], we have made the following:

CONJECTURE: $H_{(2)}^*(\Gamma \backslash X, \mathbb{E}) \simeq IH^*(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$.

This is to be verified by showing that $\mathcal{L}^* = \mathcal{L}_{(2)}^*(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$ (or equivalently $\mathcal{A}_{(2)}^*(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$) satisfies (i)–(iv) of the theorem of (3.1); we will see in (3.8) that $\Gamma \backslash_{\mathbb{Q}} X^*$ has arbitrarily fine L_2 -admissible open cover-

ings, hence (2.3(4)) holds. In [15,§6], the conjecture is verified for quotients of the ball and Hilbert modular varieties (of arbitrary dimension), for which the compactifications are spaces with isolated singularities. The general $\mathbb{Q}$ -rank one case has been proved by Borel (see [4]). We will develop in this chapter techniques of approaching the problem in higher rank, especially the case of $\mathbb{Q}$ -rank 2, and verify the conjecture for $G = \mathrm{Sp}(4)$, for $G = \mathrm{SU}(p, 2)$ when $\mathbb{E} = \mathbb{C}$, and for $G = \mathrm{Sp}(6)$.

REMARK: In view of (1.6) and (2.2), it is easy to see that it suffices to consider the case where G is (almost-)simple over $\mathbb{Q}$.

(3.3). Of the conditions in the theorem of (3.1), for $\mathcal{L}' = \mathcal{L}'_{(2)}(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$, (ii) is obvious. Also, (i) follows rather easily from [17, (3.8)] (see the proposition of (1.6), which implies that the quotient mapping

$$(p^*)': \Gamma \backslash \bar{X} \rightarrow \Gamma \backslash_{\mathbb{Q}} X^* \quad (1)$$

is, over each stratum, a stratified fiber bundle).

In order to have (iii), it suffices to show that for $y \in S'$, there exists a fundamental system of neighborhoods W of y such that for $W' = W \cap (\Gamma \backslash X)$, one has

$$H'_{(2)}(W', \mathbb{E}) = 0 \quad \text{if } i \geq j. \quad (2)$$

(3.4). We recall the determination of $H'(\mathfrak{u}_P, E)$, for any P , according to Kostant. Let $\mathfrak{h}$ be a Cartan subalgebra for P , hence also for G , containing $\mathfrak{a}_P$; let Φ^+ be a system of positive $\mathbb{C}$ -roots compatible with $_{\mathbb{R}}\Delta$, and W the Weyl group of the complex root system. Then, as a representation of the corresponding Levi subgroup, one has

$$H'(\mathfrak{u}_P, E) \simeq \bigoplus_{w \in W^P(i)} E'_w, \quad (1)$$

where E'_w is the irreducible representation with highest weight $w(\lambda + \delta) - \delta$, λ is the highest weight of E , $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, and

$$W^P(i) = \{w \in W : \text{there are exactly } i \text{ roots } \alpha \in \Phi^+ \text{ with } w^{-1}(\alpha) \notin \Phi^+, \text{ all of which occur in } \mathfrak{u}_P \otimes \mathbb{C}\}. \quad (2)$$

An element $w \in W^P(i)$ has length $l(w) = i$. Each factor E'_w has a single weight under $\mathfrak{a}_P$, namely the restriction of $w(\lambda + \delta) - \delta$. We remark that $\delta|_{\mathfrak{a}_P} = \delta_P$, and

$$w(\delta) - \delta = \sum_{\alpha \in \Phi^+, w^{-1}(\alpha) \notin \Phi^+} (-\alpha). \quad (3)$$

(3.5). The proof of (3.3(2)) for $j = j(1)$ is contained in Borel's proof for the $\mathbb{Q}$ -rank one case, so we will take this as known and proceed to the more singular strata. Before doing so, we make the following observations.

In the case that the $\mathbb{Q}$ - and $\mathbb{R}$ -rank of G are equal, we have for $P_{\Lambda_1} = P_{\Theta_1}$ that $\mathfrak{u}_P$ is of odd dimension (for X and $e(P)$ have complex structures, and A_P is one-dimensional) $2m - 1$. Thus, $j(1) = m$. We then note that the coefficient of β_1 in the restriction of a positive root is either 0 (if it does not occur in $\mathfrak{u}_P$), 1 or 2, and that 2 occurs for precisely one root. Thus, $\delta_P = m\beta_1$.

One can take $W' = W'_x(\hat{O}, t)$ in (3.3(2)), where $\hat{O}$ runs over contractible neighborhoods of a point in $\hat{e}'(P)$. The required vanishing of L_2 -cohomology follows from:

PROPOSITION: *Let $\alpha^* \in \Phi^+$ be the root whose restriction to $\mathfrak{a}_P$ is $2\beta_1$. Then if $w \in W^P(i)$,*

- i) $w^{-1}(\alpha^*) < 0$ if and only if $i \geq m$,*
- ii) if $i < m$, $w(\lambda)|_{\mathfrak{a}_P} \geq 0$ for all dominant λ , and if*

$$i > m, w(\lambda)|_{\mathfrak{a}_P} \leq 0 \text{ for all dominant } \lambda.$$

REMARKS: (i) The existence of a unique root α^* can be stated in the somewhat "a priori" form: the restriction of the highest root (which is α^*) has 2 as the coefficient of β_1 ; there is but one simple $\mathbb{C}$ -root whose inner product with it is non-zero, and this root restricts to β_1 .

(ii) Given the existence of α^* , one almost get the vanishing of (3.3(2)) for $i \geq m$ for free when $\mathbb{E} = \mathbb{C}$ – one needs only to rule out the presence of a non-zero summand with trivial weight, which would give rise to infinite dimensional L_2 -cohomology (non-closed range for d).

(3.6). In order to proceed with the proof of (iii), we need to describe, in a concrete way, a fundamental system of neighborhoods of points on the lower-dimensional strata. We will first make a related construction of L_2 -admissible open coverings of $\Gamma \backslash X$ that extend to coverings of $\Gamma \backslash \bar{X}$. We can drop for the moment the assumption that the $\mathbb{Q}$ -rank and $\mathbb{R}$ -rank of G be equal.

We begin by recalling (1.4(3)). The object is to give one distinguished open set for each $P \in \mathfrak{R}(\Gamma)$:

$$\mathfrak{B} = \{W(P) : P \in \mathfrak{R}(\Gamma)\}, \quad (1)$$

with as few non-empty intersections as possible. This is done recursively. Fix $x \in X$.

Step 1: If P has maximal parabolic $\mathbb{Q}$ -rank, r , (i.e., if P is minimal $\mathbb{Q}$ -parabolic), then we take $V'_P = e'(P)$ and

$$W(P) = W'_x(V'_P, t_P), \quad (2)$$

for $t_P \in A_P$ sufficiently large.

Step 2: If Q has parabolic $\mathbb{Q}$ -rank $r - 1$, then

$$\overline{e'(Q)} = e'(Q) \sqcup \left(\prod_P e'(P) \right), \quad (3)$$

where P runs over the set of (minimal) $\mathbb{Q}$ -parabolic subgroups in $\mathfrak{R}(\Gamma)$ for which some Γ -conjugate is properly contained in Q . We select a relatively compact open subset $\hat{O}_Q$ of $\hat{e}(Q)$, such that for $V'_Q = (p'_Q)^{-1}(\hat{O}_Q)$ we have

$$\begin{aligned} e'(Q) &= V'_Q \cup \left(\bigcup_P W'_{x_Q}(e'(P), t_P) \right) \\ &= V'_Q \cup \left(\bigcup_P \pi_Q q_Q(W_x(e(P), t_P)) \right), \end{aligned} \quad (4)$$

where the union is over the same set as in (3). For V'_Q , it is best to take

$$V'_Q = e'(Q) - \bigcup_P W'_{x_Q}(e'(P), s_P), \quad (5)$$

where $t_P < s_P \in A_P$. We see that V'_Q is, in fact, a deformation retract of $e'(Q)$. We now put

$$W(Q) = W'_x(V'_Q, t_Q) \quad (6)$$

for sufficiently large $t_Q \in A_Q$.

Inductive step: The pattern has already emerged. Given Q , such that $W(P)$ has been constructed for all P of higher parabolic $\mathbb{Q}$ -rank, select a relatively compact open set $V'_Q = (p'_Q)^{-1}(\hat{O}_Q)$ in $e'(Q)$, such that

$$e'(Q) = V'_Q \cup \left(\bigcup_P W'_{x_Q}(V'_P, t_P) \right), \quad (7)$$

where the union is now over those $P \in \mathfrak{R}(\Gamma)$ with some Γ -conjugate of P properly contained in Q (such P are necessarily of higher parabolic $\mathbb{Q}$ -rank), and define $W(Q)$ as in (6). The choice of V'_Q can be made analogous to that in (5), though the usefulness of such a description remains to be seen.

We can arrange that $W(P)$ and $W(Q)$ intersect if and only if one of $e'(P)$ and $e'(Q)$ is contained in the closure of the other, by taking the t_Q 's large enough. The L_2 -admissibility of $\mathfrak{B}$ follows from the existence of suitable cut-off functions, for which we refer to [15,(4.5),(5.4,ii)]. We shall refer to $\mathfrak{B}$ as a *distinguished covering* of $\Gamma \backslash X$. There is an analogous notion for $\Gamma \backslash \bar{X}$.

(3.7). We indicate some useful properties of the distinguished coverings in (3.6).

It is apparent that the construction of $\mathfrak{B}$ goes through verbatim if we replace the symmetric space X by any $e(Q)$. We then note:

PROPOSITION: (i) *There is a natural one-to-one correspondence between the distinguished coverings of $e'(Q)$ and those of $\hat{e}'(Q)$.*

(ii) *If G is an almost-direct product $G_1 \cdot G_2$, so that X is a product $X_1 \times X_2$; and if also $\Gamma = \Gamma_1 \cdot \Gamma_2$, then products of distinguished coverings of each factor are among the distinguished coverings of $\Gamma \backslash X$.*

(iii) *Given a distinguished covering of $\Gamma \backslash X$,*

$$\mathfrak{N}(Q) = \{W(P) \in \mathfrak{B} : \gamma P \subseteq Q \text{ for some } \gamma \in \Gamma\} \quad (1)$$

covers a neighborhood of $e'(Q)$ in $\Gamma \backslash X$. As the t 's and s 's go suitably to infinity,

$$\bar{N}(Q) = \bigcup \{\bar{W}(P) : \gamma P \subset Q \text{ for some } \gamma \in \Gamma\},$$

where $\bar{W}(P) = \bar{W}_x'(V_Q', t_Q)$, defines a fundamental system of neighborhoods of $e'(Q)$ in $\Gamma \backslash \bar{X}$. In addition, (1), taken modulo $\hat{A}_Q$ (see remark in (1.5)), determines a distinguished covering of $e'(Q)$.

REMARK:

(i) Given any fixed $Q \in {}_{\mathbb{Q}}\mathfrak{P}(G)$, one can select $\mathfrak{N}(\Gamma)$ to first include Q , and then have the property that all P occurring in (3.6(7)) are actually contained in Q .

(ii) The preceding assertion is of particular importance to us for (3.6(3)), where Q is of parabolic $\mathbb{Q}$ -rank $r - 1$. We can then arrange that

$$W(Q) \cap W(P) = W_x'(S, t_Q), \quad (2)$$

where

$$S = W_{x_Q}'(e'(P), t_P) \cap V_Q' \subset e'(Q); \quad (3)$$

moreover, the sets (2) are disjoint as P varies, and we have

$$S = W_{x_Q}'(e'(P), t_P) - \overline{W_{x_Q}'(e'(P), s_P)}, \quad (4)$$

a relatively compact region diffeomorphic to $e'(P) \times I$, where I is a bounded interval.

(3.8). Let $y \in \hat{e}''(P_{\Theta_1})$ in $\Gamma \backslash_{\mathbb{Q}} X^*$. We can now describe a fundamental system of neighborhoods of y . *The following discussion holds when the $\mathbb{Q}$ -rank and $\mathbb{R}$ -rank of G are equal, and can be adapted to the general case with a little care* (see the remark of (1.6)).

Put $Q = P_{\Lambda_1}$. By (1.6), the inverse image $F = (p^{*'})^{-1}(y)$ of y in $\Gamma/\bar{X}$ is contained in $e'(Q)$, and is, in fact, a pullback via p'_Q of a subset of $\hat{e}'(Q)$. Moreover, a dense subset of $\hat{e}'(Q)$ admits a fibration over $\hat{e}''(P_{\Theta_1})$, with fiber $\hat{e}'(P_{\Psi_1})$, and $p'_Q(F)$ is just one of the fibers. We may therefore write a cofinal set of neighborhoods of $p'_Q(F)$ in $\hat{e}'(Q)$ as

$$O \times \overline{\hat{e}'(P_{\Psi_1})}, \quad (1)$$

where O is a contractible neighborhood of y in $\hat{e}''(P_{\Theta_1})$. We then obtain explicit fundamental neighborhoods of F in $\Gamma \backslash \bar{X}$ by taking distinguished coverings of $\hat{e}'(P_{\Psi_1})$, or the restriction of ones for $\hat{e}'(Q)$, and then taking geodesic influxes via $\hat{A}_Q$ (for example, $\bar{N}(Q)$, and its restriction). We denote the pieces of such neighborhoods by $\tilde{W}(P)$, and put

$$\tilde{W}(P) = \overline{\tilde{W}(P)} \cap (\Gamma \backslash X).$$

The parabolic subgroups P which parametrize the pieces are, of course, in one-to-one correspondence with the $\Gamma_{P_{\Psi_1}}$ -conjugacy classes of $\mathbb{Q}$ -parabolic subgroups of P_{Ψ_1} ; more to the point, they are representatives of the Γ_Q -conjugacy classes of $\mathbb{Q}$ -parabolic subgroups of Q that contain a $Q_{\mathbb{Q}}$ -conjugate of P_{Θ_1} . When $s = r$, we have that $\tilde{W}(P)$ coincides with $W(P)$ of (3.6), for then $\Psi_s = \Lambda_s$.

When $s = 2$, we have a manageable description of the neighborhood base:

PROPOSITION: *Let $Q = P_{\Lambda_2}$, $P = P_{\Theta_2}$. If $y \in \hat{e}''(P)$, then y has a fundamental system of neighborhoods whose intersection with $\Gamma \backslash X$ are of the form*

$$W = \tilde{W}(Q) \cup \left(\coprod_h \tilde{W}(^h P) \right), \quad (2)$$

where h runs through a set of representatives of $\Gamma_Q \backslash Q_{\mathbb{Q}}/P_{\mathbb{Q}}$.

(3.9). Let $s = 2$; we retain the above notation. Since (3.8(2)) gives an L_2 -admissible covering of W , we may (see (2.2)) write a Mayer-Vietoris sequence

$$\begin{aligned} & H'_{(2)}{}^{-1}(\tilde{W}(Q), \mathbb{E}) \oplus \left(\bigoplus_h H'_{(2)}{}^{-1}(\tilde{W}({}^hP), \mathbb{E}) \right) \\ & \rightarrow \bigoplus_h H'_{(2)}{}^{-1}(\tilde{W}(Q) \cap \tilde{W}({}^hP), \mathbb{E}) \\ & \rightarrow H'_{(2)}(W, \mathbb{E}) \rightarrow H'_{(2)}(\tilde{W}(Q), \mathbb{E}) \oplus \left(\bigoplus_h H'_{(2)}(\tilde{W}({}^hP), \mathbb{E}) \right). \quad (1) \end{aligned}$$

We can identify the terms (see (2.4) and (2.6)):

$$H'_{(2)}(\tilde{W}(Q), \mathbb{E}) \simeq \bigoplus_{\substack{q \\ \alpha_Q^* \ni \mu > -\delta_Q}} H'^{-q}(\hat{e}'(P_{\Psi_2}), \mathbb{H}_\mu^q(\mathfrak{u}_{\Lambda_2}, \mathbb{E})), \quad (2)$$

$$H'_{(2)}(\tilde{W}({}^hP), \mathbb{E}) \simeq \bigoplus_{\alpha_P^* \ni \mu > -\delta_P} H'_\mu(\mathfrak{u}_{\Theta_2}, E), \quad (3)$$

$$H'_{(2)}(\tilde{W}(Q) \cap \tilde{W}({}^hP), \mathbb{E}) \simeq \bigoplus_{\alpha_Q^* \ni \mu > -\delta_Q} H'_\mu(\mathfrak{u}_{\Theta_2}, E), \quad (4)$$

neglecting potential infinite-dimensional summands.

The goal is to verify that $H'_{(2)}(W, \mathbb{E}) = 0$ for $i \geq j(2)$, and this is to be achieved primarily by showing that (2), (3) and (4) are zero for most $i \geq j(2) - 1$. We remark that

$$j(2) = 1/2 \dim \mathfrak{u}_{\Theta_2} + 1; \quad (5)$$

more generally, $j(s) = 1/2(\dim \mathfrak{u}_{\Theta_s} + s)$.

(3.10). We carry out the calculations in the easiest case CI (the Siegel upper half-spaces). In classical notation, the positive roots of interest are:

$$\begin{aligned} \text{in } \mathfrak{u}_{\Theta_2}: \quad & \epsilon_i - \epsilon_j \ (i \leq 2, i < j \leq r), \ \epsilon_i + \epsilon_j \ (i \leq 2, i < j \leq r), \\ & 2\epsilon_i \ (i \leq 2), \end{aligned} \quad (1)$$

$$\text{in } \mathfrak{u}_{\Lambda_2}: \quad \text{all of the above except } \epsilon_1 - \epsilon_2. \quad (2)$$

We count (1) to obtain for (3.9(5))

$$j(2) = 2r - 1. \quad (3)$$

We also get from (1):

$$\delta_P = \begin{cases} r\beta_1 + (2r-1)\beta_2 & \text{if } r > 2 \\ 2\beta_1 + \frac{3}{2}\beta_2 & \text{if } r = 2 \end{cases} \quad (4)$$

The two cases can be handled simultaneously if one replaces $\beta_r = 2\epsilon_r$ by $\beta_r/2 = \epsilon_r$ in discussing the root system C_r ; as it seems to be a helpful combinatorial device, and also allows for simultaneous discussion of C_r and BC_r , we adopt this convention throughout. From (4), we obtain immediately

$$\delta_Q = (2r-1)\beta_2. \quad (5)$$

The elements of W^P are precisely the signed permutations $w = w_{\pm k, \pm l}$, defined by the conditions $w^{-1}(\epsilon_1) = \pm \epsilon_k$, $w^{-1}(\epsilon_2) = \pm \epsilon_l$ and w^{-1} is otherwise positive and increasing. Of these, the elements of W^Q are the $w_{k,l}$ ($k < l$), $w_{k,-l}$, and $w_{-k,-l}$ ($k > l$).

We need the following calculation:

LEMMA: (i)

$$l(w_{k,l}) = \begin{cases} k+l-3 & \text{if } k < l \\ k+l-2 & \text{if } k > l \end{cases}$$

$$l(w_{k,-l}) = \begin{cases} 2r-2+k-l & \text{if } k < l \\ 2r-3+k-l & \text{if } k > l \end{cases}$$

$$l(w_{-k,l}) = \begin{cases} 2r-1+l-k & \text{if } k > l \\ 2r-2+l-k & \text{if } k < l \end{cases}$$

$$l(w_{-k,-l}) = \begin{cases} 4r-l-k-2 & \text{if } k > l \\ 4r-l-k-1 & \text{if } k < l \end{cases}$$

(ii)

$$(\delta - w(\delta))|_{\alpha_P}$$

$$= \begin{cases} (k-1)\beta_1 + (k+l-3)\beta_2 & \text{if } w = w_{k,l} \\ (k-1)\beta_1 + (2r-1+k-l)\beta_2 & \text{if } w = w_{k,-l} \\ (2r+1-k)\beta_1 + (2r-1+l-k)\beta_2 & \text{if } w = w_{-k,l} \\ (2r+1-k)\beta_1 + (4r-k-l+1)\beta_2 & \text{if } w = w_{-k,-l} \end{cases}$$

(iii) The values of $w(\lambda_i)|_{\alpha_P}$ are:

(a)	$k < l$	$i < k$	$k \leq i < l$	$i \geq l$
	$w_{k,l}$	0	$\beta_1 + \beta_2$	$\beta_1 + 2\beta_2$
	$w_{k,-l}$	0	$\beta_1 + \beta_2$	β_1
	$w_{-k,l}$	0	$-\beta_1 - \beta_2$	$-\beta_1$
	$w_{-k,-l}$	0	$-\beta_1 - \beta_2$	$-\beta_1 - 2\beta_2$
(b)	$k > l$	$i < l$	$l \leq i < k$	$i \geq k$
	$w_{k,l}$	0	β_2	$\beta_1 + 2\beta_2$
	$w_{k,-l}$	0	$-\beta_2$	β_1
	$w_{-k,l}$	0	β_2	$-\beta_1$
	$w_{-k,-l}$	0	$-\beta_2$	$-\beta_1 - 2\beta_2$

PROOF: By duality ($w_{\pm k, \pm l} \leftrightarrow w_{\mp k, \mp l}$), it is enough to verify the upper half of each formula.

(a) $w = w_{k,l}$: the positive roots with negative images under w^{-1} are $\epsilon_1 - \epsilon_j$ ($j > 1$, $w^{-1}(j) < k$) and $\epsilon_2 - \epsilon_m$ ($m > 2$, $w^{-1}(m) < l$). The sum of these roots, restricted to α_P , is

$$\begin{cases} (k-1)(\beta_1 + \beta_2) + (l-2)\beta_2 & \text{if } k < l \\ [\beta_1 + (k-2)(\beta_1 + \beta_2)] + (l-1)\beta_2 & \text{if } k > l. \end{cases} \quad (6)$$

By (3.4(3)), this gives (ii). We sum the coefficients in (6) to obtain (i).

(b) $w = w_{k,-l}$: the positive roots with negative images under w^{-1} are $\epsilon_1 - \epsilon_j$ ($j > 2$, $w^{-1}(j) < k$), $\epsilon_2 - \epsilon_m$ ($m > 2$), $\epsilon_2 + \epsilon_m$ ($w^{-1}(m) > l$), and $2\epsilon_2$. The sum of these roots, restricted to α_P , is

$$\begin{cases} (k-1)(\beta_1 + \beta_2) + (r-2)\beta_2 + (r-l)\beta_2 + (2\beta_2) & \text{if } k < l, \\ (k-2)(\beta_1 + \beta_2) + (r-2)\beta_2 + [(\beta_1 + 2\beta_2) + (r-l-1)\beta_2] \\ \quad + (2\beta_2) & \text{if } k > l. \end{cases} \quad (7)$$

This gives (i) and (ii) as in (a) above.

The tables (iii) follow immediately from the fact that $\lambda_i = \sum_{j=1}^i \epsilon_j$ and the following elementary observation:

$$\epsilon_i|_{\alpha_P} = \begin{cases} \beta_1 + \beta_2 & \text{if } i = 1, \\ \beta_2 & \text{if } i = 2, \\ 0 & \text{if } i > 2. \end{cases} \quad (8)$$

REMARK: From (ii), we see that the coefficient of β_1 is never r , and that of β_2 is never $2r - 1$. It follows that there is no infinite dimensional term in (3.9(2)–(4)).

(3.11). We use the above data to verify:

PROPOSITION 1: $H'_{(2)}(\tilde{W}({}^hP), \mathbb{E}) = 0$ for $i \geq 2r - 2$.

PROOF: By (3.4(1)) and (3.9(3)), it is enough to check that for all $w \in W^P$ with $l(w) \geq 2r - 2$, one of the coefficients in (ii) is greater than the corresponding coefficient of (3.10(4)), and the corresponding coefficient in (iii) is non-positive. The cases are

- (a) $w = w_{k,-l}$ ($k > l$): it is for the coefficient of β_2 ,
- (b) $w = w_{-k,l}$ ($k < l$): it is for either coefficient,
- (c) $w = w_{-k,l}$ ($k = l + 1$): it is for the coefficient of β_1 ,
- (d) $w = w_{-k,-l}$: it is for either coefficient.

REMARK: We can see from the lemma (3.10) that $l(w_{k,-l}) = 2r - 3$ for $k = l - 1$, but $w(\lambda + \delta)|_{\alpha_P} > -\delta_P$ for all λ . Thus $H'^{2r-3}_{(2)}(\tilde{W}({}^hP), \mathbb{E})$ never vanishes.

We can also state

PROPOSITION 2: $H'_{(2)}(\tilde{W}({}^hP) \cap \tilde{W}(Q), \mathbb{E}) = 0$ for $i \geq 2r - 1$.

PROOF: By (3.9(4)) we must compare coefficients, as in Proposition 1, but only of β_2 . Since the condition $w \in W^Q$ rules out case (c) above (where $l(w) = 2r - 2$), the desired result follows.

We recall that $\tilde{W}(Q)$ is fibered over a deformation retract of $\hat{\mathcal{E}}'(P_{\Psi_2})$ (cf. (3.9(2))), which is a locally symmetric space associated to $SL(2, \mathbb{R})$. Being an arithmetic quotient, it is, in addition, non-compact, hence is of cohomological dimension one.

PROPOSITION 3: $H'_{(2)}(\tilde{W}(Q), \mathbb{E}) = 0$ for $i \geq 2r - 1$.

PROOF: By (3.9(2)) and the above remarks, it suffices to check that for all $w \in W^Q$ with $l(w) \geq 2r - 2$, the coefficient of β_2 in (ii) is greater than that in (3.10(4)), and the coefficient of β_2 in (iii) is non-positive. From the description of W^Q after (3.10(5)) and the list in the proof of Proposition 1, we see that we are discussing only the cases $w = w_{k,-l}$ ($k > l$) and $w = w_{-k,-l}$ ($k > l$). In both instances, the coefficients of β_2 were previously seen to behave properly, so we are done.

Combining the three Propositions above with (3.9(1)), we get:

COROLLARY: $H_{(2)}^i(W, \mathbb{E}) = 0$ for $i > 2r - 1$.

(3.12). In order to complete our task, it remains to prove that $H_{(2)}^{2r-1}(W, \mathbb{E}) = 0$.

By (3.9(1)) and (3.11), we have an exact sequence:

$$\begin{aligned} H_{(2)}^{2r-2}(\tilde{W}(Q), \mathbb{E}) &\rightarrow \bigoplus_h H_{(2)}^{2r-2}(\tilde{W}(Q) \cap \tilde{W}({}^hP), \mathbb{E}) \\ &\rightarrow H_{(2)}^{2r-1}(W, \mathbb{E}) \rightarrow 0. \end{aligned} \quad (1)$$

It seems clear at this stage that we should alter our point of view from (3.9(4)), and regard $\tilde{W}(Q) \cap \tilde{W}({}^hP)$ as a $\Gamma_Q \backslash U_Q$ fibration. Now, the discussion underlying theorem (2.4) provides an argument proving the degeneration at E_2 of the Leray spectral sequence for $p'_Q{}^*$ over reasonable subsets of $e'(Q)$. Using Proposition 3 of (3.11) for $i = 2r - 1$, we can rewrite (1) as

$$\begin{aligned} \bigoplus_w H^1(\tilde{V}(Q), \mathbb{E}'_w) &\rightarrow \bigoplus_w \bigoplus_h H^1(\tilde{V}(Q) \cap \tilde{V}({}^hP), \mathbb{E}'_w) \\ &\rightarrow H_{(2)}^{2r-1}(W, \mathbb{E}) \rightarrow 0. \end{aligned} \quad (2)$$

where w runs over the elements of $W^Q(2r - 3)$ for which $w(\lambda + \delta)|_{\mathfrak{a}_Q} > 0$, $\mathbb{E}'_w$ is the corresponding summand of $\mathbb{H}^{2r-3}(\mathfrak{u}_Q, E)$, and $\tilde{V}(Q) \subset \hat{e}'(P_{\Psi_2})$ denotes the projection of $\tilde{W}(Q)$ under p'_Q and modulo the geodesic action of $\hat{A}_Q$, etc.

The first mapping respects the w -summands. Thus, we wish to show that for each w that occurs, the mapping

$$H^1(\tilde{V}(Q), \mathbb{E}'_w) \rightarrow \bigoplus_h H^1(\tilde{V}(Q) \cap \tilde{V}({}^hP), \mathbb{E}'_w) \quad (3)$$

is surjective. We can reformulate the problem as follows. Let $S = \hat{e}'(P_{\Psi_2})$, which we identify as an algebraic curve. Let $\bar{S}$ be its smooth completion (obtained by adjoining cusps), and let j denote the inclusion of S in $\bar{S}$. Via a deformation retract, we see that (3) can be identified as the mapping

$$H^1(S, \mathbb{E}'_w) \rightarrow H^0(\bar{S}, R^1j_*\mathbb{E}'_w). \quad (4)$$

* The case at hand is an instance of the following algebraic fact: For any pair of parabolic subgroups $P \subset Q$, the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{u}_P/\mathfrak{u}_Q, H^q(\mathfrak{u}_Q, E)) \Rightarrow H^{p+q}(\mathfrak{u}_P, E)$$

degenerates at E_2 . This follows from Kostant's theorem, for example.

It is surjective if and only if $H^2(\bar{S}, j_*\mathbb{E}'_w) = 0$, i.e. if and only if $\mathbb{E}'_w$ is non-trivial.

From (3.4), we recall that $\mathbb{E}'_w$ corresponds to an irreducible representation E'_w of the Levi subalgebra $\mathfrak{l}_{Q,x}$, with highest weight $w(\lambda + \delta) - \delta$. Since $\mathfrak{l}_{Q,x}$ is, as a Lie algebra, a quotient of the direct sum $\mathfrak{l}_{P,x} \oplus \mathfrak{l}_{P',x}$ (where $P' = P_{\Psi_2}$), the restriction of E'_w to $\mathfrak{l}_{P',x}$ splits into isomorphic irreducible constituents. We conclude that (4) is surjective if and only if $w(\lambda + \delta) - \delta$ is non-zero on the intersection of the Cartan subalgebra with $\mathfrak{l}_{P',x} \otimes \mathbb{C}$.

Since ${}_{\mathbb{R}}\Delta = {}_{\mathbb{C}}\Delta$ in the case at hand, we can describe the condition for (4) as:

$$\langle w(\lambda + \delta) - \delta, \beta_1 \rangle \neq 0. \quad (5)$$

Writing (with our usual convention for the case $r = 2$ in force)

$$w(\lambda + \delta) - \delta = \sum n_i \beta_i, \quad (6)$$

we see that (5) becomes

$$2n_1 - n_2 \neq 0, \quad (7)$$

a condition we can check directly. From the discussion in (3.10), we can see that we are concerned with the elements $w_{k, -(k+1)}$ ($k = 1, \dots, r-1$). From the tables, we see that if $w = w_{k, -(k+1)}$ and

$$\begin{aligned} \lambda &= \sum m_i \lambda_i, \\ w(\lambda + \delta) - \delta &= \left(-k + 1 + \sum_{i \geq k} m_i \right) \beta_1 + (-2r + 2 + m_k) \beta_2 + \dots \end{aligned} \quad (8)$$

We calculate (7):

$$2n_1 - n_2 = 2(r - k) + m_k + 2 \sum_{i > k} m_k \geq 2.$$

This finishes the verification of (3.3(2)) for the case CI, along the second (codimension $j(2)$) singular stratum.

(3.13). We turn next to condition (iv) of the theorem of (3.1). We begin by discussing in general terms some issues surrounding duality and L_2 cohomology.

Let M be an arbitrary Riemannian manifold of (real) dimension m , and $\mathbb{E}$ be as in (2.1). In what follows, we assume M to be oriented (otherwise, a twist by the orientation sheaf will be required).

If $\phi \in L_2^i(M, \mathbb{E})$ is in the domain of d , then $*\phi \in L_2^{m-i}(M, \mathbb{E}^*)$ is in the domain of the weak closed extension of the formal adjoint δ of d (the one with maximal domain). Thus, $L_2^i(M, \mathbb{E}^*)$ becomes a chain complex of Hilbert spaces, when δ is taken as differential, isomorphic to $L_2^i(M, \mathbb{E})$ after a reindexing.

Suppose that

$$d: L_2^{i-1}(M, \mathbb{E}) \rightarrow L_2^i(M, \mathbb{E})$$

has closed range. Then, of course, the same is true for

$$\delta: L_2^{m-i+1}(M, \mathbb{E}^*) \rightarrow L_2^{m-i}(M, \mathbb{E}^*).$$

As is well-known, it follows that

$$\delta*: L_2^{m-i}(M, \mathbb{E}^*) \rightarrow L_2^{m-i+1}(M, \mathbb{E}^*)$$

also has closed range. The operator δ^* , which we also denote by ${}_0d$, is the strong closure of d on smooth forms of compact support in M ; the cohomology of $L_2^i(M, \mathbb{E}^*)$ with respect to ${}_0d$ will be denoted ${}_0H_{(2)}^i(M, \mathbb{E}^*)$.

REMARK: If M is the interior of a *complete* Riemannian manifold-with-boundary, then an L_2 form smooth to the boundary is in the domain of ${}_0d$ if and only if its restriction to the boundary vanishes. (This includes the assertion that if the boundary is empty, one has ${}_0d = d$.)

We recall that the assumption of closed range is equivalent to the assertion that the cohomology in the corresponding degree is isomorphic to the space of strictly harmonic L_2 forms, namely those satisfying the appropriate “boundary condition” (for d , it is that ϕ be in the domain of d^*). Now, the space of harmonic forms for d is mapped by $*$ isometrically onto those for δ . Moreover, the conditions on a form to be harmonic are the same for an operator and its adjoint. We therefore obtain:

PROPOSITION: If $dL_2^k(M, \mathbb{E})$ is closed in $L_2^{k+1}(M, \mathbb{E})$ for $k = i - 1$ and $k = i$, then the restriction of $*$ to harmonic forms induces an isomorphism

$$H_{(2)}^i(M, \mathbb{E}) \simeq {}_0H_{(2)}^{m-i}(M, \mathbb{E}^*).$$

(3.14). Let $\bar{M}$ be a topological compactification of M , as in (2.3). We

wish to use the preceding proposition, if possible, to establish an isomorphism

$$H_c^i(W, \mathcal{L}_{(2)}^\bullet(\overline{M}, \mathbb{E})) \simeq H_{(2)}^{m-i}(W \cap M, \mathbb{E}^*) \quad (1)$$

for (certain) open subsets W of $\overline{M}$. This is something to be determined along the (topological) boundary of W , which we do locally.

We define a complex of sheaves ${}_0\mathcal{L}_{(2)}^\bullet(\overline{W}, \mathbb{E})$ on $\overline{W}$ as follows. To each open subset V of $\overline{W}$ associate the space of forms

$$\phi \in L_2^\bullet(V \cap W, \mathbb{E})$$

for which there is a sequence of C^∞ forms $\{\phi_j\}$ in $V \cap W$ whose support is disjoint from the boundary of W , such that $\phi_j \rightarrow \phi$ and $d\phi_j \rightarrow d\phi$ (in L_2 norm). As before, these sheaves are fine if $\overline{M}$ admits enough L_2 -admissible coverings. If this is the case, there are natural isomorphisms

$$\begin{aligned} H^*(\overline{W}, {}_0\mathcal{L}_{(2)}^\bullet(\overline{W}, \mathbb{E})) &\simeq H^*(\Gamma(\overline{W}, {}_0\mathcal{L}_{(2)}^\bullet(\overline{W}, \mathbb{E}))) \\ &\simeq {}_0H_{(2)}^*(W \cap M, \mathbb{E}). \end{aligned} \quad (2)$$

The following is then immediate.

PROPOSITION: *Let $\overline{M}$ be as above. Suppose that every point of $\overline{W} - W$ has a fundamental system of neighborhoods V in $\overline{W}$ for which*

$$H^*(V, {}_0\mathcal{L}_{(2)}^\bullet(\overline{W}, \mathbb{E})) = 0. \quad (3)$$

Then ${}_0H_{(2)}^(W \cap M, \mathbb{E}) \simeq H_c^*(\overline{W}, \mathcal{L}_{(2)}^\bullet(W, \mathbb{E}))$.*

This, when coupled with the proposition of (3.13), provides a criterion for (1):

COROLLARY: *If the above condition on $\overline{M}$ and W are satisfied,*

$$H_c^i(W, \mathcal{L}_{(2)}^\bullet(\overline{M}, \mathbb{E})) \simeq H_{(2)}^{m-i}(W \cap M, \mathbb{E}^*).$$

(3.15). Of course, we still need a way to determine when (3.14(3)) holds. We next show that if the boundary of W sufficiently resembles a geometric boundary, then the criterion will be met.

Let I be the Euclidean interval $(0, 1)$, N be any complete Riemannian manifold-with-corners, and put $M = I \times N$. We impose the zero boundary condition on $\{0\} \times N$, and whatever on $I \times \partial N$; that is, we

consider $L_2^*(M, \mathbb{E})$, for any $\mathbb{E}$, as a complex of Hilbert spaces with differential

$$d' = d_I + \sigma_I d_N, \quad (1)$$

where σ_I is a sign. Let K^* denote this complex.

PROPOSITION: $H^*(K^*) = 0$.

PROOF: Let, first, ϕ be a smooth form of bounded support that vanishes in a neighborhood of $\{0\} \times N$. Such ϕ are dense in the graph norm for d' . Let r be the variable on I . We may write

$$\phi = d(B\phi) + B(d\phi), \quad (2)$$

where

$$(B\phi)(r) = \int_0^r \phi L dr. \quad (3)$$

We can see that $B\phi$ is also a smooth form of bounded support, vanishing in a neighborhood of $\{0\} \times N$. Moreover, B extends to a bounded operator on K^* . By taking limits, we see that (2) persists on the domain of d' , i.e., on K^* , from which the desired result follows.

(3.16). Because the open sets of (3.8) have non-convex corners, and for other reasons as well, we desire to construct a fundamental system of neighborhoods of $\overline{e'(Q)}$ whose restrictions to $\Gamma \backslash X$ have smooth boundary. To do this, we must first compare distinguished coordinates associated to $Q \in {}_{\mathbb{Q}}\mathfrak{P}(G)$ and those of its parabolic subgroups.

Fix $x \in X$, and let $P \subset Q$. Any point $x' \in X$ can be written

$$x' = p \cdot a \cdot x, \quad (1)$$

where $p \in {}^0P_{\mathbb{R}}$, and $a \in A_{P,x}$ is uniquely determined. We convert (1) into a corresponding expression relative to Q . Writing, as in (1.3(14)),

$$A_{P,x} = A_{Q,x} \times B_{P,Q,x}, \quad (2)$$

we accordingly decompose a as

$$a = a_1 a_2. \quad (3)$$

We see that we then have

$$x' = (pa_2) \cdot a_1 \cdot x, \quad (4)$$

with $(pa_2) \in {}^0Q$ and $a_1 \in A_{Q,x}$.

We need to express a_1^β , for $\beta \in \Delta_Q$, in terms of the $a^{\alpha/s}$ ($\alpha \in \Delta_P$). Toward this end, we observe that the factors in (2) are defined by

$$A_{Q,x} = \bigcap_{\alpha \in \Delta_P - \Delta_Q} \ker \alpha, \quad (5)$$

and

$$B_{P,Q,x} = \bigcap_{\beta \in \Delta_Q} \ker \lambda_\beta, \quad (6)$$

where λ_β is the fundamental dominant weight dual to β ; the formula (6) is just the assertion that the Lie algebra of $B_{P,Q,x}$ is orthogonal to $\mathfrak{a}_{Q,x}$. From (3) and (6), we obtain

$$a_1^{\lambda_\beta} = a^{\lambda_\beta} \quad (\beta \in \Delta_Q) \quad (7)$$

If we now express λ_β in terms of the α 's in Δ_Q , we get the desired formula. This involves a submatrix of the inverse of the Cartan matrix, namely the one that corresponds to $\Delta_Q \times \Delta_P$.

If Q is maximal, the formula is particularly simple. Let $\Delta_Q = \{\beta\}$. We write

$$\lambda_\beta = m_\beta \beta + \sum_{\alpha \neq \beta} m_\alpha \alpha, \quad (8)$$

where $m_\alpha \geq 0$ for all α , and $m_\beta > 0$. (In fact, if the $\mathbb{Q}$ -root system is irreducible, $m_\alpha > 0$ for all α .) From (7) and (8), we obtain:

$$a_1^\beta = a^\beta \prod_{\alpha \neq \beta} (a^\alpha)^{m_\alpha/m_\beta}. \quad (9)$$

Thus,

PROPOSITION: *Let Q be a maximal $\mathbb{Q}$ -parabolic subgroup containing P . Then for $a, t \in \hat{A}_P$ and $\{\beta\} = \Delta_Q$, $a^\beta > t^\beta$ if and only if*

$$a_1^\beta > t^\beta \prod_{\alpha \neq \beta} (a^\alpha)^{m_\alpha/m_\beta}.$$

(3.17). We continue to take $\underline{Q}$ maximal, and recall the covering (3.7(1)) of a neighborhood $\bar{N}(Q)$ of $e'(Q)$. Its projection onto $e'(Q)$ defines a distinguished covering

$$\mathfrak{B} = \{q'_Q W(P)\} \quad (1)$$

of $e'(Q)$. Since $W(P)$ is a geodesic influx from a subset of $e'(P)$ (see (3.6), (1.5(2))), there is a corresponding projection of $W(P)$ onto $\hat{A}_P$, which defines the distinguished coordinate a_P . We put

$$g_P = \prod_{\alpha \neq \beta} (a_P^\alpha)^{m_\alpha/m_\beta}; \quad (2)$$

here we are keeping the notation of the proposition of (3.16). From its definition, we can see that g_P is constant along orbits of the geodesic action of $\hat{A}_Q$, i.e., g_P descends to a function on $q'_Q W(P)$.

Though it can also be deduced from the geometry, we can see directly that on $W(P_1) \cap W(P_2)$ the functions g_{P_1} and g_{P_2} have the same growth (i.e., their ratio takes values in a compact subset of $(0, \infty)$). We first observe:

LEMMA: *Let $P \subset P'$. Then on $W(P) \cap W(P')$, the distinguished coordinate $a_{P'}$, on $W(P')$ and the restriction of the $\hat{A}_P$ -coordinate of $W(P)$ are mutually bounded.*

PROOF: It is a question of comparing a^α and a_1^α (as in (3.16(3))), with P' replacing Q) for $a \in \hat{A}_P$ and $\alpha \in \Delta_{P'}$. From equation (3.16(7)), we have $(a/a_1)^{\lambda_\alpha} = 1$. On the other hand, if $\alpha \notin \Delta_{P'}$, then a^α is bounded from above (and, of course, also from below) on $W(P) \cap W(P')$. It follows rather easily from (3.16(8)) that $(a/a_1)^\alpha$ takes values in a compact subset of $(0, \infty)$ for $\alpha \in \Delta_{P'}$, as desired.

Thus, if $P \subset P' \subset Q$, we can see directly from (2) that g_P and $g_{P'}$ are mutually bounded. Although we can have arranged the inclusion in Q (Remark, (3.7)), the general criterion for $W(P) \cap W(P') \neq \emptyset$ is that $\gamma P \subset P'$ (or vice versa) for some $\gamma \in \Gamma_Q$. To see that this Γ_Q -conjugation does not really matter, we begin by recalling that

$$W(P) = W'_x(V', t)$$

for some $V' \subset e'(P)$. We can rewrite this as

$$W(P) = W_{\gamma x}(V'', \gamma t) \quad (3)$$

(see (1.3(5))), where V'' is the result of transporting V' to $e'(\gamma P)$ via γ .

We change notation here for convenience, writing P instead of γP . Choose $p \in P$ so that $px = \gamma x$, and write

$$p = {}^0pb, \quad \text{with } {}^0p \in {}^0P \text{ and } b \in A_{P,x}. \quad (4)$$

We compute, for $q \in {}^0P$:

$$\sigma_{\gamma x}(q(\gamma x)_P) = \sigma_{p_x}(q(px)_P) = qp_x, \quad (5)$$

$$\sigma_x(q(\gamma x)_P) = \sigma_x(q^0 p_x)_P = q^0 p_x. \quad (6)$$

Thus, we get:

PROPOSITION: *With notation as above,*

$$\sigma_{\gamma x} = b \circ \sigma_x.$$

It follows that the function a_P is canonically defined, up to a constant multiple, independent of the choice of basepoint for X .

(3.18). Since the covering (3.17(1)) is L_2 -admissible, there is a partition of unity $\{f_P\}$ on $e'(Q)$ subordinate to $\mathfrak{B}$ with each df_P of bounded Riemannian norm. We put

$$g = \sum_P f_P g_P. \quad (1)$$

As f_P and g_P are constant along the fibers of p'_Q so is g . Clearly, the restriction of g to $q'_Q W(P)$ has the same growth as g_P . Identifying $\hat{A}_Q \approx (0, \infty)$, we regard g as an $\hat{A}_Q$ -valued function, and let $\tilde{g}$ be the corresponding Γ_Q -invariant function on $e(Q)$.

By the construction of g , we can see that for $t_0 \in \hat{A}_Q$ sufficiently large, $t_0 \tilde{g} \circ \sigma_x$ takes its values near enough to $e(Q)$ that it descends to define a cross-section over $e'(Q)$ to the geodesic action of $\hat{A}_Q$. By abuse of notation, we denote this cross-section by $t_0 g \sigma'_x$. We can now assert:

PROPOSITION: (i) *The system of collars $\{W'(e'(Q), t_0 g \sigma'_x, t)\}$ and the collection of neighborhoods $\{N(Q)\}$ are equivalent systems of deleted neighborhoods of $e'(Q)$.*

(ii) *$\{\bar{W}'(e'(Q), t_0 g \sigma'_x, t)\}$ is a fundamental system of neighborhoods of $e'(Q)$.*

(iii) *The inclusion of $W'(e'(Q), t_0 g \sigma'_x, t)$ in the neighborhood $N(Q)$ (for t sufficiently large) induces an isomorphism on L_2 -cohomology.*

PROOF: Statements (i) and (ii) are obvious from the construction. To see (iii), we can compare the spectral sequences (2.2(2)) of the coverings $\mathfrak{N}(Q)$ of (3.7(1)) and the induced covering of $W'(e'(Q), t_0 g \sigma'_x, t)$. That the E_2 terms for the two are isomorphic can be deduced from (2.6).

(3.19). We write down the desired neighborhoods, and give a formula for the stalks of $\mathcal{H}'\mathcal{L}_{(2)}(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$ along $\hat{e}''(\Theta_s)$. Let $Q = Q_{\Lambda}$. We have now that fundamental neighborhoods of $e'(Q)$, expressed in distinguished coordinates, are quotients of

$$\{(a, z, u) \in \hat{A}_Q \times \hat{e}(Q) \times U_Q : a^\beta > t^\beta g(z)\} \quad (\beta = \beta_s). \quad (1)$$

We make the change of variables

$$r = a^\beta / g(z). \quad (2)$$

Then (1) becomes

$$\{(r, z, u) \in \hat{A}_Q \times \hat{e}(Q) \times U_Q : r^\beta > t^\beta\} \simeq \hat{A}_Q(t) \times \hat{e}(Q) \times U_Q. \quad (3)$$

In the new coordinates, the metric (2.10(5)) has the explicit formula

$$ds^2 = (dr/r + dg/g)^2 + dz^2 + \sum_{\beta} r^{-2\beta} g(z)^{-2\beta} du_{\beta}^2(z); \quad (4)$$

by comparing (3.17(2)) with the formula for dz^2 ((2.10(5)) for $\hat{e}(Q)$ rather than X), we see that this is quasi-isometric to

$$(dr/r)^2 + dz^2 + \sum_{\beta} r^{-2\beta} g(z)^{-2\beta} du_{\beta}^2(z) \quad (5)$$

(cf. (2.10(8))). Similarly, the metric on $\mathbb{E}$ is quasi-isometric to

$$\sum_{\beta} r^{-2\beta} g(z)^{-2\beta} h_{\beta}(z), \quad (6)$$

if u is constrained to lie in a compact subset of U_Q .

We recall that $\hat{e}(Q) \simeq \hat{e}(Q_{\Theta_s}) \times \hat{e}(Q_{\Psi_s})$, and drop the subscript s . We can now state:

THEOREM: (i) *The intersection with $\Gamma \backslash X$ of an open set in a fundamental system of neighborhoods of a point of $\hat{e}''(Q_{\Theta})$ is, with the coordinates of (3), of the form*

$$W' \simeq \pi(A_Q(t) \times (O \times \hat{e}'(Q_{\Psi})) \times U_Q), \quad (7)$$

where O is a disc in $\hat{e}''(Q_{\Theta})$.

(ii) $L_2^\cdot(W', \mathbb{E})$ is naturally embedded in

$$\bigoplus_{\mu \in \alpha_Q^*} \left[L_2^0(\Gamma_{Q_\Psi} \setminus {}^0Q_\Psi \times O, \mathbb{C}; \omega_\mu) \otimes \Lambda_\mu^\cdot(E) \right. \\ \left. \otimes \wedge^* \mathfrak{p}_M^* \hat{\otimes} L_2^\cdot(A_Q(t), \mathbb{C}; h_\mu) \right]$$

where

$$\hat{\omega}_\mu = \hat{g}^{-2(\mu + \delta_Q)},$$

and $\hat{g}$ is the pullback of the restriction of g to $\hat{e}'(Q_\Psi)$.

(iii) There is a “Künneth formula”:

$$H_{(2)}^\cdot(W', \mathbb{E}) \simeq \bigoplus_{k, \mu} \left(H_{(2)}^\cdot(\hat{e}'(Q_\Psi), \mathbb{H}_\mu^k(\mathfrak{u}_Q, E); \omega_\mu) \right. \\ \left. \otimes H_{(2)}^\cdot(A_Q(t), \mathbb{C}; h_\mu)[-k] \right) \quad (8)$$

if in every term of the right-hand side, one of the two factors of the tensor product is finite-dimensional.

PROOF: (i) follows immediately from the proposition in (1.6), and we may use (6), in conjunction with (5), to get (ii). To obtain (iii), we first note from (4) and (6) that O splits off, up to quasi-isometry, as a Euclidean disc factor, so we may replace it by a point. We then wish to apply the proposition of (2.5), with

$$K_\mu^\cdot = L_2^\cdot(A_Q(t), \mathbb{C}; h_\mu), \quad L_\mu^\cdot \subset L_2^0(\Gamma_{Q_\Psi} \setminus {}^0Q_\Psi, \mathbb{C}; \hat{\omega}_\mu) \\ \otimes \Lambda_\mu^\cdot(E) \otimes \wedge^* \mathfrak{p}_M^*.$$

Let B be one of the homotopy operators to the projection of $L^\cdot$ onto the sub-complex

$$\bigoplus_{k, \mu} L_2^\cdot(\hat{e}'(Q_\Psi), \mathbb{H}_\mu^k(\mathfrak{u}_Q, E); \omega_\mu)[-k]; \quad (9)$$

it is viewed a priori as an unbounded operator. By considering the restriction of weights from A_{Q_Ψ} to A_Q , we see from (2.5(6)) that

$$B(L_\mu^\cdot) \subset \bigoplus_{\gamma \geq \mu} L_\gamma^\cdot. \quad (10)$$

Since g is bounded from below, we see that $\gamma \geq \mu$ implies $\omega_\gamma \leq \omega_\mu$. It follows, first, that B here is in fact a bounded operator, and then that

the hypothesis (2.5(6)) is satisfied. Since we are on a complete manifold-with-boundary, the strict decomposition (2.5(4)) of d holds. The proposition of (2.5) now gives that $H_{(2)}^*(W', \mathbb{E})$ is the cohomology of the complex

$$\bigoplus_{k, \mu} \left(L_2^*(A_Q(t), \mathbb{C}; h_\mu) \hat{\otimes} L_2^*(\hat{e}'(Q_\Psi), \mathbb{H}_\mu^k(u_Q, E); \omega_\mu)[-k] \right), \quad (11)$$

which is, as written, a direct sum of Hilbert space tensor products of complexes. The cohomology of each summand can be computed by the Künneth formula (see [15, (2.36)]), if (say) d has closed range on one of its factors, giving (8); this is the case under the hypothesis in (iii).

(3.20). We now, at long last, turn our attention to condition (iv) from (3.1), for $\mathcal{L}^* = \mathcal{L}_{(2)}^*(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$. By the proposition in (3.13), this condition is an immediate consequence of condition (iii) for $\mathbb{E}^*$, once we know that the criterion of (3.14) is satisfied. It thus suffices to see how the boundary of an open domain $W' \subset \Gamma \backslash X$ of (3.19(7)) comes in to the boundary (in the sense of (1.6)!) components of $\Gamma \backslash_{\mathbb{Q}} X^*$, and to show that we can locally factor out, up to quasi-isometry, a Euclidean interval from W' (see (3.15)). From (3.19(3), (5)), we see that there is a quasi-isometric factor of the interval $(1, 2) \subset A_Q$ in W' . It remains to see that a neighborhood of a point in some $\hat{e}''(Q_{\Theta'})$ ($\Theta' \supset \Theta$) is a product with this interval. But this is just the assertion that the geodesic action of A_Q descends to $\Gamma \backslash_{\mathbb{Q}} X^*$ nearby. This gives:

PROPOSITION: *The “dual support condition” (iv) of (3.1) is satisfied by $\mathcal{L}_{(2)}^*(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E})$ along a stratum $S^{j(s)}$ if (iii) is satisfied by $\mathcal{L}_{(2)}^*(\Gamma \backslash_{\mathbb{Q}} X^*, \mathbb{E}^*)$ there, and the cohomology sheaves have finite-dimensional stalks.*

REMARK: We write, in the case $_{\mathbb{Q}}\Delta =_{\mathbb{R}}\Delta$, rank two, $Q = Q_{\Lambda_2}$, $Q' = Q_{\Lambda_1}$, $P = Q_\phi$

$$\begin{aligned} \tilde{W}(P) &= \pi(O_P \cdot U_P \cdot \hat{A}_P(t) \circ x) \\ &= \pi(O_P \cdot U_P \cdot \hat{A}_Q(t) \circ \hat{A}_{Q'}(t) \circ x), \end{aligned} \quad (1)$$

where O_P maps onto a small open ball in $\hat{e}'(P)$. Projecting this onto $e'(Q')$ gives

$$\pi_{Q'}(O_P \cdot U_P \cdot \hat{A}_Q(t) \circ x_{Q'}). \quad (2)$$

We obtain the following explicit description:

(i) A point $y \in \hat{e}'(Q')$ is in the closure of $\tilde{W}(P)$ if and only if

$$y \in p'_{Q'} \pi_{Q'}(O_P \cdot U_P \cdot \hat{A}_Q(t) \circ x_{Q'}).$$

(ii) If y is at the boundary of the above set in $\hat{e}'(Q')$, then near y , $\tilde{W}(P)$ looks like

$$N \times (\Gamma_{U_{Q'}} \setminus U_{Q'}) \times I \times \hat{A}_{Q'}(t), \quad (3)$$

where N is a contractible subset of $\hat{e}'(Q')$, and I is a bounded interval

$$\{a \in \hat{A}_Q(t) : t \leq a < t'\}.$$

The proposition, coupled with our previous calculations ((3.10)–(3.12)), gives:

THEOREM: *The Conjecture (3.2) is true for arithmetic quotients of the genus two Siegel upper half-space ($G_{\mathbb{Q}} = Sp(4, \mathbb{Q})$).*

(3.21). The method of (3.9) seems a bit cumbersome for general use. We will develop, in the remaining sections of this chapter, an alternate approach based on (3.19(8)), and apply it to $G = SU(p, 2)$.

We begin by discussing the link of the component $\hat{e}''(Q_{\Theta_i})$ of the stratum $S^{(s)}$. From (1.6), we expect it to be some quotient of the $(\Gamma_{U_{\Lambda_i}} \setminus U_{\Lambda_i})$ -fibration over $\hat{e}'(\overline{Q_{\Psi_i}})$. We will be able to decide which quotient by examining what happens to the closure of the boundary of the set W' of (3.19(7)), or more simply $W = \pi^{-1}(W')$ in $\bar{X}$ under the mapping onto ${}_{\mathbb{Q}}X^*$.

For any $\Upsilon \subseteq \Psi_s$, let R_{Υ} denote the corresponding parabolic subgroup of M_{Ψ_i} . Then for any $y \in \hat{e}(Q_{\Theta_i})$, $e(R_{\Upsilon}) \times \{y\}$ is embedded at the boundary of $\hat{e}(Q_{\Lambda_i})$ as a subset of $\hat{e}(Q_{\Upsilon \cup \Theta_i})$, and its contribution to the link comes from an influx of its U_{Λ_s} -fibration into $e(Q_{\Upsilon \cup \{\beta_s\} \cup \Theta_i})$.

In forming the space ${}_{\mathbb{Q}}X^*$ from $\bar{X}$, one first takes the quotient of $e(Q_{\Theta_i})$ by ${}_{\mathbb{Q}}A_{\kappa(\Theta)}$, where $\kappa(\Theta)$ is the largest set Θ_i contained in Θ [17,(3.6(4))]. (When

$$\Theta = \Upsilon \cup \{\beta_s\} \cup \Theta_s,$$

we get $t \leq s - 1$.) This induces the quotient of $e(R_{\Upsilon})$ by ${}_{\mathbb{Q}}A_{\Upsilon_i}$, where $\Upsilon_i = \Theta_i \cap \Psi_s$. In other words, we see here (modulo the U_{Λ_s} -fibration) the first step in constructing a Satake compactification of $\hat{e}(Q_{\Psi_i})$, namely the one corresponding to $\Xi = \{\beta_{s-1}\}$ (cf. (1.6)).

By construction, near $e(R_{\Upsilon})$, the section used to define W is constant on $U_{R_{\Upsilon}}$ -orbits (3.17(2), 3.18(1)), and therefore also in the limit. With this observation, we can assert:

THEOREM: (i) *The link of $S^{j(s)}$ is the closure of the boundary of W' in $\Gamma \backslash_{\mathbf{Q}} X^*$.*

(ii) *It admits a stratified mapping into the Satake compactification of $\hat{e}'(Q_{\Psi_s})$ corresponding to $\Xi = \{\beta_{s-1}\}$.*

(iii) *The fiber over the points of the stratum $\hat{e}''(R_{\Gamma_i})$ is a compact quotient of $U_{\Lambda_i}/(U_{\Lambda_i} \cap U_{\Theta_i})$.*

(3.22). For convenience, we put $Y_s = \hat{e}'(Q_{\Psi_s})$, and let Y_s^* denote the Satake compactification in (ii) above. We describe how one might go about interpreting the first factor on the right-hand side of (3.19(8)), viz. the weighted L_2 cohomology group on Y_s , as a cohomology group on Y_s^* .

Let $P = Q_{\Theta}$, $Q = Q_{\Lambda_s}$, and $R = Q_{\Psi_s}$. We have

$$W^P = W_R \cdot W^Q,$$

where W_R is the Weyl group of R . For $w \in W^Q$ we let, as before, E'_w be the irreducible constituent of $H^*(u_Q, E)$ of highest weight $\mu = w(\lambda + \delta) - \delta$. If R' is a parabolic subgroup of M_R , then

$$H^*(u_{R'}, H^*(u_Q, E))$$

has constituents E''_{τ} with $\tau \in (W_R)^{R'}$; the highest weight of E''_{τ} is the restriction of

$$\tau w(\lambda + \delta) - \delta. \quad (1)$$

For any $w \in W^P$, we write

$$w(\lambda + \delta) = \sum n_i(w, \lambda) \beta_i. \quad (2)$$

PROPOSITION: *On W^P , the coefficient $n_i(w, \lambda)$ is constant on right W_R cosets if $i \geq s$.*

PROOF: Since W_R is generated by simple reflections corresponding to Ψ , the application of $\tau \in W_R$ changes only the coefficients $n_i(w, \lambda)$ for $i \leq s-1$.

The weighting factor from (3.19(8)) can be written asymptotically in a corner as

$$\hat{\omega}_{\mu} = [a^{(\beta_1 + 2\beta_2 + \dots + (s-1)\beta_{s-1})/s}]^{-2(\mu + \delta_Q)} = [a^{\xi_{s-1}}]^{-2(\mu + \delta_Q)}, \quad (3)$$

where ξ_{s-1} is the last fundamental dominant weight of the restricted

root system A_{s-1} of M_R . On the other hand, β_s is the only simple root outside Ψ_s that restricts non-trivially in M_R , and one sees easily that restriction to be $-\xi_{s-1}$. Thus, in view of the Proposition, the weighting is exactly offset by the restriction of β_s (this is actually a general phenomenon). In practice, it means that in making a computation, we can just truncate the sum (2) after β_{s-1} . (This is not really so surprising: cf. (3.9).)

(3.23). We now restrict ourselves to the case $s = 2$. Here Y_2^* is obtained from Y_2 by adjoining a finite number of points (cusps). We can analyze the weighted L_2 sheaf $\mathcal{L}_{(2)}^\bullet(Y_2^*, \mathbb{E}'_w; \hat{\omega}_\mu)$ on Y_2^* . For $w \in W^Q$, $\tau \in W_R$, we have τw contributing in degree $l(\tau)$ to the $\mathbb{E}'_w$ cohomology of a deleted neighborhood of the corresponding cusp, and $\mathbb{E}'_w$ is itself shifted by $l(w)$.

Let $G = \mathrm{Sp}(2r, \mathbb{R})$ again, and take $\mathbb{E} = \mathbb{C}$. We recall from (3.10) that W^Q is comprised of the elements $w_{k,l}$ ($k < l$), $w_{k,-l}$ and $w_{-k,-l}$ ($k > l$). Also W_R has two elements, 1 and $\tau = w_{2,1}$. The elements of W^P with $n_2(w) = n_2(w, 0) > 0$ (then $k < l$) are listed with the calculation of $n_1(w)$:

$$\begin{aligned} n_1(w_{k,l}) &= r - k + 1 > 0, \\ n_1(\tau w_{k,l}) &= n_1(w_{l,k}) = r - l + 1 > 0; \\ n_1(w_{k,-l}) &= r - k + 1 > 0, \\ n_1(\tau w_{k,-l}) &= n_1(w_{-l,k}) = -r + l - 1 < 0. \end{aligned} \tag{1}$$

We get immediately from (1) that for $k < l$,

$$\mathcal{L}_{(2)}^\bullet(Y_2^*, \mathbb{E}'_w; \hat{\omega}_\mu) \text{ is quasi-isomorphic to } \begin{cases} Rj_* \mathbb{E}'_w & \text{if } w = w_{k,l} \\ j_* \mathbb{E}'_w & \text{if } w = w_{k,-l} \end{cases}, \tag{2}$$

where $j: Y_2 \rightarrow Y_2^*$ denotes the inclusion. The calculation for general initial coefficients $\mathbb{E}$ is similar.

The information in (1) and (2), coupled with the non-triviality of $E_{w_{k,-(k+1)}}$, provides an alternate argument proving the vanishing of the local L_2 cohomology at or above degree $j(2) = 2r - 1$. Furthermore, by the truncation property of intersection homology [10,(2.4)], we obtain:

PROPOSITION: *Let L^* be the link of $S^{j(2)}$ in the case $G = \mathrm{Sp}(2r, \mathbb{R})$. Then in degrees less than $2r - 1$,*

$$\begin{aligned} IH^*(L^*, \mathbb{E}) &\simeq \bigoplus_{k < l} H^*(Y_2, \mathbb{E}'_{w_{k,l}})[-l(w_{k,l})] \\ &\quad \oplus \bigoplus_{k < l} IH^*(Y_2^*, \mathbb{E}'_{w_{k,-l}})[-l(w_{k,-l})]. \end{aligned}$$

REMARK: It would be nice to understand this formula directly, via the geometry of L^* . (N.B. $-\mathbb{E}'_w$ depends on $\mathbb{E}$.)

(3.24). From (3.19(8)), we can reformulate our conjecture in the following explicit form:

CONJECTURE: *Let $w \in W^Q$. Then:*

- i) *If $n_s(w, \lambda) = 0$, then $H_{(2)}^*(Y_s, \mathbb{E}'_w) = 0$ (the weight here is trivial).*
- ii) *If $n_s(w, \lambda) > 0$, then $H_{(2)}^{i-l(w)}(Y_s, \mathbb{E}'_w; \hat{\omega}_\mu)$ vanishes whenever $i \geq j(s)$.*

In the rank two case, one could appeal to [18] to obtain (i) when $s = 2$. The reason for this is that the global L_2 -cohomology $H_{(2)}^*(\Gamma \backslash X, \mathbb{E})$ is known to be finite dimensional [18,(4.5)], whereas in the spectral sequence of the covering W (3.6(1)) – see also (iii) of the proposition in (3.18) – the only possible infinite-dimensional contributions come from the cusps, hence survive in the abutment. However, it is likely that one can see (i) directly, via the methods used to verify the rest of the conjecture.

The real groups which give rise to Y_2 all have type- A behavior (the restricted root system is of type A_1) and are described by the following chart.

$$\begin{array}{llllllll} G_{\mathbb{R}}: & AIII & BI & CI & DI & DIII & EIII & EVII \\ M_{\mathbb{R}}: & SL(2, \mathbb{C}) & BI & AI & DI & AII & DI & DI \end{array}$$

(see [14,pp.30–32]; we regard DII there as a degenerate case of DI).

(3.25). Let $G = SU(p, q)$. We will verify the conjecture of (3.24) for $s = 2$ when $\mathbb{E} = \mathbb{C}$. (The discussion extends to some, but not all, other coefficients $\mathbb{E}$.) We put, as usual, $n = p + q - 1$ (the $\mathbb{C}$ -rank of G). The roots in $\mathfrak{u}_p$ are listed below, grouped according to their restrictions to $\mathfrak{a}_p$:

$$\begin{aligned} \beta_1: & \quad \epsilon_1 - \epsilon_2, \epsilon_n - \epsilon_{n+1}, \\ 2\beta_2: & \quad \epsilon_2 - \epsilon_n, \\ \beta_1 + 2\beta_2: & \quad \epsilon_1 - \epsilon_n, \epsilon_2 - \epsilon_{n+1}, \\ 2\beta_1 + 2\beta_2: & \quad \epsilon_1 - \epsilon_{n+1}, \\ \beta_1 + \beta_2: & \quad \epsilon_1 - \epsilon_j \ (2 < j < n), \epsilon_i - \epsilon_{n+1} \ (2 < i < n), \\ \beta_2: & \quad \epsilon_2 - \epsilon_j \ (2 < j < n), \epsilon_i - \epsilon_n \ (2 < i < n). \end{aligned} \tag{1}$$

By deleting the first row, we obtain a list of the roots in $\mathfrak{u}_Q$. We can see that

$$\delta_p = n\beta_1 + (2n - 2)\beta_2 \tag{2}$$

and

$$j(2) = 2n - 2. \quad (3)$$

W^P consists of the elements of W (permutations) $w_{s,t}^{k,l}$ defined by the conditions $w^{-1}(\epsilon_1) = \epsilon_k$, $w^{-1}(\epsilon_2) = \epsilon_l$, $w^{-1}(\epsilon_n) = \epsilon_s$, $w^{-1}(\epsilon_{n+1}) = \epsilon_t$, and w^{-1} is increasing on the remaining ϵ_j 's. The subset W^Q of W^P is defined by the inequalities $k < l$ and $s < t$. We have again $W^P = W_R W^Q$, where now $W_R \simeq (\mathbb{Z}/2\mathbb{Z})^2$, with generators $\tau^+ = w_{n,n+1}^{2,1}$ and $\tau_- = w_{n+1,n}^{1,2}$. We note that $\tau^+ w_{s,t}^{k,l} = w_{s,t}^{l,k}$ and $\tau_- w_{s,t}^{k,l} = w_{t,s}^{k,l}$.

The positive roots in (1) that have negative images under w^{-1} are:

$$\begin{aligned} \text{i)} & \epsilon_1 - \epsilon_j, \text{ for } (k-1) \text{ values of } j, \\ \text{ii)} & \epsilon_1 - \epsilon_{n+1}, \text{ for } (n+1-t) \text{ values of } i, \\ \text{iii)} & \epsilon_2 - \epsilon_j, \text{ for } \begin{cases} (l-1) \text{ values of } j \text{ if } k > l \\ (l-2) \text{ values of } j \text{ if } k < l, \end{cases} \\ \text{iv)} & \epsilon_t - \epsilon_n, \text{ for } \begin{cases} (n+1-s) \text{ values of } i \text{ if } t < s \\ (n-s) \text{ values of } i \text{ if } t > s. \end{cases} \end{aligned} \quad (4)$$

From this, we can see that for $w = w_{s,t}^{k,l} \in W^Q$

$$l(w) = 2n + k + l - s - t - 2 - d(w), \quad (5)$$

where $0 \leq d(w) \leq 4$ and $d(w)$ counts when repetitions of $\epsilon_1 - \epsilon_n$, $\epsilon_1 - \epsilon_{n+1}$, $\epsilon_2 - \epsilon_n$, $\epsilon_2 - \epsilon_{n+1}$ occur in (4) (when, respectively, $k > s$, $k > t$, $l > s$, $l > t$). However, one checks that there is a single formula:

$$(\delta - w(\delta))|_{\alpha_P} = (n + k - t)\beta_1 + (2n + k + l - s - t - 2)\beta_2, \quad (6)$$

valid moreover for all $w \in W^P$, or

$$\begin{aligned} w(\delta)|_{\alpha_P} &= (t-k)\beta_1 + (s+t-k-l)\beta_2 \\ &= (t-k)\beta_1 + (2n-2-l(w)-d(w))\beta_2. \end{aligned} \quad (7)$$

We note that α_P is defined by the equations

$$\epsilon_j = 0 \quad \text{if } 2 < j < n; \quad \epsilon_2 = -\epsilon_n = \beta_2; \quad \epsilon_1 = -\epsilon_{n+1} = \beta_1 + \beta_2. \quad (8)$$

This enables us to calculate $w(\lambda_i)|_{\alpha_P}$ as the sum of the following contributions:

$$\begin{aligned} \text{i)} & \beta_2, \quad \text{when } l \leq i, \\ \text{ii)} & -\beta_2, \quad \text{when } s \leq i, \\ \text{iii)} & \beta_1 + \beta_2, \quad \text{when } k \leq i, \\ \text{iv)} & -\beta_1 - \beta_2, \quad \text{when } t \leq i. \end{aligned} \quad (9)$$

LEMMA: Let $w \in W^P$.

- i) If $t > k$, $n_1(w, \lambda) > 0$ for all λ ; if $t < k$, $n_1(w, \lambda) < 0$ for all λ .
- ii) For $\lambda = \sum m_i \lambda_i$,

$$n_2(w, \lambda) = (s + t - k - l) + \sum m_i (e_i(k) + e_i(l) - e_i(s) - e_i(t)), \quad (10)$$

where

$$e_i(m) = \begin{cases} 1 & \text{if } m \leq i \\ 0 & \text{if } m > i. \end{cases}$$

PROOF: From (7) and (9), we get (ii) directly, and also the formula

$$n_1(w, \lambda) = (t - k) + \sum m_i (e_i(k) - e_i(t)). \quad (11)$$

Since $e_i(m) \geq e_i(m')$ whenever $m < m'$, we obtain (i).

(3.26). We need to determine which elements $w \in W^Q$ enter in the conjecture (3.24). We have that $\hat{\varrho}'(R)$ is associated to $G_{\mathbb{R}} = \mathrm{SL}(2, \mathbb{C})$, so is three-dimensional. If we have ruled out infinite dimensionality, we need therefore consider only $w \in W^Q$ with $l(w) \geq 2n - 5$, or, by (3.25(5)),

$$(s + t - k - l) + d(w) \leq 3. \quad (1)$$

Also, the condition $n_2(w, \lambda) > 0$ can be written, by (3.25(10)), as

$$(s + t - k - l) + (\sigma(k, \lambda) + \sigma(l, \lambda) - \sigma(s, \lambda) - \sigma(t, \lambda)) > 0, \quad (2)$$

where $\sigma(j, \lambda) = \sum_{i \geq j} m_i$ is a non-increasing function of j . It may be useful to observe that

$$n_2(w, \lambda) = \sigma(k, \lambda - \delta) + \sigma(l, \lambda - \delta) - \sigma(s, \lambda - \delta) - \sigma(t, \lambda - \delta). \quad (3)$$

We can compute the highest weight of the constituents of E'_w on $(\mathfrak{m}_R)_{\mathbb{C}} \simeq [\mathfrak{sl}(2, \mathbb{C})]^2$. This amounts to calculating the two integers

$$\nu_1 = \langle w(\lambda + \delta) - \delta, \alpha_1 \rangle, \quad \nu_2 = \langle w(\lambda + \delta) - \delta, \alpha_n \rangle.$$

For the first one,

$$\begin{aligned}
 \nu_1 &= \langle \lambda + \delta, w^{-1}(\alpha_1) \rangle - 1 \quad (\delta = \sum \lambda_i) \\
 &= \langle \sum (m_i + 1) \lambda_i, \epsilon_k - \epsilon_l \rangle - 1 \\
 \nu_1 &= \sum_{k \leq i < l} (m_i + 1) - 1 = \sigma(k, \lambda + \delta) - \sigma(l, \lambda + \delta) - 1 \quad (4)
 \end{aligned}$$

By an identical calculation,

$$\nu_2 = \sum_{s \leq i < t} (m_i + 1) - 1 = \sigma(s, \lambda + \delta) - \sigma(t, \lambda + \delta) - 1. \quad (5)$$

In particular,

LEMMA: *The constituents of E'_w are non-trivial unless $\mathbb{E} = \mathbb{C}$, and $l - k = t - s = 1$.*

(3.27). Let $\mathbb{E} = \mathbb{C}$, and suppose that $w \in W^Q$ satisfies (3.26(1)) and

$$n_2(w) = s + t - k - l > 0. \quad (1)$$

Then one easily checks that $d(w) = 1$ and $n_2(w) = 2$, or $d(w) = 1$ and $n_2(w) = 2$ are the only possibilities. These give $l(w) = 2n - 5$.

PROPOSITION: *Suppose that $l(w) \geq 2n - 5$ and $n_2(w, \lambda) > 0$. Then E'_w is non-trivial.*

PROOF. By the lemma of (3.26), if the assertion were false, then the original representation E is trivial, and $l = k + 1$, $t = s + 1$. But then (1) implies $2(s - k) = 2$, i.e., $s = k + 1$, which is, of course, impossible.

We can now complete the argument. We need to show that for w as above, $H_{(2)}^3(Y_2^*, \mathbb{E}'_w; \hat{\omega}_\mu) = 0$. There is certainly a mapping

$$j_! \mathbb{E}'_w \rightarrow \mathcal{L}_{(2)}^*(Y_2^*, \mathbb{E}'_w; \hat{\omega}_\mu),$$

which is a quasi-isomorphism on Y_2 . We obtain a surjection

$$H_c^3(Y_2, \mathbb{E}'_w) \rightarrow H_{(2)}^3(Y_2^*, \mathbb{E}'_w).$$

But $H_c^3(Y_2, \mathbb{E}'_w) = 0$ unless the coefficients are trivial. Thus we have:

THEOREM: *The conjecture (3.2) is true for arithmetic quotients of the symmetric space of $SU(p, 2)$ and constant coefficients.*

(3.28). For good measure, we check directly that condition (i) of (3.24) holds.

Now, if $n_2(w) = 0$, we have $s + t - k - l = 0$ by (3.25(10)). From (3.26(4),(5)), we get

$$\nu_2 - \nu_1 = t - s - l + k = 2(k - s) \neq 0.$$

Let $\alpha \subset \mathfrak{sl}(2, \mathbb{R})$ denote the subspace of diagonal matrices. Then $\alpha_{\mathbb{C}} \subset \mathfrak{sl}(2, \mathbb{C})$ is the Cartan subalgebra induced by the standard one for $\mathfrak{su}(p, q)$. It is, in fact, a fundamental Cartan subalgebra (see [18,(1.7)]), with $i\alpha$ as its compact factor. Also, $\mathfrak{sl}(2, \mathbb{C})$ is embedded in its complexification $[\mathfrak{sl}(2, \mathbb{C})]^2$ as

$$\{(M, \bar{M}) : M \in \mathfrak{sl}(2, \mathbb{C})\}, \quad (1)$$

where $\bar{M}$ is now the conjugate of M with respect to the real form $\mathfrak{su}(2)$, and complex conjugation is given by

$$(M, N) \mapsto (\bar{N}, \bar{M}). \quad (2)$$

On $\alpha_{\mathbb{C}}$, (2) is just the negative of usual complex conjugation.

We finish by showing:

PROPOSITION: *If $\nu_1 \neq \nu_2$, $H_{(2)}^*(\hat{e}'(R), \mathbb{E}_{\nu}) = 0$.*

PROOF: We make use of an argument involving $(\mathfrak{g}, K)$ -cohomology. A necessary condition for the non-vanishing of $H_{(2)}^*(\hat{e}'(R), \mathbb{E}_{\nu})$ is the equality of the infinitesimal character of E_{ν} and that of an irreducible unitary representation of $\mathfrak{sl}(2, \mathbb{C})$, and this requires conjugate self-contragredience; this can be expressed here as

$$\nu = \bar{\nu}, \quad (3)$$

in view of (2) (see [18,(2.2),(5.6)]). The desired conclusion follows.

(3.29). I have heard it suggested that the method of (3.23) is limited to the rank two case. We will show here that this impression is wrong by proving the conjecture of (3.2) in a rank three example (admittedly the simplest one).

We consider $G = \mathrm{Sp}(6, \mathbb{R})$, for which $r = 3$. In view of (3.12) and (3.20), we need only check the vanishing condition (iii) of (3.1) at the zero-dimensional stratum S^6 (as $j(3) = 6$ here), in order to verify the conjecture in this case.

w (as signed permutation) $w(1), w(2), w(3)$		$l(w)$	$w(\delta)$	$w(\lambda_1)$	$w(\lambda_2)$	$w(\lambda_3)$
		coefficients of $\beta_1, \beta_2, \beta_3,$				
I	1, 2, 3	0	3, 5, 6	1, 1, 1	1, 2, 2	1, 2, 3
II	1, 2, -3	1	3, 5, 4	1, 1, 1	1, 2, 2	1, 2, 1
III	1, -3, 2	2	3, 4, 2	1, 1, 1	1, 1, 0	1, 2, 1
IV	1, -3, -2	3	3, 2, 0	1, 1, 1	1, 1, 0	1, 0, -1
V	-3, 1, 2	3	2, 3, 0	0, 0, -1	1, 1, 0	1, 2, 1
VI	-3, 1, -2	4	2, 1, -2	0, 0, -1	1, 1, 0	1, 0, -1
VII	-3, -2, 1	5	1, -1, -4	0, 0, -1	0, -1, -2	1, 0, -1
VIII	-3, -2, -1	6	-1, -3, -6	0, 0, -1	0, -1, -2	-1, -2, -3

In the Künneth formula (3.19(8)), the space

$$Y_3 = \hat{e}'(\mathcal{Q}) \mathcal{Q} = \mathcal{Q}_{\Lambda_3}$$

is locally symmetric for $\mathrm{SL}(3, \mathbb{R})$, hence is of real dimension 5. The set $W^{\mathcal{Q}}$ consists of eight elements, tabulated along with their “vital statistics” in Table 1.

Suppose that E , a representation of $\mathrm{Sp}(6)$, has highest weight

$$\lambda = m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3$$

By examining the signs of the coefficients of β_3 in $w(\lambda + \delta)$, we use the above table to reduce the conjecture to showing that the corresponding weighted L_2 cohomology groups on Y vanish in the following cases:

- II) $H_{(2)}^5$;
- III) $H_{(2)}^4, H_{(2)}^5$;
- IV) $H_{(2)}'$: A) for all i if $m_1 = m_3$,
B) for $i \geq 3$ if $m_1 > m_3$;
- V) $H_{(2)}'$: A) for all i if $m_1 = m_3$,
B) for $i \geq 3$ if $m_1 < m_3$.

We dispose of most of the above by just calculating the highest weight $w(\lambda + \delta) - \delta$ of the corresponding representation E'_w . We have:

$$w(\lambda + \delta) - \delta = \sum_{i=1}^3 [(m_i + 1)w(\lambda_i) - \lambda_i].$$

This yields in the respective instances:

$$\text{II) } m_1 \lambda_1 + m_2 \lambda_2 + [m_3 \lambda_3 - (m_3 + 1)(2\beta_3)],$$

which restricts to (as $\lambda_3 \mapsto 0$ and $\beta_3 \mapsto -\lambda_2$ in $\mathrm{SL}(3)$)

$$m_1 \lambda_1 + [m_2 + 2m_3 + 2]\lambda_2:$$

$$\text{III) } m_1 \lambda_1 + [(m_2 + 1)(\lambda_1 - \beta_3) - \lambda_2] + [m_3 \lambda_3 - (m_3 + 1)(2\beta_3)],$$

which restricts to

$$[m_1 + m_2 + 1]\lambda_1 + [m_2 + 2m_3 + 2]\lambda_2;$$

$$\text{IV) } m_1\lambda_1 + [(m_2 + 1)(\lambda_1 - \beta_3) - \lambda_2] \\ + [(m_3 + 1)(2\lambda_1 - \lambda_3) - \lambda_3],$$

which restricts to

$$[m_1 + m_2 + 2m_3 + 3]\lambda_1 + m_2\lambda_2;$$

$$\text{V) } [(m_1 + 1)(-\beta_3) - \lambda_1] + [(m_2 + 1)(\lambda_1 - \beta_3) - \lambda_2] \\ + [(m_3 + 1)(\lambda_3 - 2\beta_3) - \lambda_3],$$

which restricts to

$$m_2\lambda_1 + [m_1 + m_2 + 2m_3 + 3]\lambda_2.$$

From this, we observe:

(i) For *unweighted* L_2 cohomology

$$H_{(2)}^*(Y_3, \mathbb{E}'_w) = 0,$$

as E'_w is not isomorphic to its conjugate-contragredient (i.e., the coefficients of λ_1 and λ_2 are unequal) [18], *except* in case III if $m_1 = 2m_3 + 1$.

(ii) As E'_w is never the trivial representation of $\text{SL}(3)$, it follows from [20] (the vanishing of cohomology below the rank) that the *reduced* unweighted groups $\overline{H}_{(2)}^0$ and $\overline{H}_{(2)}^1$, hence by duality $\overline{H}_{(2)}^5$ and $\overline{H}_{(2)}^4$, are trivial.

Now, (i) takes care of cases IVA and VA at once, since there is then trivial weighting. In the remaining cases, we will see that the weighted and unweighted L_2 -cohomology groups are isomorphic for the degrees that we are concerned with. We may then appeal to (i) or (ii) as the case dictates (Remark: it is obvious for elementary topological reasons that $H_{(2)}^5 = 0$ if E'_w is non-trivial. This settles case II.)

(3.30). The comparison of weighted and unweighted L_2 -cohomology on Y_3 is to be done on the Satake compactification Y_3^* of (3.22). The latter has two singular strata, one of dimension two, and one of dimension zero. The associated Kostant subsets of the Weyl group of $\text{SL}(3)$ – as a subset of the Weyl group of $\text{Sp}(6)$ – will be denoted $\{\sigma_0, \sigma_1, \sigma_2\}$ and $\{\tau_0, \tau_1, \tau_2\}$ respectively, with the subscripts indicating length ($\sigma_0 = \tau_0 =$

1). We extend our table from (3.29):

III) σ	$\sigma w(\delta)$	$\sigma w(\lambda_1)$	$\sigma w(\lambda_2)$	$\sigma w(\lambda_3)$
$\sigma_0: 1, 2, 3$	3, 4, 2	1, 1, 1	1, 1, 0	1, 2, 1
$\sigma_1: 2, 1, 3$	1, 4, 2	0, 1, 1	0, 1, 0	1, 2, 1
$\sigma_2: 2, 3, 1$	-2, 1, 2	0, 1, 1	-1, 0, 0	-1, 0, 1
$\tau_0: 1, 2, 3$	3, 4, 2	1, 1, 1	1, 1, 0	1, 2, 1
$\tau_1: 1, 3, 2$	3, 1, 2	1, 1, 1	1, 0, 0	1, 0, 1
$\tau_2: 3, 1, 2$	1, -1, 2	0, 0, 1	0, -1, 0	1, 0, 1
IV) σ_0	3, 2, 0	1, 1, 1	1, 1, 0	1, 0, -1
σ_1	-1, 2, 0	0, 1, 1	0, 1, 0	-1, 0, -1
σ_2	-2, 1, 0	0, 1, 1	-1, 0, 0	-1, 0, -1
τ_0	3, 2, 0	1, 1, 1	1, 1, 0	1, 0, -1
τ_1	3, 1, 0	1, 1, 1	1, 0, 0	1, 0, -1
τ_2	-1, -3, 0	0, 0, 1	0, -1, 0	-1, -2, -1
V) σ_0	2, 3, 0	0, 0, -1	1, 1, 0	1, 2, 1
σ_1	1, 3, 0	0, 0, -1	0, 1, 0	1, 2, 1
σ_2	-3, -1, 0	-1, -1, -1	-1, 0, 0	-1, 0, 1
τ_0	2, 3, 0	0, 0, -1	1, 1, 0	1, 2, 1
τ_1	2, -1, 0	0, -1, -1	1, 0, 0	1, 0, 1
τ_2	1, -2, 0	0, -1, -1	0, -1, 0	1, 0, 1

It is perhaps worthwhile stating that we will be taking the point of view that the *unweighted* L_2 norm is a deviation from the weighted, and not the other way around. This is because the coefficients n_1 and n_2 of β_1 and β_2 resp. in $\sigma w(\lambda + \delta)$ determine the asymptotics for the weighted cohomology; we must subtract off

$$n_3 \lambda_3 = n_3 (\beta_1 + 2\beta_2)/3$$

to get those for the unweighted cohomology.

For the elements w under consideration in (3.29), there is a natural inclusion

$$\mathcal{L}_{(2)}^\bullet(Y_3^*, \mathbb{E}'_w) \rightarrow \mathcal{L}_{(2)}^\bullet(Y_3^*, \mathbb{E}'_w; \hat{\omega}_\mu), \quad (1)$$

for $\hat{\omega}_\mu$ then vanishes at infinity. We compare the cohomology sheaves of both sides of (1). These are computable by our usual methods, and the local L_2 -cohomology on Y_3^* becomes given by Künneth formulas analogous to (3.19(8)), as follows.

Along the two-dimensional (top singular) stratum, we get for a deleted neighborhood W' on Y_3

$$H_{(2)}^\bullet(W', \mathbb{E}'_w; \omega) \simeq \bigoplus_{l, \nu} \left(H_\nu^l(\mathfrak{v}, \mathbb{E}'_w) \otimes H_{(2)}^\bullet(\mathbb{R}^+, \mathbb{C}; h_{\nu, \omega})[-l] \right), \quad (2)$$

where, for $\omega = \hat{\omega}_\mu$ or $\omega = 1$, $h_{\nu, \omega}$ is an exponential weight that depends on the weight ω . Since the behavior of the L_2 -cohomology on $\mathbb{R}^+$ depends only on the sign of the weight (see (2.4)), we see that we need only compare the coefficient n_1 in each $\sigma_i w(\lambda + \delta)$ to its unweighted counterpart, i.e., determine whether subtracting $n_3 \beta_1 / 3$ makes any difference in the sign of n_1 . By using the above tables, one checks that it does not. It follows that (1) is a quasi-isomorphism outside the zero-dimensional stratum.

The following elementary fact will be used:

LEMMA: Let $\mathcal{X}_1^* \hookrightarrow \mathcal{X}_2^*$ be complexes of sheaves such that the inclusion is a quasi-isomorphism outside a finite set Σ . If

$$H^i(\mathcal{X}_1^*|_\Sigma) \rightarrow H^i(\mathcal{X}_2^*|_\Sigma)$$

is an isomorphism for $i > i_0$, and surjective for $i = i_0$, then the same is true for

$$H^i(\mathcal{X}_1^*) \rightarrow H^i(\mathcal{X}_2^*).$$

On the zero-dimensional stratum of Y_3^* , we have the following analogue of (2):

$$H_{(2)}^*(W', \mathbb{E}'_w; \omega) \simeq \bigoplus_{l, \nu} \left(H_{(2)}^*(Z', \mathbb{H}'_\nu(\mathfrak{v}, E'_w); g_\omega) \otimes H_{(2)}^*(A^+, \mathbb{C}; h_{\nu, \omega})[-l] \right), \quad (3)$$

where Z' is the base of the fibration of the regular locus of the link in Y_3^* , so is locally symmetric for $\mathrm{SL}(2, \mathbb{R})$, and is thus 2-dimensional. We must first compare the truncations caused by the coefficients of β_2 , i.e., the sign of n_2 versus that of $n_2 - (2n_3/3)$ in $\tau_i w(\lambda + \delta)$. These we tabulate (in the order given)

	τ_0	τ_1	τ_2
III)	+ +	+ $\mathrm{sgn}(m_1 - 2m_2 - 1)$	- -
IV)	+ +	+ +	- -
V)	+ +	- -	- -

We obtain the desired vanishing as follows:

(V B): The criterion of the lemma is satisfied with $i_0 = 3$, by reason of dimension. For the same reason as in (3.24), we know that $H_{(2)}^*(Y_3, \mathbb{E}'_w; \omega_\mu)$ must be finite dimensional.

(IV B): Here, there is an a priori danger of cohomology in dimension $\dim_{\mathbb{R}} Z' + l(\tau_1) = 3$. However, one verifies that $(E'_w)_{\tau_1}'$ is always a non-trivial representation of $\mathrm{SL}(2)$, i.e.,

$$\langle \tau_1 w(\lambda + \delta) - \delta, \beta_1 \rangle = m_1 + 2m_2 + 2m_3 + 4 \neq 0.$$

(III): Here, the value of $\langle \tau_1 w(\lambda + \delta) - \delta, \beta_1 \rangle$ is the same as above. Even if $m_1 = 2m_2 + 1$, one sees that the conditions of the lemma hold for $i_0 = 3$. We now can apply (3.29, ii).

To summarize:

PROPOSITION: *In cases IV and V, the mapping*

$$H'_{(2)}(Y_3, \mathbb{E}'_w) \rightarrow H'_{(2)}(Y_3, \mathbb{E}'_w; \hat{\omega}_\mu)$$

is an isomorphism for $i \geq 3$; for $i \geq 4$ in case III.

COROLLARY: *In the ranges indicated above,*

$$H'_{(2)}(Y_3, \mathbb{E}'_w; \hat{\omega}_\mu) = 0.$$

This completes our proof of the conjecture for $\mathrm{Sp}(6)$.

Appendix

(A.1). The following observation should turn out to be useful. We write $_{\mathbb{R}}\alpha$ for $\alpha_{P_\emptyset}$.

PROPOSITION: *If we write*

$$\delta|_{_{\mathbb{R}}\alpha} = \sum_{i=1}^r n_i \beta_i \tag{1}$$

(with our convention for the C_r root system (see (3.10)) in force), then $n_i = j(i)$.

REMARK: The assertion about n_1 was known to Borel, in conjunction with [4].

PROOF: We observe first that

$$\delta|_{\alpha_{\Psi_2}} = \sum_{i=2}^r n_i \beta_i$$

can be identified with the quantity (1) for the space $Y^{j(1)}$. Since $j(i) - j(1)$ is the codimension of $Y^{j(i)}$ in $Y^{j(1)}$, it suffices by induction to check that

- (i) $n_1 = j(1)$,
- (ii) the sum of the restriction to ${}_{\mathbb{R}}\alpha$ of the roots in $\mathfrak{u}_{\Theta_1}$ is a multiple of $\sum_{i=1}^r \beta_i$.

We have (i) already from (3.5). To see (ii), we observe that the first fundamental dominant weight ξ_1 of the restricted root system is given by

$$\xi_1 = \sum_{i=1}^r \beta_i. \quad (2)$$

It therefore suffices to verify that

$$(ii') \quad \langle \rho(\delta_1), \beta_i \rangle = 0 \quad \text{if } i \neq 1,$$

where ρ denotes restriction, and δ_1 is the half-sum of the roots in $\mathfrak{u}_{\Theta_1}$. Let δ' be the half-sum of the positive roots in the subsystem spanned by Θ_1 . As δ_1 is clearly fixed by complex conjugation, we have for any simple $\mathbb{C}$ -root α ,

$$\begin{aligned} \langle \rho(\delta_1), \rho(\alpha) \rangle &= \langle \delta_1, \alpha \rangle = \langle \delta - \delta', \alpha \rangle \\ &= 1 - 1 = 0 \quad \text{if } \rho(\alpha) \neq \beta_1, \end{aligned}$$

as desired.

One can use the known description of the restriction from ${}_{\mathbb{R}}\Delta$ to ${}_{\mathbb{Q}}\Delta$ to conclude in the general case:

COROLLARY: If ${}_{\mathbb{Q}}\Delta = \{\gamma_1, \dots, \gamma_q\}$, and

$$\delta|_{{}_{\mathbb{Q}}\alpha} = \sum_{1 \leq i \leq q} c_i \gamma_i,$$

then

$$j(i) = c_i.$$

PROOF: In each connected component Δ_j of ${}_{\mathbb{R}}\Delta$, there is precisely one simple $\mathbb{R}$ -root, which we denote $\beta_{k(i,j)}$, restricting to γ_i [1,(2.9)]. Thus, in the notation of the Proposition,

$$c_i = \sum_j n_{k(i,j)}.$$

This shows that c_i equals the sum of the codimensions of the corresponding boundary components on each factor; i.e., the codimension of $Y^{(i)}$ (see [1,p.471]).

(A.2). There is good evidence (see [18]) that the proper setting for the discussion of our conjecture is the arithmetic quotients of symmetric spaces $X = G_{\mathbb{R}}/K$ for which G and K have the same (absolute, i.e. $\mathbb{C}$ -) rank – let us then say that X is an *equal-rank symmetric space* – in which case X is even-dimensional. Borel has extended the conjecture to assert:

CONJECTURE: let $\Gamma \backslash X^*$ be a Satake compactification (see [17]) such that every boundary component of X^* (including the improper one, viz. X) is equal-rank. Then the L_2 complex $\mathcal{L}_{(2)}(\Gamma \backslash X^*, \mathbb{E})$ is quasi-isomorphic to the middle perversity intersection chain complex with coefficients in $\mathbb{E}$; i.e., there is for local reasons on $\Gamma \backslash X^*$, an isomorphism

$$H_{(2)}^*(\Gamma \backslash X, \mathbb{E}) \simeq IH^*(\Gamma \backslash X^*, \mathbb{E}).$$

We wish to give the reader some idea of what the extended conjecture says explicitly, in terms of the classification of symmetric spaces.

Let G be an algebraic group defined over $\mathbb{Q}$. If ${}_{\mathbb{Q}}\Delta$ denotes a set of simple $\mathbb{Q}$ -roots for G , the distinct Satake compactifications are in one-to-one correspondence with non-empty subsets Ξ of ${}_{\mathbb{Q}}\Delta$ (subject to a possibly non-vacuous condition [17:(3.4)]), and the types of boundary components are parametrized by the Ξ -connected subsets of ${}_{\mathbb{Q}}\Delta$ (see [17:(2.10),(3.3)]). We recall that a set Υ is Ξ -connected if the graph of $\Xi \cup \Upsilon$ (as a subset of an inner product space) is connected.

We assume, once again, that the $\mathbb{Q}$ -structure of G is “standard” so that the scope of the conjecture can be determined by means of the classification of non-compact symmetric spaces (equivalently, semi-simple Lie algebras over $\mathbb{R}$) – see [11:p.518] or [14:pp.30–32] – as opposed to that of semi-simple algebraic groups over $\mathbb{Q}$ [19]. Then, ${}_{\mathbb{Q}}\Delta$ is also a set of simple $\mathbb{R}$ -roots for G .

PROPOSITION: *Under the above restrictions, the Satake compactifications of symmetric spaces that satisfy the hypotheses of the extended conjecture are:*

- (a) *the Baily-Borel Satake compactification, where Ξ consists of a simple root at one end of the Dynkin diagram of ${}_{\mathbb{Q}}\Delta$, in the Hermitian cases: AIII, BI/DI ($r = 2$), CI, DIII, EIII, EVII.*
- (b) *an analogous one, with Ξ a single root from an end of the diagram, for: BI (the end other than the one from (a) when $r = 2$), CII,*
- (c) *all three Satake compactifications for: BI ($r = 2$), CI ($r = 2$), G.*
- (d) *the unique Satake compactification for FII (where $r = 1$).*

N.B. – Case (c) overlaps (a) and (b) a little, and we have checked the conjecture for all three compactifications for CI.

PROOF (outline):

1. By going down the list, one can see that the irreducible symmetric spaces that are not equal-rank are: $\underline{\text{AI}}$ ($r > 1$), $\underline{\text{AII}}$, $\underline{\text{DI}}$ (r odd), $\underline{\text{EI}}$, $\underline{\text{EIV}}$.
2. One checks that the case in (a)–(d) do satisfy the hypotheses of the conjecture.
3. It is immediate that if $\Psi \subset \Xi$, then every Ψ -connected set is Ξ -connected.
4. In every case where Ξ has exactly one element, other than those listed in (a)–(d), there is a Ξ -connected set giving rise to a boundary component of one of the following types: $\underline{\text{AI}}$ ($r = 2$), $\underline{\text{AII}}$ ($r = 1$), $\underline{\text{DI}}$ ($r = 1$).

REMARK: It was tempting to try, as the hypothesis of the generalized conjecture, the less restrictive statement that all boundary components be even dimensional. Although some of the symmetric spaces in 1. above are even-dimensional, it is easy to check that they all admit odd dimensional boundary components, so one would get no additional examples with standard $\mathbb{Q}$ -structure. However, if one allows other $\mathbb{Q}$ -structures, e.g., restrictions of scalars, it is possible to come up with counterexamples to the larger conjecture; Borel realized that for the $\mathbb{Q}$ -rank one group

$$G = R_{\mathbb{Q}[\iota]/\mathbb{Q}} \text{SO}(3, 1),$$

giving rise to arithmetically-defined quotients of $H^3 \times H^3$ (H^3 here denotes real hyperbolic 3-space), compactified with point cusps, the L_2 cohomology with $\mathbb{C}$ coefficients is infinite dimensional in certain degrees (no closed range for d) – hence cannot be the intersection homology of a compact space – as follows by the methods of [4] or [15].

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