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## ON ABSOLUTELY EXTREMAL POINTS

S. Glasner and D. Maon

### Abstract

Given three doubly asymptotic points  $x, y, z$  in a minimal flow  $X$ , we construct an affine embedding  $\varphi: X \rightarrow Q$  such that  $\varphi(x) = \frac{1}{2}(\varphi(y) + \varphi(z))$ . Thus  $x$  is not absolutely extremal. We produce an example of a metric minimal flow  $X$  with the property that for every  $x \in X$  a triple  $x, y, z$  as above exists, thereby showing that no point of  $X$  is absolutely extremal.

### Introduction

We recall the definitions of affine embedding and absolute extremality for flows, introduced in [1]. If  $(T, X)$  is a flow ( $T$  is a self homeomorphism of the compact space  $X$ ) and  $(T, Q)$  an affine flow (i.e.,  $Q$  is a compact convex set and  $T$  an affine homeomorphism) then an equivariant continuous map  $\varphi: X \rightarrow Q$  is called an *affine embedding* if  $\overline{\text{co}} \varphi(X) = Q$ . A point  $x \in X$  is called *absolutely extremal* if for every affine embedding  $\varphi: X \rightarrow Q$   $\varphi(x)$  is in  $\partial Q$ , the set of extreme points of  $Q$ .

Suppose  $(T, X)$  is metric and minimal (i.e., every orbit is dense) then for every affine embedding  $\varphi: X \rightarrow Q$  the set  $\{x \in X: \varphi(x) \in \partial Q\}$  is a dense  $G_\delta$ . It was shown in [1] that if  $(T, X)$  is metric and minimal then every distal point of  $X$  is absolutely extremal. Again under our assumptions on  $(T, X)$  the set of distal points is either empty or a dense  $G_\delta$ . These facts led the first author to ask in [1] whether every minimal metric flow must have absolutely extremal points.

The easiest examples where non-absolutely extremal points exist are given by certain almost automorphic flows where the flow  $X$  is presented as a set of sequences in  $l^\infty(Z)$  and the identity map of  $X$  into  $Q = \overline{\text{co}}(X) \subset l^\infty(Z)$  gives a natural affine embedding [1]. Some doubly asymptotic points of  $X$  turns out to be non-extreme in  $Q$ . In this note, we show that in any minimal flow a point with two doubly asymptotic points is not absolutely extremal. ( $x, y$  are doubly asymptotic if  $\lim_{|n| \rightarrow \infty} d(T^n x, T^n y) = 0$ ). We construct a minimal metric flow, every point of which has a

continuum of doubly asymptotic points; thus providing an example of a metric minimal flow no point of which is absolutely extremal.

The principle of construction is due to Grillenberger (see e.g. [4]), who first showed how to define a minimal set with some desired property as an intersection of a family of subshifts of finite type. A continuous version of Grillenberger's construction and applications of this method are described in [3] and [2]. The present paper can be considered as a sequel to [1] and we refer the reader to [1] for further motivation.

### Section 1. An affine embedding associated with doubly asymptotic points

**1.1 PROPOSITION:** *Let  $(X, T)$  be an infinite metric minimal flow,  $x_0, y_0, z_0 \in X$ , doubly asymptotic points. Then there exists an affine embedding  $\varphi: X \rightarrow Q$  such that  $\varphi(x_0) = \frac{1}{2}(\varphi(y_0) + \varphi(z_0))$ .*

**PROOF:** In  $C^*(X)$  we let  $V$  be the weak \* closed linear space spanned by the set

$$V_0 = \left\{ \delta_{T_{x_0}^n} - \frac{1}{2}(\delta_{T_{y_0}^n} + \delta_{T_{z_0}^n}) : n \in \mathbb{Z} \right\}.$$

We let  $\pi: C^*(X) \rightarrow E = C^*(X)/V$  be the quotient map and define  $\varphi: X \rightarrow E$  by  $\varphi(x) = \pi(\delta_x) = \delta_x + V$ . Put  $Q = \overline{\text{co}}(\varphi(X))$  and let

$$W = \left\{ \eta \in C^*(X) : \eta = \sum_{n \in \mathbb{Z}} a_n \left( \delta_{T_{x_0}^n} - \frac{1}{2}(\delta_{T_{y_0}^n} + \delta_{T_{z_0}^n}) \right), \right. \\ \left. \times \sum_{n \in \mathbb{Z}} |a_n| < \infty \right\}.$$

We claim that  $V = W$ ; to see this let  $\sum_{n \in \mathbb{Z}} |a_n| < \infty$  be given. Put

$$\eta_N = \sum_{|n| \leq N} a_n \left( \delta_{T_{x_0}^n} - \frac{1}{2}(\delta_{T_{y_0}^n} + \delta_{T_{z_0}^n}) \right)$$

and let  $\eta$  be the infinite sum. For every  $f \in C(X)$  we have

$$|f(\eta_N) - f(\eta)| \leq \|f\|_2 \sum_{|n| > N} |a_n| \rightarrow 0.$$

Hence  $\eta_N \rightarrow \eta$  and  $\eta \in V$ . Thus  $W \subset V$ ; since  $V_0 \subset W$  it is enough to show that  $W$  is weak \* closed. By Krein-Šmulyan's theorem it suffices to

show that  $W \cap B_r$  is weak \* closed, where  $B_r = \{\nu \in C^*(X) : \|\nu\| \leq r\}$ . Since  $B_r$  is metrizable we can deal with sequences. So let

$$\eta^k = \sum_{n \in Z} a_n^k \left( \delta_{T_{x_0}^n} - \frac{1}{2} (\delta_{T_{y_0}^n} + \delta_{T_{z_0}^n}) \right)$$

be a sequence in  $W \cap B_r$  with  $\eta^k \rightarrow \eta \in V$ . We have

$$\text{Sup}_k \|\eta^k\| = \text{Sup}_k \text{Sup} \{ |f(\eta^k)| : \|f\| = 1 \} = \text{Sup}_k 2 \sum_{n \in Z} |a_n^k| < r.$$

Using a diagonal process we can choose a subsequence  $\eta^{k_i}$  such that for each  $n$   $a_n^{k_i} \rightarrow b_n$ . For convenience we denote this subsequence also by  $\eta^k$ , thus we now assume  $a_n^k \rightarrow b_n$  for every  $n$ . Using Fatou's lemma (in  $l_1(Z)$ ) we have

$$\sum_{n \in Z} |b_n| \leq \underline{\lim} \sum_{n \in Z} |a_n^k| < r.$$

Put

$$\tilde{\eta} = \sum_{n \in Z} b_n \left( \delta_{T_{x_0}^n} - \frac{1}{2} (\delta_{T_{y_0}^n} + \delta_{T_{z_0}^n}) \right)$$

and let  $f \in C(X)$  and  $\epsilon > 0$  be given. Choose  $\delta > 0$  such that  $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$  and  $N$  with

$$|n| > N \Rightarrow d(T^n y_0, T^n z_0), d(T^n z_0, T^n x_0), d(T^n y_0, T^n x_0) < \delta.$$

Then

$$\begin{aligned} |f(\eta^k - \tilde{\eta})| &= \left| \sum_{|n| \leq N} (a_n^k - b_n) [f(T^n x_0) - \frac{1}{2} (f(T^n y_0) + f(T^n z_0))] \right. \\ &\quad \left. + \sum_{|n| > N} (a_n^k - b_n) [f(T^n x_0) - \frac{1}{2} (f(T^n y_0) + f(T^n z_0))] \right| \\ &\leq 2 \|f\| \sum_{|n| \leq N} |a_n^k - b_n| + \epsilon \sum_{|n| > N} (|a_n^k| + |b_n|) \\ &\leq 2 \|f\| \sum_{|n| \leq N} |a_n^k - b_n| + \epsilon (\|\eta^k\| + \|\tilde{\eta}\|). \end{aligned}$$

It follows that  $\eta^k \rightarrow \tilde{\eta}$  so that  $\tilde{\eta} = \eta$  is in  $W$  and  $V = W$ .

Clearly  $\varphi$  is continuous and equivariant from  $X$  into the affine flow  $Q$ . If  $\varphi(x) = \varphi(y)$  then  $\delta_x - \delta_y \in V$ . But as  $V = W$ , every non-zero

measure in  $V$  is supported by at least three points. Thus  $x = y$  and  $\varphi$  is one to one. Finally

$$\begin{aligned}\varphi(x_0) &= \delta_{x_0} + V = \delta_{x_0} - \left( \delta_{x_0} - \frac{1}{2}(\delta_{y_0} + \delta_{z_0}) \right) + V \\ &= \frac{1}{2}(\delta_{y_0} + \delta_{z_0}) + V = \frac{1}{2}(\varphi(y_0) + \varphi(z_0)).\end{aligned}$$

This completes the proof.  $\square$

## Section 2. A metric minimal flow every point of which has a continuum of asymptotic points

Let  $\Omega = [0, 1]^Z$  denote the compact metric space of two sided  $[0, 1]$  valued sequences with the metric  $d(x, y) = \sup_{n \in Z} 2^{-|n|} |x_n - y_n|$ . For a closed  $W \subset [0, 1]^n$  and  $i \in Z$  we let

$$C_i(W) = \{x \in \Omega : \forall j \in Z, x[i + jn, i + (j + 1)n - 1] \in W\}$$

and  $C(W) = \bigcup_{i=1}^n C_i(W)$ . We define inductively a sequence  $n_k$  and closed sets  $W_k \subset W_{k-1}^{n_k}$  as follows. Let  $W_0 = [0, 1]$ . Given  $W_{k-1}$  we choose an arbitrary but fixed  $2^{-k}$ -net  $\{u_1, u_2, \dots, u_{l_k}\}$  of  $W_{k-1}^2$ , where the metric on a finite dimensional cube  $[0, 1]^n$  is  $d(w, v) = \sup_{1 \leq i \leq n} |w_i - v_i|$ . Let  $n_k = 100l_k$  and define

$$\begin{aligned}W_k &= \{w \in W_{k-1}^{n_k} : \text{there exist odd indices } 1 \leq i_1, i_2, \dots, i_{l_k} \leq n_k \\ &\quad \text{such that } w_{i_j} w_{i_j+1} = u_j \text{ for } j = 1, 2, \dots, l_k \\ &\quad \text{where } w = w_1 w_2 \dots w_{n_k}, w_{i_j} \in W_{k-1}\}.\end{aligned}$$

We call the set  $\{i_1, i_2, \dots, i_{l_k}\}$  a  $u$ -set for  $w$ . Put  $X = \bigcap_{k=1}^{\infty} C(W_k)$ .

**2.1 PROPOSITION:** *Let  $T$  be the shift on  $X$ , then  $(X, T)$  is a minimal flow.*

**PROOF:** Follows directly from the way  $X$  was defined.

**2.2 PROPOSITION:** *For every  $k$ ,  $W_k$  is pathwise connected.*

**PROOF:** Assume  $W_{k-1}$  is connected. Let  $w, w' \in W_k$ ,  $w = w_1 w_2 \dots w_{n_k}$ ,  $w' = w'_1 w'_2 \dots w'_{n_k}$ ,  $w_i, w'_i \in W_{k-1}$ . Assume first that there exists a  $u$ -set  $A = \{i_1, i_2, \dots, i_{l_k}\}$  common to  $w$  and  $w'$ . We let  $w(t) = w_1(t) w_2(t) \dots w_{n_k}(t)$  be defined by  $w_{i_j}(t) w_{i_j+1}(t) \equiv u_j$  if  $i_j \in A$  and where for all other  $i$ 's  $w_i(t)$  is a path in  $W_{k-1}$  connecting  $w_i$  and  $w'_i$ . Clearly  $w(t) \in W_k$  for every  $t \in [0, 1]$ . For the general case let  $A = \{i_1, i_2, \dots, i_{l_k}\}$

and  $A' = \{i'_1, i'_2, \dots, i'_k\}$  be  $u$ -sets for  $w$  and  $w'$  respectively. Choose  $A'' = \{i''_1, i''_2, \dots, i''_k\} \subset \{1, 2, \dots, n_k\}$  a set of odd indices disjoint from  $A \cup A'$  and define  $v = v_1 v_2 \dots v_{n_k}$ ,  $v' = v'_1 v'_2 \dots v'_{n_k}$  as follows

$$v_i v_{i+1} = \begin{cases} u_j & \text{if } i = i_j \text{ or } i = i'_j \\ w_i w_{i+1} & \text{otherwise} \end{cases}$$

$$v'_i v'_{i+1} = \begin{cases} u_j & \text{if } i = i'_j \text{ or } i = i''_j \\ w'_i w'_{i+1} & \text{otherwise.} \end{cases}$$

Clearly  $v, v' \in W_k$ . Now as  $A$  is a common  $u$ -set for  $w$  and  $v$ ,  $A''$  a common  $u$ -set for  $v$  and  $v'$  and  $A'$  a common  $u$ -set for  $v'$  and  $w'$ , we conclude by the first part of the proof, that there exists a path in  $W_k$  connecting  $w$  and  $w'$ .  $\square$

**DEFINITION:** Let  $K$  be a natural number  $1 \leq r, s \leq l_k$ ,  $r \neq s$ . A *chain* from  $u_r$  to  $u_s$  is a set  $\{j_0, j_1, \dots, j_l\}$  of indices such that  $j_0 = r$ ,  $j_l = s$  and  $d(u_{j_n}, u_{j_{n+1}}) < 2^{-k}$ ,  $0 \leq n < l$ .

For every  $r$  and  $s$  as above, the existence of a chain from  $u_r$  to  $u_s$  follows from the fact that  $W_k$  is pathwise connected.

**DEFINITION:** For  $x \in X$  there exists by definition a sequence of integers  $\{t_k\}$  such that  $t_k \leq 0 < t_k + m_k$  (where  $m_k$  is the length of sequences in  $W_k$ ) and such that for every  $k$   $x \in C_{t_k}(W_k)$ . It is easy to see that one can choose  $\{t_k\}$  so that  $\forall k$   $t_{k-1} \equiv t_k \pmod{m_{k-1}}$ . Such a sequence  $\{t_k\}$  will be called a *block partition* for  $x$ .

**2.3 PROPOSITION:** Let  $x \in X$ ,  $\{t_k\}$  a block partition for  $x$  and  $w_0 \in W_{k_0}$  for some  $k_0$ . Then there exists  $y \in X$  such that

- (1)  $y[t_{k_0}, t_{k_0} + m_{k_0} - 1] = w_0$
- (2)  $y$  is doubly asymptotic to  $x$ .

**PROOF:** We define  $y[t_k, t_k + m_k - 1]$  by induction on  $k$ . Put  $y[t_{k_0}, t_{k_0} + m_{k_0} - 1] = w_0$ . Let  $x[t_k, t_k + m_k - 1] = w_1 w_2 \dots w_{n_k} = w$ ,  $w_i \in W_{k-1}$ ,  $i = 1, \dots, n_k$ , and suppose  $x[t_{k-1}, t_{k-1} + m_{k-1} - 1]$  is  $w_n$ . Let  $A$  be a  $u$ -set for  $w$ . If  $n, n-1 \notin A$  define  $y[i] = x[i]$  for  $t_k \leq i \leq t_k + m_k - 1$ ,  $i \notin [t_{k-1}, t_{k-1} + m_{k-1} - 1]$  and then clearly  $y[t_k, t_k + m_k - 1] \in W_k$ .

If  $n = i_j \in A$ , let  $m$ ,  $1 \leq m \leq n_k - 1$  be an odd integer such that  $m \notin A$ . There exists an  $s$ ,  $1 \leq s \leq l_k$  such that  $d(u_s, w_m w_{m+1}) < 2^{-k}$ . Let  $j_0, j_1, \dots, j_l$  be a chain from  $u_r$  to  $u_s$ ,  $j_0 = r$ ,  $j_l = s$ ; thus for  $i_{j_t} \in A$   $w_{i_t} w_{i_t+1} = u_{j_t}$ ,  $t = 1, \dots, l$ . Put  $w' = w'_1 w'_2 \dots w'_{n_k}$  where for an odd  $i$

$$w'_i w'_{i+1} = \begin{cases} u_{j_{t-1}} & \text{if } i = i_{j_t}, 0 < t \leq l \\ u_s & \text{if } i = m \\ w_i w_{i+1} & \text{otherwise} \end{cases}$$

Since for every  $t$ ,  $d(u_{j_{t-1}}, u_{j_t}) < 2^{-k}$  and also  $d(u_s, w_m w_{m+1}) < 2^{-k}$  we have  $d(w, w') < 2^{-k}$ . Let  $y[t_k, t_k + m_k - 1] = w'_1 w'_2 \dots w'_{n-1} y[t_{k-1}, t_{k-1} + m_{k-1} - 1] w'_{n+1} \dots w'_{n_k}$ , then clearly  $y[t_k, t_k + m_k - 1] \in W_k$ . If  $n - 1 = i_r \in A$  the construction is similar.

There are now three possibilities:

- (1)  $t_k \rightarrow \infty$ ,  $t_k + m_k \rightarrow \infty$ , in which case  $y$  is now fully defined.
- (2) There exists  $k_0$  such that for  $k \geq k_0$ ,  $t_k = t_{k_0}$ . In this case define  $y[-\infty, t_{k_0} - 1] = x[-\infty, t_{k_0} - 1]$ .
- (3) There exists  $k_0$  such that for  $k \geq k_0$ ,  $t_k + m_k = t_{k_0} + m_{k_0}$ .

In this case define  $y[t_{k_0} + m_{k_0} + 1, \infty] = x[t_{k_0} + m_{k_0} + 1, \infty]$ . By definition of  $y$  we have for  $i < t_k$  and  $i > t_k + m_k$ ,  $|y[i] - x[i]| < 2^{-k}$ . Thus  $y$  and  $x$  are asymptotic.  $\square$

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