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ON ABSOLUTELY EXTREMAL POINTS

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Abstract

Given three doubly asymptotic points x, y, z in a minimal flow X, we construct an affine embedding $\varphi: X \to Q$ such that $\varphi(x) = \frac{1}{2}(\varphi(y) + \varphi(z))$. Thus x is not absolutely extremal. We produce an example of a metric minimal flow X with the property that for every $x \in X$ a triple x, y, z as above exists, thereby showing that no point of X is absolutely extremal.

Introduction

We recall the definitions of affine embedding and absolute extremality for flows, introduced in [1]. If (T, X) is a flow (T is a self homeomorphism of the compact space X) and (T, Q) an affine flow (i.e., Q is a compact convex set and T an affine homeomorphism) then an equivariant continuous map $\varphi: X \to Q$ is called an *affine embedding* if $\overline{\text{co}}\, \varphi(X) = Q$. A point $x \in X$ is called *absolutely extremal* if for every affine embedding $\varphi: X \to Q$ $\varphi(x)$ is in ∂Q, the set of extreme points of Q.

Suppose (T, X) is metric and minimal (i.e., every orbit is dense) then for every affine embedding $\varphi: X \to Q$ the set $\{x \in X: \varphi(x) \in \partial Q\}$ is a dense G_δ. It was shown in [1] that if (T, X) is metric and minimal then every distal point of X is absolutely extremal. Again under our assumptions on (T, X) the set of distal points is either empty or a dense G_δ. These facts led the first author to ask in [1] whether every minimal metric flow must have absolutely extremal points.

The easiest examples where non-absolutely extremal points exist are given by certain almost automorphic flows where the flow X is presented as a set of sequences in $l^\infty(Z)$ and the identity map of X into $Q = \overline{\text{co}}(X) \subset l^\infty(Z)$ gives a natural affine embedding [1]. Some doubly asymptotic points of X turns out to be non-extreme in Q. In this note, we show that in any minimal flow a point with two doubly asymptotic points is not absolutely extremal. (x, y are doubly asymptotic if $\lim_{|n| \to \infty} \mathrm{d}(T^n x, T^n y) = 0$). We construct a minimal metric flow, every point of which has a continuum of doubly asymptotic points; thus providing an example of a metric minimal flow no point of which is absolutely extremal.

The principle of construction is due to Grillenberger (see e.g. [4]), who first showed how to define a minimal set with some desired property as an intersection of a family of subshifts of finite type. A continuous version of Grillenberger's construction and applications of this method are described in [3] and [2]. The present paper can be considered as a sequel to [1] and we refer the reader to [1] for further motivation.

Section 1. An affine embedding associated with doubly asymptotic points

1.1 PROPOSITION: *Let (X, T) be an infinite metric minimal flow, $x_0, y_0, z_0 \in X$, doubly asymptotic points. Then there exists an affine embedding $\varphi: X \to Q$ such that $\varphi(x_0) = \frac{1}{2}(\varphi(y_0) + \varphi(z_0))$.*

PROOF: In $C^*(X)$ we let V be the weak * closed linear space spanned by the set

$$V_0 = \left\{ \delta_{T^n_{x_0}} - \tfrac{1}{2}\left(\delta_{T^n_{y_0}} + \delta_{T^n_{z_0}}\right) : n \in Z \right\}.$$

We let $\pi: C^*(X) \to E = C^*(X)/V$ be the quotient map and define $\varphi: X \to E$ by $\varphi(x) = \pi(\delta_x) = \delta_x + V$. Put $Q = \overline{\mathrm{co}}(\varphi(X))$ and let

$$W = \left\{ \eta \in C^*(X) : \eta = \sum_{n \in Z} a_n \left(\delta_{T^n_{x_0}} - \tfrac{1}{2}\left(\delta_{T^n_{y_0}} + \delta_{T^n_{z_0}}\right), \right.\right.$$

$$\left. \times \sum_{n \in Z} |a_n| < \infty \right\}.$$

We claim that $V = W$; to see this let $\sum_{n \in Z} |a_n| < \infty$ be given. Put

$$\eta_N = \sum_{|n| \leqslant N} a_n \left(\delta_{T^n_{x_0}} - \tfrac{1}{2}\left(\delta_{T^n_{y_0}} + \delta_{T^n_{z_0}}\right)\right)$$

and let η be the infinite sum. For every $f \in C(X)$ we have

$$|f(\eta_N) - f(\eta)| \leqslant \|f\| 2 \sum_{|n| > N} |a_n| \to 0.$$

Hence $\eta_N \to \eta$ and $\eta \in V$. Thus $W \subset V$; since $V_0 \subset W$ it is enough to show that W is weak * closed. By Krein–Šmulyan's theorem it suffices to show that $W \cap B_r$ is weak * closed, where $B_r = \{\nu \in C^*(X) : \|\nu\| \leqslant r\}$. Since B_r is metrizable we can deal with sequences. So let

$$\eta^k = \sum_{n \in Z} a_n^k \left(\delta_{T^n_{x_0}} - \tfrac{1}{2} \left(\delta_{T^n_{y_0}} + \delta_{T^n_{z_0}} \right) \right)$$

be a sequence in $W \cap B_r$ with $\eta^k \to \eta \in V$. We have

$$\operatorname*{Sup}_k \|\eta^k\| = \operatorname*{Sup}_k \operatorname{Sup}\{ |f(\eta^k)| : \|f\| = 1\} = \operatorname*{Sup}_k 2 \sum_{n \in Z} |a_n^k| < r.$$

Using a diagonal process we can choose a subsequence η^{k_l} such that for each n $a_n^{k_l} \to b_n$. For convenience we denote this subsequence also by η^k, thus we now assume $a_n^k \to b_n$ for every n. Using Fatou's lemma (in $l_1(Z)$) we have

$$\sum_{n \in Z} |b_n| \leqslant \underline{\lim} \sum_{n \in Z} |a_n^k| < r.$$

Put

$$\tilde{\eta} = \sum_{n \in Z} b_n \left(\delta_{T^n_{x_0}} - \tfrac{1}{2} \left(\delta_{T^n_{y_0}} + \delta_{T^n_{z_0}} \right) \right)$$

and let $f \in C(X)$ and $\epsilon > 0$ be given. Choose $\delta > 0$ such that $d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ and N with

$$|n| > N \Rightarrow d(T^n y_0, T^n z_0),\ d(T^n z_0, T^n x_0),\ d(T^n y_0, T^n x_0) < \delta.$$

Then

$$\begin{aligned} |f(\eta^k - \tilde{\eta})| = \Bigg| &\sum_{|n| \leqslant N} (a_n^k - b_n)\left[f(T^n x_0) - \tfrac{1}{2}\left(f(T^n y_0) + f(T^n z_0)\right)\right] \\ &+ \sum_{|n| > N} (a_n^k - b_n)\left[f(T^n x_0) - \tfrac{1}{2}\left(f(T^n y_0) + f(T^n z_0)\right]\right] \Bigg| \\ &\leqslant 2\|f\| \sum_{|n| \leqslant N} |a_n^k - b_n| + \epsilon \sum_{|n| > N} \left(|a_n^k| + |b_n| \right) \\ &\leqslant 2\|f\| \sum_{|n| \leqslant N} |a_n^k - b_n| + \epsilon \left(\|\eta^k\| + \|\tilde{\eta}\| \right). \end{aligned}$$

It follows that $\eta^k \to \tilde{\eta}$ so that $\tilde{\eta} = \eta$ is in W and $V = W$.

Clearly φ is continuous and equivariant from X into the affine flow Q. If $\varphi(x) = \varphi(y)$ then $\delta_x - \delta_y \in V$. But as $V = W$, every non-zero measure in V is supported by at least three points. Thus $x=y$ and φ is one to one. Finally

$$\varphi(x_0)=\delta_{x_0}+V=\delta_{x_0}-\left(\delta_{x_0}-\tfrac{1}{2}\left(\delta_{y_0}+\delta_{z_0}\right)\right)+V$$

$$=\tfrac{1}{2}\left(\delta_{y_0}+\delta_{z_0}\right)+V=\tfrac{1}{2}\left(\varphi(y_0)+\varphi(z_0)\right).$$

This completes the proof. □

Section 2. A metric minimal flow every point of which has a continuum of asymptotic points

Let $\Omega=[0,1]^Z$ denote the compact metric space of two sided $[0,1]$ valued sequences with the metric $d(x,y)=\sup_{n\in Z} 2^{-|n|}|x_n-y_n|$. For a closed $W\subset[0,1]^n$ and $i\in Z$ we let

$$C_i(W)=\{x\in\Omega:\forall j\in Z,\ x[i+jn,\ i+(j+1)n-1]\in W\}$$

and $C(W)=\bigcup_{i=1}^{n} C_i(W)$. We define inductively a sequence n_k and closed sets $W_k\subset W_{k-1}^{n_k}$ as follows. Let $W_0=[0,1]$. Given W_{k-1} we choose an arbitrary but fixed 2^{-k}-net $\{u_1,u_2,\dots,u_{l_k}\}$ of W_{k-1}^2, where the metric on a finite dimensional cube $[0,1]^n$ is $d(w,v)=\operatorname{Sup}_{1\leqslant i\leqslant n}|w_i-v_i|$. Let $n_k=100l_k$ and define

$$\begin{aligned}W_k=\{w\in W_{k-1}^{n_k}:\ &\text{there exist odd indices } 1\leqslant i_1,i_2,\dots,i_{l_k}\leqslant n_k\\ &\text{such that } w_{i_j}w_{i_j+1}=u_j \text{ for } j=1,2,\dots,l_k\\ &\text{where } w=w_1w_2\dots w_{n_k},\ w_{i_j}\in W_{k-1}\}.\end{aligned}$$

We call the set $\{i_1,i_2,\dots,i_{l_k}\}$ a *u-set for w*. Put $X=\bigcap_{k=1}^{\infty} C(W_k)$.

2.1 PROPOSITION: *Let T be the shift on X, then (X,T) is a minimal flow.*

PROOF: Follows directly from the way X was defined.

2.2 PROPOSITION: *For every k, W_k is pathwise connected.*

PROOF: Assume W_{k-1} is connected. Let $w,w'\in W_k$, $w=w_1w_2\dots w_{n_k}$, $w'=w_1'w_2'\dots w_{n_k}'$, $w_i,w_i'\in W_{k-1}$. Assume first that there exists a u-set $A=\{i_1,i_2,\dots,i_{l_k}\}$ common to w and w'. We let $w(t)=w_1(t)w_2(t)\dots w_{n_k}(t)$ be defined by $w_{i_j}(t)w_{i_j+1}(t)\equiv u_j$ if $i_j\in A$ and where for all other i's $w_i(t)$ is a path in W_{k-1} connecting w_i and w_i'. Clearly $w(t)\in W_k$ for every $t\in[0,1]$. For the general case let $A=\{i_1,i_2,\dots,i_{l_k}\}$

and $A' = \{i'_1, i'_2, \ldots, i'_{l_k}\}$ be u-sets for w and w' respectively. Choose $A'' = \{i''_1, i''_2, \ldots, i''_{l_k}\} \subset \{1, 2, \ldots, n_k\}$ a set of odd indices disjoint from $A \cup A'$ and define $v = v_1 v_2 \ldots v_{n_k}$, $v' = v'_1 v'_2 \ldots v'_{n_k}$ as follows

$$v_i v_{i+1} = \begin{cases} u_j & \text{if } i = i_j \text{ or } i = i''_j \\ w_i w_{i+1} & \text{otherwise} \end{cases}$$

$$v'_i v'_{i+1} = \begin{cases} u_j & \text{if } i = i'_j \text{ or } i = i''_j \\ w'_i w'_{i+1} & \text{otherwise.} \end{cases}$$

Clearly $v, v' \in W_k$. Now as A is a common u-set for w and v, A'' a common u-set for v and v' and A' a common u-set for v' and w'', we conclude by the first part of the proof, that there exists a path in W_k connecting w and w'. □

DEFINITION: Let K be a natural number $1 \leqslant r, s \leqslant l_k$, $r \neq s$. A *chain* from u_r to u_s is a set $\{j_0, j_1, \ldots, j_l\}$ of indices such that $j_0 = r$, $j_l = s$ and $d(u_{j_n}, u_{j_{n+1}}) < 2^{-k}$, $0 \leqslant n < l$.

For every r and s as above, the existence of a chain from u_r to u_s follows from the fact that W_k is pathwise connected.

DEFINITION: For $x \in X$ there exists by definition a sequence of integers $\{t_k\}$ such that $t_k \leqslant 0 < t_k + m_k$ (where m_k is the length of sequences in W_k) and such that for every k $x \in C_{t_k}(W_k)$. It is easy to see that one can choose $\{t_k\}$ so that $\forall k$ $t_{k-1} \equiv t_k (\text{mod } m_{k-1})$. Such a sequence $\{t_k\}$ will be called a *block partition for x*.

2.3 PROPOSITION: *Let $x \in X$, $\{t_k\}$ a block partition for x and $w_0 \in W_{k_0}$ for some k_0. Then there exists $y \in X$ such that*

(1) $y[t_{k_0}, t_{k_0} + m_{k_0} - 1] = w_0$

(2) *y is doubly asymptotic to x.*

PROOF: We define $y[t_k, t_k + m_k - 1]$ by induction on k. Put $y[t_{k_0}, t_{k_0} + m_{k_0} - 1] = w_0$. Let $x[t_k, t_k + m_k - 1] = w_1 w_2 \ldots w_{n_k} = w$, $w_i \in W_{k-1}$, $i = 1, \ldots, n_k$, and suppose $x[t_{k-1}, t_{k-1} + m_{k-1} - 1]$ is w_n. Let A be a u-set for w. If $n, n-1 \notin A$ define $y[i] = x[i]$ for $t_k \leqslant i \leqslant t_k + m_k - 1$, $i \notin [t_{k-1}, t_{k-1} + m_{k-1} - 1]$ and then clearly $y[t_k, t_k + m_k - 1] \in W_k$.

If $n = i_r \in A$, let m, $1 \leqslant m \leqslant n_k - 1$ be an odd integer such that $m \notin A$. There exists an s, $1 \leqslant s \leqslant l_k$ such that $d(u_s, w_m w_{m+1}) < 2^{-k}$. Let $j_0, j_1, \ldots, j_l$ be a chain from u_r to u_s, $j_0 = r$, $j_l = s$; thus for $i_{j_t} \in A$ $w_{i_{j_t}} w_{i_{j_t}+1} = u_{j_t}$, $t = 1, \ldots, l$. Put $w' = w'_1 w'_2 \ldots w'_{n_k}$ where for an odd i

$$w'_i w'_{i+1} = \begin{cases} u_{j_{t-1}} & \text{if } i = i_{j_t} \ 0 < t \leqslant l \\ u_s & \text{if } i = m \\ w_i w_{i+1} & \text{otherwise} \end{cases}$$

Since for every t, $d(u_{J_{t-1}}, u_{J_t}) < 2^{-k}$ and also $d(u_s, w_m w_{m+1}) < 2^{-k}$ we have $d(w, w') < 2^{-k}$. Let $y[t_k, t_k + m_k - 1] = w_1' w_2' \dots w_{n-1}' y[t_{k-1}, t_{k-1} + m_{k-1} - 1] w_{n+1}' \dots w_{n_k}'$, then clearly $y[t_k, t_k + m_k - 1] \in W_k$. If $n - 1 = i_r \in A$ the construction is similar.

There are now three possibilities:

(1) $t_k \to \infty$, $t_k + m_k \to \infty$, in which case y is now fully defined.

(2) There exists k_0 such that for $k \geqslant k_0$, $t_k = t_{k_0}$. In this case define $y[-\infty, t_{k_0} - 1] = x[-\infty, t_{k_0} - 1]$.

(3) There exists k_0 such that for $k \geqslant k_0$, $t_k + m_k = t_{k_0} + m_{k_0}$.

In this case define $y[t_{k_0} + m_{k_0} + 1, \infty] = x[t_{k_0} + m_{k_0} + 1, \infty]$. By definition of y we have for $i < t_k$ and $i > t_k + m_k$, $|y[i] - x[i]| < 2^{-k}$. Thus y and x are asymptotic. □

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