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THE FIRST EIGENVALUE OF THE LAPLACIAN FOR A POSITIVELY CURVED HOMOGENEOUS RIEMANNIAN MANIFOLD

Hajime Urakawa

§0. Introduction

The purpose of this paper is to compute the first eigenvalue of the Laplacian for a certain positively curved homogeneous Riemannian manifold.

By a theorem of A. Lichnérowicz and M. Obata, if the Ricci curvature Ric_M of an n-dimensional compact Riemannian manifold M satisfies $\mathrm{Ric}_M \geqslant n-1$, then the first eigenvalue $\lambda_1(M)$ of the Laplacian of M is bigger than or equal to n, and the equality holds if and only if M is the standard sphere S^n of constant curvature one. Moreover, due to [L.Z.], [L.T], [C], [B.B.G], the following eigenvalue pinching theorem is known:

THEOREM: *Let M be a compact, n-dimensional Riemannian manifold whose sectional curvature $K_M \geqslant 1$. Then, there exists a constant $C(n) > 1$ depending only on n such that $C(n)n \geqslant \lambda_1(M) \geqslant n$ only if M is homeomorphic to S^n.*

On the other hand, due to [B], [W], [A.W], [B.B 1,2] the classification of compact homogeneous Riemannian manifolds with positive sectional curvature is known. Therefore, it would be interesting to know the first eigenvalues λ_1(M) of these positively curved homogeneous manifolds. In this paper, we give a comparatively sharp estimate of $\lambda_1(M)$ for such manifolds and as an application we determine $\lambda_1(M)$ of 7-dimensional positively curved homogeneous Riemannian manifolds $\mathrm{SU}(3)/T_{k,l}$, and the manifold $F_4/\mathrm{Spin}(8)$ of flags in the Cayley plane (cf. Theorem 2.1). Moreover, in the appendix, we give a complete list of $\lambda_1(M)$ of all compact simply connected irreducible Riemannian symmetric spaces, which was already given in [N1] for the classical cases. As a further application, we obtain a complete list of compact simply connected irreducible Riemannian symmetric spaces (cf. Theorem A.1) so that the identity map is stable as as a harmonic map (cf. [Sm]).

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§1. Homogeneous Riemannian manifolds with positive curvature

In this section, following [A.W], [W], [B.B 1,2], we prepare the results of classifying simply connected homogeneous Riemannian manifolds with positive curvature.

Let G be a compact connected Lie group, and H a closed subgroup. Let g be a Lie algebra of G, and h the subalgebra of g corresponding to H.

Definition 1.1 (cf. [A.W]): The pair (G, H) satisfies *Condition* (II) if there exists an $\mathrm{Ad}(G)$-invariant inner product $(\cdot,\cdot)_0$ on g such that the orthogonal complement v of h in g has an orthogonal decomposition $v = v_1 \oplus v_2$ with the following properties:

(i) $[v_1, v_2] \subset v_2$, $[v_1, v_1] \subset \mathrm{h} \oplus v_1$, $[v_2, v_2] \subset \mathrm{h} \oplus v_1$, and

(ii) for $X = X_1 + X_2$, $Y = Y_1 + Y_2$ with $X_i, Y_i \in v_i$, $i = 1, 2$, $[X, Y] = 0$ and $X \wedge Y \neq 0$ imply $[X_1, Y_1] \neq 0$.

Putting $\mathrm{k} := \mathrm{h} \oplus v_1$, k is a subalgebra of g and (g, k) is a symmetric pair of rank one (cf. [B.B 1, p. 58]). Furthermore, the connected Lie subgroup K of G corresponding to k is closed (cf. [B.B 1, p. 44]). In fact, the center z of g is included in k (cf. [A.W, p 97]). Then, we have the decompositions $\mathrm{g} = \mathrm{g}' \oplus \mathrm{z}$, and $\mathrm{k} = \mathrm{k}' \oplus \mathrm{z}$, where g' is the semi-simple part of g, and k' is a subalgebra of k. Since $(\mathrm{g}', \mathrm{k}')$ is a symmetric pair with the semi-simple Lie algebra g', the connected subgroup K' in G corresponding to k' is closed in the connected Lie group G' corresponding to g' (cf. [He, p. 179, the last part of the proof of Proposition 3.6]). Since G' is compact, K' is closed in G. Since the center Z of G is closed, the Lie group $Z \cdot K'$ is closed in G. Therefore its identity component K is closed.

Definition 1.2 (cf. [A.W]): For $-1 < t < \infty$, we define an $\mathrm{Ad}(H)$-invariant inner product $(\cdot,\cdot)_t$ on $v = v_1 \oplus v_2$ by setting

$$(X_1 + X_2, Y_1 + Y_2)_t := (1+t)(X_1, Y_1)_0 + (X_2, Y_2)_0,$$

for $X_i, Y_i \in v_i$, $i = 1, 2$, and let g_t be a G-invariant Riemannian metric on G/H induced from $(\cdot,\cdot)_t$.

Moreover let h be a G-invariant Riemannian metric on G/K induced by the inner product $(\cdot,\cdot)_0$ on v_2. Then, the natural projection π; $G/H \to G/K$ induces a Riemannian submersion π; $(G/H, g_t) \to (G/K, h)$ with totally geodesic fibers for all $-1 < t < \infty$ (cf. [B.B.B]).

Note that in case $v_1 = \{0\}$, $(G/H, g_0) = (G/K, h)$ is a Riemannian symmetric space of rank one, and in case $v_2 = \{0\}$, Condition (II) implies the one such that the normally homogeneous Riemannian manifold $(G/H, g_0)$ has positive curvature.

THEOREM 1.3 (cf. [A.W, Theorem 2.4], [H, Corollary 2.2]): *Let (G, H) be a pair satisfying Condition (II), and $v_1 \neq \{0\}$, and $v_2 \neq \{0\}$. Let g_t, $-1 < t < \infty$ the G-invariant metric on G/H given in Definition* 1.2. *Then the Riemannian manifold $(G/H, g_t)$, $-1 < t < 0$, has positive curvature.*

THEOREM 1.4 (cf. [W], [B.B 1,2], [B]). *All compact simply connected homogeneous spaces G/H which are not homeomorphic to S^n and have positively curved G-invariant Riemannian metrics are displayed in the following table*:

(I) In the case of normally homogeneous spaces,

	G/H
(1)	$SU(n+1)/S(U(n)\times U(1)) = P^n(\mathbb{C})$, $n \geqslant 2$
(2)	$Sp(n+1)/Sp(n)\times Sp(1) = P^n(H)$, $n \geqslant 2$
(3)	$F_4/\mathrm{Spin}(9) = P^2(\mathrm{Cay})$
(4)	$Sp(2)/SU(2)$

(II) In the case of Condition (II) with $v_1 \neq \{0\}$ and $v_2 \neq \{0\}$,

	G/H	G/K
(5)	$Sp(n)/Sp(n-1)\times T^1 \approx P^{2n-1}(\mathbb{C})$, $n \geqslant 2$	$Sp(n)/Sp(n-1)\times Sp(1)$
(6)	$SU(5)/Sp(2)\times T^1$	$SU(5)/S(U(4)\times U(1))$
(7)	$SU(3)/T^2$	$SU(3)/S(U(2)\times U(1))$
(8)	$SU(3)/T^1$	$SU(3)/S(U(2)\times U(1))$
(9)	$U(3)/T^2 \approx SU(3)/T^1$	$SU(3)/S(U(2)\times U(1))$
(10)	$SU(3)\times SU(2)/T^1\times\overline{SU(2)}$	$SU(3)/S(U(2)\times U(1))$
(11)	$Sp(3)/SU(2)\times SU(2)\times SU(2)$	$Sp(3)/Sp(2)\times Sp(1)$
(12)	$F_4/\mathrm{Spin}(8)$	$F_4/\mathrm{Spin}(9)$

REMARK 1: Here we denote by T^k, $k = 1, 2$, k-dimensional tori. In case of (7), T^2 is a maximal torus in $SU(3)$. In cases of (8), (9), the embeddings of T^1 and T^2 are given in [A.W], [B.B2] or §2. In case of (10), $T^1 \times \overline{SU(2)}$ is defined by $\{(t\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}, x);\ t \in T^1,\ x \in SU(2)\}$ which is a closed subgroup in $SU(3)\times SU(2)$. Here T^1 is a torus in $SU(3)$

whose diagonal entries are $(e^{-2i\eta}, e^{i\eta}, e^{i\eta})$, $\eta \in \mathbb{R}$, and $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ is an element in $SU(3)$, x being in $SU(2)$.

REMARK 2: The examples (4), (5) and (6) are due to [B], and the ones (7) ~ (12) to [W], [A.W]. The inclusion $SU(2) \hookrightarrow Sp(2)$ in (4) is not canonical (cf. [B]). In the cases (5) and (6) the normally homogeneous Riemannian metric g_0 has positive curvature.

REMARK 3: The pairs (G, H) satisfying Condition (II) are classified in [B.B1, p. 59]. All simply connected homogeneous spaces G/H satisfying Condition (II) which are not homeomorphic to S^n appear in the following table Theorem 1.4.

§2. The first eigenvalue of the Laplacian

In this section, we prove the following theorem:

THEOREM 2.1: *Let G/H be a homogeneous space as in Theorem* 1.4. *Let $(\cdot,\cdot)_0$ be the* Ad(G)*-invariant inner product on* g *given by*

$$(X, Y)_0 = -B(X, Y), \quad X, Y \in \mathfrak{g},$$

for (1) ~ (12), *except* (9), *where B is the Killing form of* g. *For* (9), *we define $(\cdot,\cdot)_0$ by*

$$(X, Y)_0 = -6\mathrm{Trace}(XY), \quad X, Y \in \mathfrak{u}(3).$$

We define the inner product $(\cdot,\cdot)_t$, $-1 < t \leqslant 0$, on the orthogonal complement $\mathfrak{v}$ of h *in* g *with respect to $(\cdot,\cdot)_0$ as in Definition* 1.2 *for* (5) ~ (12), *and we consider only $(\cdot,\cdot)_0$ for the cases* (1) ~ (4). *Then, we can estimate the first eigenvalue $\lambda_1(g_t)$ of the G-invariant Riemannian metric g_t on G/H corresponding to $(\cdot,\cdot)_t$ as follows*:

	G/H	$\lambda_1(g_t)$, $-1 < t \leqslant 0$
(1)	$SU(n+1)/S(U(n) \times U(1))$	$\lambda_1(g_0) = 1$
(2)	$Sp(n+1)/Sp(n) \times Sp(1)$	$\lambda_1(g_0) = \frac{n+1}{n+2}$
(3)	$F_4/\mathrm{Spin}(9)$	$\lambda_1(g_0) = \frac{2}{3}$
(4)	$Sp(2)/SU(2)$	$\frac{5}{12} \leqslant \lambda_1(g_0)$
(5)	$Sp(n)/Sp(n-1) \times T^1$	$\frac{2n+1}{4(n+1)} \leqslant \lambda_1(g_t) \leqslant \frac{n}{n+1}$
(6)	$SU(5)/Sp(2) \times T^1$	$\frac{12}{25} \leqslant \lambda_1(g_t) \leqslant 1$
(7)	$SU(3)/T^2$	$\frac{4}{9} \leqslant \lambda_1(g_t) \leqslant 1$
(8)	$SU(3)/T^1$	$\lambda_1(g_t) = 1$
(9)	$U(3)/T^2$	$\frac{1}{2} \leqslant \lambda_1(g_t) \leqslant 1$
(10)	$SU(3) \times SU(2)/T^1 \times \overline{SU(2)}$	$\frac{3}{8} \leqslant \lambda_1(g_t) \leqslant 1$
(11)	$Sp(3)/SU(2) \times SU(2) \times SU(2)$	$\frac{7}{16} \leqslant \lambda_1(g_t) \leqslant \frac{3}{4}$
(12)	$F_4/\mathrm{Spin}(8)$	$\lambda_1(g_t) = \frac{2}{3}$

REMARK: The cases (1) ~ (3) are known, see [C.W].

We prepare for the proof of Theorem 2.1 with Lemma 2.2.

LEMMA 2.2: *Under the assumptions of Theorem* 2.1, *the first eigenvalue* $\lambda_1(g_t)$ *of* $(G/H, g_t)$, $-1 < t \leqslant 0$, *can be estimated as*

$$\lambda_1(g_0) \leqslant \lambda_1(g_t) \leqslant \lambda_1(G/K, h), \quad -1 < t \leqslant 0,$$

where $\lambda_1(G/K, h)$ *is the first eigenvalue of* $(G/K, h)$.

PROOF: Since π; $(G/H, g_t) \to (G/K, h)$ is a Riemannian submersion with totally geodesic fibers, the (positive) Laplacian Δ_{g_t}, Δ_h of $(G/H, g_t)$, $(G/K, h)$ satisfy

$$\Delta_{g_t}(f \cdot \pi) = (\Delta_h f) \cdot \pi, \quad f \in C^\infty(G/K)$$

(cf. [B.B.B, p. 188]). Thus the spectrum Spec(Δ_{g_t}) includes Spec(Δ_h), in particular, $\lambda_1(g_t) \leqslant \lambda_1(G/K, h)$ for all t.

For the remaining inequality, we put $p = \dim(v_1)$ and $q = \dim(v_2)$. Let $\{X_i\}_{i=1}^p$, $\{Y_i\}_{i=1}^q$ be orthonormal bases of v_1, v_2, respectively. Then, since $\{X_i/\sqrt{t+1}\}_{i=1}^p$, $\{Y_i\}_{i=1}^q$ are orthonormal with respect to $(\cdot,\cdot)_t$, the Laplacian Δ_{g_t} of $(G/H, g_t)$ can be expressed (cf. [M.U, p. 476]) as

$$\Delta_{g_t} = -\frac{1}{t+1}\hat{\lambda}\left(\sum_{i=1}^p X_i^2\right) + \hat{\lambda}\left(\sum_{i=1}^q Y_i^2\right),$$

in particular,

$$\Delta_{g_0} = -\hat{\lambda}\left(\sum_{i=1}^p X_i^2\right) - \hat{\lambda}\left(\sum_{i=1}^q Y_i^2\right),$$

where $\hat{\lambda}$ is the canonical isomorphism of the algebra of Ad(H)-invariant polynomials of $v = v_1 \oplus v_2$ into the space of G-invariant differential operators on G/H. Therefore we obtain

$$\Delta_{g_t} = \Delta_{g_0} + \left(1 - \frac{1}{t+1}\right)\hat{\lambda}\left(\sum_{i=1}^p X_i^2\right). \tag{2.1}$$

Here because of $-1 < t \leqslant 0$, the operator $P := (1 - \frac{1}{t+1})\hat{\lambda}\left(\sum_{i=1}^p X_i^2\right) = \frac{t}{t+1}\hat{\lambda}\left(\sum_{i=1}^p X_i^2\right)$ is non-negative, i.e., $\int_{G/H}(Pf)f$

$dv_{g_t} \geqslant 0$ for $f \in C^{\infty}(G/H)$, where dv_{g_t} is the volume element of $(G/H,\ g_t)$. Note that

$$dv_{g_t} = (t+1)^{p/2} dv_{g_0}. \tag{2.2}$$

Therefore, using (2.1), (2.2) and the Mini-Max Principle (cf. [B.U, Proposition 2.1]), we obtain $\lambda_1(g_t) \geqslant \lambda_1(g_0)$, $-1 < t \leqslant 0$. Q.E.D.

PROOF OF THEOREM 2.1: The case (8) will be shown in Lemma 2.3. The upper estimate of $\lambda_1(g_t)$ can be obtained by the inequality $\lambda_1(g_t) \leqslant \lambda_1(G/K,\ h)$ in Lemma 2.2 and Theorem 2.1, (1) ~ (3). For the lower estimate we use the inequalities

$$\lambda_1(G,\ g) \leqslant \lambda_1(g_0) \leqslant \lambda_1(g_t), \quad -1 < t \leqslant 0.$$

Here, $\lambda_1(G,\ g)$ is the first eigenvalue of $(G,\ g)$ whose metric g is the bi-invariant one induced from the inner product $(\cdot,\cdot)_0$ on g. The computations of $\lambda_1(G,\ g)$ are accomplished in the appendix and note that $\lambda_1(U(n+1),\ g) = \frac{1}{2}$ (cf. [T, p. 307, Remark 2] or a direct computation, see Appendix). Here, the metric g on $U(n+1)$ is the bi-invariant Riemannian metric on $U(n+1)$ which is induced from the inner product $(X,\ Y)_0 := -2(n+1)\,\mathrm{Trace}(XY)$, $\quad X,\ Y \in \mathrm{u}(n+1)$.

Case (8). A 1-dimensional torus $H = T^1$ in $G = SU(3)$ is conjugate in $SU(3)$ to

$$T_{k,l} = \left\{ \begin{pmatrix} e^{2\pi i k\theta} & & \\ & e^{2\pi i l\theta} & \\ & & e^{-2\pi i(k+l)\theta} \end{pmatrix};\quad \theta \in \mathbb{R} \right\},$$

where $k,\ l$ are integers. We know by Lemma 3.1 and Theorem 3.2 in [A.W], that the pair $(SU(3),\ T^1)$ satisfies Condition (II) if and only if T^1 is conjugate in $SU(3)$ to $T_{k,l}$ with $kl(k+l) \neq 0$. Moreover, since $T_{k,l} = T_{mk,ml}$, $m \in \mathbb{Z} - (0)$, we can assume without loss of generality that $H = T^1 = T_{k,l}$ where $kl \neq 0$ an $k,\ l$ are relatively prime.

Let $K = S(U(2) \times U(1)) = \left\{ \begin{pmatrix} x & \\ & \det x^{-1} \end{pmatrix};\quad x \in U(2) \right\}$, and g, k, h the corresponding Lie algebras of $G = SU(3)$, K, H, respectively. Let $(\cdot,\cdot)_0$ be the inner product on g defined by

$$(X,\ Y)_0 = -B(X,\ Y) = -6\,\mathrm{Trace}(XY), \quad X,\ Y \in \mathrm{g} = \mathrm{su}(3),$$

and we put $v_1 := \mathrm{h}^{\perp} \cap \mathrm{k}$, $v_2 := \mathrm{k}^{\perp}$, and $v = \mathrm{h}^{\perp} = v_1 \oplus v_2$, where $\mathrm{k}^{\perp}$, $\mathrm{h}^{\perp}$ are the orthogonal complements of k, h in g with respect to $(\cdot,\cdot)_0$, respectively. We define the $\mathrm{Ad}(H)$-invariant inner product $(\cdot,\cdot)_t$, $-1 < t < \infty$, on $v = v_1 \oplus v_2$ as in Definition 1.2 and let g_t be the G-invariant

Riemannian metric on $G/H = SU(3)/T_{k,l}$ induced by $(\cdot,\cdot)_t$. Then, we have:

LEMMA 2.3: *Assume that $kl(k+l) \neq 0$. Then the first eigenvalue $\lambda_1(g_t)$ of $(SU(3)/T_{k,l}, g_t)$, $-1 < t \leqslant 0$, is given by $\lambda_1(g_t) = 1$ for every $-1 < t \leqslant 0$.*

PROOF: We already know that

$$\lambda_1(g_0) \leqslant \lambda_1(g_t) \leqslant \lambda_1(G/H, h) = 1.$$

So we only have to show $\lambda_1(g_0) = 1$. For this, we use Theorem 1 in [U] which tells us that the eigenvalues of the Laplacian of $(SU(3)/T_{k,l}, g_0)$ are given by

$$f(n_1, n_2) := \tfrac{1}{9}\left(m_1^2 + m_2^2 - m_1 m_2 + 3m_1\right),$$

where $m_1 := n_1 + n_2$, $m_2 := n_2$, and n_1 and n_2 run over the set of nonnegative integral satisfying $S^{k,l}_{n_1,n_2} \neq 0$. Here $S^{k,l}_{n_1,n_2}$ is the number of all the integer solutions (p', q, r) of the equations:

$$\begin{cases} kn_1 - ln_2 - (2k+l)p' + (-k+l)q + (k+2l)r = 0, \quad \text{and} \\ 0 \leqslant p' \leqslant n_1,\ 0 \leqslant q \leqslant n_2, \quad \text{and } 0 \leqslant r \leqslant p' + (n_2 - q). \end{cases}$$

Notice that, since our metric g_0 in this paper is 6 times the Riemannian metric in [U], the eigenvalues of $(SU(3)/T_{k,l}, g_0)$ are $\frac{1}{6}$ of the ones of the paper.

Then, we can easily check that

$$f(n_1, n_2) \geqslant 1 = f(1, 1),$$

except the cases $(n_1, n_2) = (1, 0)$ or $(0, 1)$. However, $S^{k,l}_{1,0} = S^{k,l}_{0,1} = 0$ due to the assumption $kl(k+l) \neq 0$. Therefore, we have the desired result. Q.E.D.

Appendix: The first eigenvalues of symmetric spaces

The table of the first eigenvalues of the Laplacian of compact simply connected irreducible Riemannian symmetric spaces has been already given by [N1] for the classical cases. In this appendix, we give a complete list including the exceptional cases.

At first let G be a compact simply connected simple Lie group, g its Lie algebra, and g the bi-invariant Riemannian metric on G induced from the inner product $(\cdot,\cdot)$ on g given by

$$(X, Y) = -B(X, Y), \quad X, Y \in \mathrm{g}, \tag{A.1}$$

where B is the Killing form of g. We denote by the same notation the inner product on g* canonically induced fron $(\cdot,\cdot)$ on g. Then it is known (cf. [Su]) that the spectrum of (G, g) can be given Freudenthal's formula as follows:

$$\begin{cases} \text{the eigenvalues:} & (\lambda + 2\rho, \lambda), \\ \text{their multiplicities;} & d_\lambda^2, \end{cases}$$

where 2ρ is the sum of all positive roots of the complexification $g^{\mathbf{C}}$ of g relative to a maximal abelian subalgebra t of g, and d_λ is the dimension of the irreducible unitary representation of G with highest weight λ, and λ varies over the set D(G) of all dominant weights in the dual t* of t. Therefore, we get

$$\lambda_1(G, g) = \min\{(\tilde{\omega}_i + 2\rho, \tilde{\omega}_i); \quad 1 \leqslant i \leqslant l\},$$

where $\{\tilde{\omega}_i\}_{i=1}^{l}$ are the fundamental weights of g corresponding to the fundamental root system $\{\alpha_1, \dots, \alpha_l\}$ of g.

By the last table in [Bo], we know D(G), 2ρ, and the inner product $(\cdot,\cdot)$ in t*, so we get the following table of the first eigenvalue of the Laplacian of (G, g):
In this Table A.1, the symbol X means that the identity map of (G, g) is unstable.

Next, the spectrum of the Laplacian of an irreducible Riemannian symmetric space G/K of compact type is given as follows. Let G be a compact simply connected simple Lie group, K the corresponding closed subgroup of G. Let g, k be the Lie algebras of G, K, respectively, and $g = k \oplus p$, the Cartan decomposition. We give the inner product $(\cdot,\cdot)$ on p by the restriction of(A.1), and let h be the G-invariant Riemannian metric on G/K induced from $(\cdot,\cdot)$. Then, it is known (cf. [Su]) that the spectrum of the Laplacian of $(G/K, h)$ is given by

$$\begin{cases} \text{the eigenvalues;} & (\lambda + 2\delta, \lambda), \\ \text{their multiplicities;} & d_\lambda. \end{cases}$$

Here, λ varies over the set $D(G, K)$ of highest weights of all spherical representations of (G, K), which is determined by [Su] as follows.

Let $a \subset p$ be a maximal abelian subspace in p, and h, a maximal abelian subalgebra of g containing a. Let π be a δ-fundamental root system, say $\pi = \{\beta_l, \dots, \beta_l\}$, $l = \dim(h)$, $\pi_0 = \{\beta \in \pi;\ \beta|_a \equiv 0\}$, and 2δ is the sum of all positive roots of $(g^{\mathbf{C}}, h)$ relative to π. We denote by $\{\mu_1, \dots, \mu_l\}$ the fundamental weights of g corresponding to π, and put

TABLE A.1. The first eigenvalue of the Laplacian of a compact simply connected simple Lie group.

type of G	$\lambda_1(G, g)$	
$A_l,\ l \geqslant 1$	$\dfrac{l(l+2)}{2(l+1)^2}$	X
$B_l,\ l \geqslant 2$	$\min\left\{\dfrac{l}{2l-1}, \dfrac{l(2l+1)}{8(2l-1)}\right\}$	
	$\frac{5}{12},\ l=2$	X
	$= \frac{21}{40},\ l=3$	
	$\dfrac{l}{2l-1},\ l \geqslant 4$	
$C_l,\ l \geqslant 2$	$\dfrac{2l+1}{4l+4}$	X
$D_l,\ l \geqslant 3$	$\min\left\{\dfrac{2l-1}{4l-4}, \dfrac{l(2l-1)}{16(l-1)}\right\}$	
	$\frac{15}{32},\ l=3$	X
	$= \dfrac{2l-1}{4l-4},\ l \geqslant 4$	
E_6	$\frac{13}{18}$	
E_7	$\frac{57}{72}$	
E_8	1	
F_4	$\frac{2}{3}$	
G_2	$\frac{1}{2}$	

$q = \dim(\mathfrak{a})$. Define

$$M_i,\ 1 \leqslant i \leqslant q, = \begin{cases} 2\mu_i, & \text{if } p\beta_i = \beta_i \text{ and } (\beta_i, \pi_0) = \{0\}, \\ \mu_i, & \text{if } p\beta_i = \beta_i \text{ and } (\beta_i, \pi_0) \neq \{0\}, \\ \mu_i + \mu_{i'}, & \text{if } p\beta_i = \beta_{i'} \text{ and } \beta_i \neq \beta_{i'}, \end{cases}$$

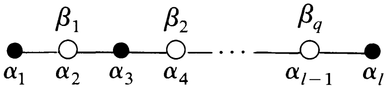
where p is Satake's involution. Then we have

$$D(G, K) = \left\{ \sum_{i=1}^{q} m_i M_i;\ m_i \geqslant 0,\ m_i \in \mathbb{Z},\ i = 1, \ldots, q \right\}.$$

Then, since $(M_i + 2\delta, M_i) \geqslant 0$, we have

$$\lambda_1(G/K, h) = \min\{(M_i + 2\delta, M_i);\quad i = 1, \ldots, q\}.$$

Let $\{\alpha_1, \ldots, \alpha_l\}$, $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_l\}$, 2ρ be the fundamental root system, the corresponding fundamental weights, the sum of all positive roots, in the last table in [Bo], respectively. Here, we should notice that the order in [Bo] is not in general the σ-order. Now $\{\beta_1,\ldots,\beta_q\}$, $q=\dim \mathfrak{a}$, in the δ-fundamental system $\pi=\{\beta_1,\ldots,\beta_l\}$ is given in the table in [Wr, p. 30–32] which is written as $\{\alpha_1,\ldots,\alpha_{l_+}\}$. And $\{\beta_{q+1},\ldots,\beta_l\}$ are some black circles in the Satake diagram π. Ignoring the distinction between black circles and white circles in the Satake diagram, we get the Dynkin diagram $\{\alpha_1,\ldots,\alpha_l\}$.

Example. AII 

, $l=2q+1$,

Then, define a one-to-one mapping ϕ from $\{\alpha_1,\ldots,\alpha_l\}$ onto $\{\beta_1,\ldots,\beta_l\}$ sending $\alpha_i (1\leqslant i\leqslant l)$ to β_{i^*} $(1\leqslant i^*\leqslant l)$ which has the same position as α_i in the diagram. In the above example AII, we get $\phi(\alpha_{2i})=\beta_i$, $1\leqslant i\leqslant q$. Then ϕ can be extended to an automorphism of $\mathfrak{g}$ due to Theorem 1 in [Seminaire "Sophus Lie" 1954/55, Ex. 11-04]. Under the identification of $\mathfrak{g}$ and $\mathfrak{g}^*$ with respect to the inner product $(\cdot,\cdot)$, the automorphism ϕ of $\mathfrak{g}$ and $\mathfrak{g}^*$ preserves the inner products $(\cdot,\cdot)$ of $\mathfrak{g}$ and $\mathfrak{g}^*$, $\phi(\tilde\omega_i)=\mu_{i^*}$ if $\phi(\alpha_i)=\beta_{i^*}$, and $\phi(\rho)=\delta$ by definition of $\tilde\omega_i$, μ_i, ρ and δ. In the above example AII, we get

$$M_i=\mu_i=\phi(\tilde\omega_{2i}),\quad \text{and } (M_i+2\delta,\ M_i)=(\tilde\omega_{2i}+2\rho,\ \tilde\omega_{2i}),$$

for $i=1,\ldots,q$,

Therefore, we have a list of the first eigenvalues $\lambda_1(G/K,\ h)$ and the M_i's of all simply connected irreducible Riemannian symmetric spaces $(G/K,\ h)$ of compact type:
Here in the Table A.2, $\tilde N$ means the universal covering of N and X means that the identity map of $(G/K,\ h)$ is unstable.

As an application we can discuss the stability or unstability of the identity map of all compact simply connected irreducible Riemannian symmetric spaces. The identity map of a compact Riemannian manifold $(M,\ g)$ onto itself is stable as a harmonic map (cf. [Sm]), if all the eigenvalues of the Jacobi operator coming from the second variation of a one parameter family of harmonic maps are non-negative. In case of an Einstein manifold $(M,\ g)$, i.e., $\mathrm{Ric}_g=cg$, where Ric_g is the Ricci tensor of $(M,\ g)$, $(M,\ g)$ is stable if and only if its first eigenvalue $\lambda_1(M,\ g)$ of the Laplacian on $C^\infty(M)$ satisfies $\lambda_1(M,\ g)\geqslant 2c$ (cf. [Sm, Proposition 2.1]).

Since a compact simply connected Lie group $(G,\ g)$ whose metric g is induced from the inner product (A.1) satisfies (cf. [K.N, p. 204]) $\mathrm{Ric}_g=\frac{1}{4}g$, we have:

$$\text{the identity map of } (G,\ g) \text{ is stable if and only if } \lambda_1(G,\ g)\geqslant \tfrac{1}{2}.$$

TABLE A.2. The first eigenvalue of the Laplacian of a simple connected irreducible Riemannian symmetric space of compact type.

type of G/K	G/K	M_i, $1 \leqslant i \leqslant q$	$\lambda_1(G/K, h)$
A I, $q \geqslant 2$	$SU(q+1)/SO(q+1)$	$M_i = 2\phi(\tilde{\omega}_i)$, $1 \leqslant i \leqslant q$	$\frac{(q+3)q}{(q+1)^2}$
A II, $q \geqslant 1$	$SU(2q+2)/Sp(q+1)$	$M_i = \phi(\tilde{\omega}_{2i})$, $1 \leqslant i \leqslant q$	$\frac{(2q+3)q}{2(q+1)^2}$ X
A III, $\frac{l}{2} \geqslant q \geqslant 2$ $q \geqslant 2$	$SU(l+1)/S(U(l+1-q) \times U(q))$ $SU(2q)/S(U(q) \times U(q))$	$M_i = \phi(\tilde{\omega}_i) + \phi(\tilde{\omega}_{l-i+1})$, $1 \leqslant i \leqslant q$ $\begin{cases} M_i = \phi(\tilde{\omega}_i) + \phi(\tilde{\omega}_{l-i+1}), 1 \leqslant i \leqslant q-1 \\ M_q = 2\phi(\tilde{\omega}_q) \end{cases}$	1 1
A IV, $l \geqslant 1$	$SU(l+1)/S(U(l) \times U(1))$	$M_1 = \phi(\tilde{\omega}_1) + \phi(\tilde{\omega}_l)$	1
B I, $l, \geqslant q \geqslant 2$	$SO(2l+1)/SO(2l+1-q) \times SO(q)$ $l > q$ $SO(2q+1)/SO(q+1) \times SO(q)$	$\begin{cases} M_i = 2\phi(\tilde{\omega}_i), 1 \leqslant i \leqslant q-1, \\ M_q = \phi(\tilde{\omega}_q) \end{cases}$ $M_i = 2\phi(\tilde{\omega}_i)$, $1 \leqslant i \leqslant q$	$\min\left\{\frac{2l+1}{2l-1}, \frac{-q^2+(2l+1)q}{4l-2}\right\}$ $= \begin{cases} 1, q=2, l \geqslant 2, \\ \frac{6}{5}, q=3, l=3, \\ \frac{2l+1}{2l-1}, \text{otherwise} \end{cases}$
B II, $l \geqslant 2$	$SO(2l+1)/SO(2l)$	$M_1 = \phi(\tilde{\omega}_1)$	$\frac{l}{2l-1}$ X
C I, $q \geqslant 3$	$Sp(q)/U(q)$	$M_i = 2\phi(\tilde{\omega}_i)$, $1 \leqslant i \leqslant q$	1
C II, $\frac{l-1}{2} \geqslant q \geqslant 1$ $q \geqslant 2$	$Sp(l)/Sp(l-q) \times Sp(q)$ $Sp(2q)/Sp(q)Sp(q)$	$M_i = \phi(\omega_{2i})$, $1 \leqslant i \leqslant q$	$\frac{l}{l+1}$ X $\frac{2q}{2q+1}$ X

TABLE A.2. (continued)

type of G/K	G/K	M_i, $1 \leqslant i \leqslant q$	$\lambda_1(G/K, h)$
D I, $l-2 \geqslant q \geqslant 2$	$SO(2l)/SO(2l-q)\times SO(q)$	$\begin{cases} M_i = 2\phi(\omega_i),\ 1 \leqslant i \leqslant q-1, \\ M_q = \phi(\omega_q) \end{cases}$	$\min\left\{\frac{l}{l-1}, \frac{-q^2+2lq}{4l-4}\right\} = \begin{cases} 1,\ q=2, \\ \frac{l}{l-1},\ q \geqslant 3, \end{cases}$
$q \geqslant 2$	$SO(2q+2)/SO(q+2)\times SO(q)$	$\begin{cases} M_i = 2\phi(\tilde{\omega}_i),\ 1 \leqslant i \leqslant q-1, \\ M_q = \phi(\tilde{\omega}_q)+\phi(\tilde{\omega}_{q+1}) \end{cases}$	$\min\left\{\frac{q+1}{q}, \frac{q+2}{4}\right\} = \begin{cases} 1,\ q=2, \\ \frac{5}{4},\ q=3, \\ \frac{q+1}{q},\ q \geqslant 4, \end{cases}$
$q \geqslant 2$	$SO(2q)/SO(q)\times SO(q)$	$M_i = 2\phi(\tilde{\omega}_i)$, $1 \leqslant i \leqslant q$	$\min\left\{\frac{q}{q-1}, \frac{q^2}{4q-4}\right\} = \begin{cases} 1,\ q=2, \\ \frac{9}{8},\ q=3, \\ \frac{q}{q-1},\ q \geqslant 4, \end{cases}$
D II, $l \geqslant 2$	$SO(2l)/SO(2l-1)$	$M_1 = \phi(\tilde{\omega}_1)$	$\frac{2l-1}{4l-4}$ X
D III, $q \geqslant 2$	$SO(4q)/U(2q)$	$\begin{cases} M_i = \phi(\tilde{\omega}_{2i}),\ 1 \leqslant i \leqslant q-1, \\ M_1 = 2\phi(\tilde{\omega}_{2q}) \end{cases}$	1
$q \geqslant 2$	$SO(4q+2)/U(2q+1)$	$\begin{cases} M_i = \phi(\tilde{\omega}_{2i}),\ 1 \leqslant i \leqslant q-1, \\ M_q = \phi(\tilde{\omega}_{2q})+\phi(\tilde{\omega}_{2q+1}) \end{cases}$	1
E I	$\widetilde{E_6/Sp(4)}$	$M_i = 2\phi(\tilde{\omega}_i)$, $1 \leqslant i \leqslant 6$	$\frac{14}{9}$
E II	$\widetilde{E_6/SU(2)\cdot SU(6)}$	$M_1 = 2\phi(\tilde{\omega}_2)$, $M_2 = 2\phi(\tilde{\omega}_4)$, $M_3 = \phi(\tilde{\omega}_3)+\phi(\tilde{\omega}_5)$, $M_4 = \phi(\omega_1)+\phi(\tilde{\omega}_6)$	$\frac{3}{2}$

E IV	E_6/F_4	$M_1 = \phi(\tilde{\omega}_1),\ M_2 = \phi(\tilde{\omega}_6)$	$\frac{13}{18}$	X
E V	$\widetilde{E_7/SU(8)}$	$M_i = 2\Phi(\tilde{\omega}_i),\ I \leqslant i \leqslant 7$	$\frac{5}{3}$	
E VI	$\widetilde{E_7/SO(12)\cdot SU(2)}$	$M_1 = 2\phi(\tilde{\omega}_1),\ M_2 = 2\phi(\tilde{\omega}_3),$ $M_3 = \phi(\tilde{\omega}_4),\ M_4 = \phi(\tilde{\omega}_6)$	$\frac{14}{9}$	
E VII	$\widetilde{E_7/E_6\cdot SO(2)}$	$M_1 = 2\phi(\tilde{\omega}_7),$ $M_2 = \phi(\tilde{\omega}_6),\ M_3 = \phi(\tilde{\omega}_1)$	1	
E VIII	$E_8/SO(16)$	$M_i = 2\phi(\tilde{\omega}_i),\ 1 \leqslant i \leqslant 8$	$\frac{31}{15}$	
E IX	$\widetilde{E_8/E_7\cdot SU(2)}$	$M_1 = 2\phi(\tilde{\omega}_8),\ M_2 = 2\phi(\tilde{\omega}_7),$ $M_3 = \phi(\tilde{\omega}_6),\ M_4 = \phi(\tilde{\omega}_1)$	$\frac{8}{5}$	
F I	$\widetilde{F_4/Sp(3)\cdot SU(2)}$	$M_i = 2\phi(\tilde{\omega}_i),\ 1 \leqslant i \leqslant 4$	$\frac{13}{9}$	
F II	$F_4/\mathrm{Spin}(9)$	$M_1 = \phi(\tilde{\omega}_4)$	$\frac{2}{3}$	X
G	$G_2/SU(2)\times SU(2)$	$M_i = 2\phi(\tilde{\omega}_i),\ i = 1, 2$	$\frac{7}{6}$	

Moreover we know the Ricci tensor Ric_h of a simply connected irreducible Riemannian symmetric space $(G/K, h)$ of compact type satisfies $\mathrm{Ric}_h = \frac{1}{2} h$ (cf. [T.K, p. 213]), so we have:

the identity map of $(G/K, h)$ is stable if and only if $\lambda_1(G/K, h) \geqslant 1$.

Together with the Tables A.1, A.2, we obtain:

THEOREM A.1: (1) *Let G be a compact simply connected simple Lie group, g a bi-invariant Riemannian metric on G. Then, the identity map of* (G, g) *is unstable if and only if the type of G is one of the following*: A_l, $l \geqslant 1$, B_2, C_l, $l \geqslant 2$ *and* D_3. (2) *Let* $(G/K, h)$ *be a simply connected irreducible Riemannian symmetric space of compact type. Then, the identity map is unstable if and only if the type of* G/K *if one of the following*: *AII, BII, CII, DII, EIV and FII, that is,* $SU(2q+2)/Sp(q+1)$, $q \geqslant 1$, *the unit sphere* S^n, $n \geqslant 3$, *the quaternion Grassmann manifolds* $Sp(l)/Sp(l-q) \times Sp(q)$, $l-q \geqslant q \geqslant 1$, E_6/F_4, *and the Cayley projective space* $F_4/\mathrm{Spin}(9)$.

REMARK: The classical irreducible Riemannian symmetric spaces with stable or unstable identity map have been known in [Sm, Proposition 2.13], and also see [N2]. However it should be noticed that the statement (3.1) in [N2] is false.

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