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The infinitesimal M. Noether theorem for singularities

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THE INFINITESIMAL M. NOETHER THEOREM
FOR SINGULARITIES

Hubert Flenner

Introduction

In 1882 M. Noether [25] has shown that for a general surface of degree $d > 4$ in $\mathbb{P}^3 = \mathbb{P}_\mathbb{C}^3$ each curve in $S$ is the intersection of $S$ with some hypersurface $S'$ in $\mathbb{P}^3$. Recently Carlson-Green-Griffiths-Harris [7] have given an infinitesimal version of this result: If $S$ is a smooth hypersurface of degree $d > 4$ in $\mathbb{P}^3$ and $C$ is a curve on $S$ such that for each first order deformation $\tilde{S}$ of $S$ the curve $C$ can be lifted to a first order deformation $\tilde{C} \subseteq \tilde{S}$ then $C = S \cap S'$ with some hypersurface $S' \subseteq \mathbb{P}^3$.

The purpose of this paper is to derive a similar result for singularities. Moreover we obtain with our methods, that isolated Gorenstein singularities $(X, 0)$ of dimension $d \geq 3$ with vanishing tangent functor $T_{X, 0}^{d-2}$ are almost factorial, i.e. each divisor $D \subseteq X$ is set theoretically given by one equation, or – equivalently – the divisor class group $\text{Cl}(\mathcal{O}_{X, 0})$ is a torsion group, see [27], [10]. By a result of Huneke [20] and Buchweitz [6] the assumption on the vanishing of $T_{X, 0}^{d-2}$ is always satisfied for isolated Gorenstein singularities which are linked to complete intersections.


We remark that throughout this paper we work in characteristic 0.

§1. The Main Lemma

Let $k$ be a field of characteristic 0 and $A = k[[T]]_{m}/\alpha$ a normal complete $k$-algebra with an isolated singularity of dimension $d \geq 3$. We set $X = \text{Spec}(A)$, $U := X \setminus \{m_A\}$. By $\Omega^1_X$ resp. $\Omega^1_U$, we denote the sheaf associated to the module of differentials $\Omega^1_A = \prod_{1 \leq i \leq n} A \cdot dT_i/A \cdot d(\alpha)$.

The logarithmic derivation $d \log : \mathcal{O}_U^\times \rightarrow \Omega^1_U$ induces a map $\text{Pic}(U) = H^1(U, \mathcal{O}_U^\times) \rightarrow H^1(U, \Omega^1_U)$. Since $A$ has isolated singularity we have $\text{Cl}(A) \cong \text{Pic}(U)$, see [10], (18.10) (b), and we obtain a map

$$\xi : \text{Cl}(A) \rightarrow H^1(U, \Omega^1_U).$$

In this section we will show:
Main Lemma 1.1: If depth \( A \geq 3 \) then \( \text{Ker}(\xi) \) is the torsion of \( \text{Cl}(A) \). In particular, if \( H^1(U, \Omega^1_U) \) vanishes then \( \text{Cl}(A) \) is a torsion group and \( A \) is almost factorial.

If \( k \subseteq K \) is a subfield and if \( A_K := A \otimes K, \ X_K := \text{Spec}(A_K), \ U_K := X_K \setminus \{ m_{A_K} \} \), then \( \text{Cl}(A) \subseteq \text{Cl}(A_K) \) and \( \text{Cl}(A) = \text{Cl}(A_K) \) by [24] if \( k \) and \( K \) are algebraically closed. Therefore by standard arguments we can easily reduce our assertion to the case \( k = \mathbb{C} \), which we shall henceforth assume. Before proving (1.1) in this case we need three lemmata:

Lemma 1.2: Let \( E \) be a complete algebraic \( \mathbb{C} \)-scheme. Then the canonical mapping induced by the logarithmic derivation

\[
\left( \text{Pic}(E)/\text{Pic}^*(E) \right) \otimes \mathbb{Z} \mathbb{C} \to H^1(E, \Omega^1_E)
\]

is injective.

Proof: If \( E \) is in addition smooth, then (1.2) is well known and follows from the Lefschetz-theorem on \((1, 1)\) sections, see [12], p. 163. In the general case, let \( f : E' \to E \) be a resolution of singularities of \( E \) and consider the following diagram:

\[
\begin{array}{ccc}
\left( \text{Pic}(E)/\text{Pic}^*(E) \right) \otimes \mathbb{Z} \mathbb{C} & \to & H^1(E, \Omega^1_E) \\
\downarrow \varphi & & \downarrow \\
\text{Pic}(E')/\text{Pic}^*(E') \otimes \mathbb{Z} \mathbb{C} & \to & H^1(E', \Omega^1_{E'})
\end{array}
\]

By [13], Exp. XII, Théorème 1.1 the map \( \text{Pic}(E) \to \text{Pic}(E') \) is of finite type. It follows that \( f^* : \text{Pic}(E)/\text{Pic}^*(E) \to \text{Pic}(E')/\text{Pic}^*(E') \) is injective, since \( \ker(f^*) \) is a torsion free discrete group scheme of finite type and so vanishes. Hence in the diagram \( \varphi \) is injective, from which the general case follows.

Lemma 1.3: Let \( A = \mathbb{C}\{X\}_n/\mathbb{A} \) be a normal (convergent) analytic algebra of dimension \( d \geq 3 \) with isolated singularity and set \( X = \text{Spec}(A), \ U = X \setminus \{ m_A \} \). Let \( X' \to X \) be a resolution of singularities of \( X \) such that \( E = \pi^{-1}(m_A) = E_1 \cup \ldots \cup E_k \) is a divisor with normal crossings. Then \( H^1(X', \Omega^1_{X'}) \) is a \( \mathbb{C} \)-vectorspace of rank \( k \).

Proof: The groups \( H^1(E(X'), \Omega^1_{X'}) \), \( H^{d-1}(X', \Omega^{d-1}_{X'}) \) are finite dimensional and dual to each other as the reasoning in the proof of prop. (2.2) in [18] shows. Let \( \pi^{an} : (X'^{an}, E'^{an}) \to (X^{an}, 0) \) be the corresponding analytic map. Then \( H^{d-1}(X', \Omega^{d-1}_{X'}) \cong H^{d-1}(E'^{an}, \Omega^{d-1}_{X'^{an}}) \), since \( (X'_{(n)}) \) indicates the \( n \)th infinitesimal neighbourhood of \( E \) in \( X' \)

\[
H^{d-1}(X'_{(n)}, \Omega^{d-1}_{X'_{(n)}}) \cong H^{d-1}(E'^{an}, \Omega^{d-1}_{X'^{an}_{(n)}})
\]
by the GAGA-theorems and since in the algebraic as well as in the analytic situation the comparison theorem holds. By Oshawa [26]

\[ H^{2d-2}(E^{an}, \mathbb{C}) \cong \bigsqcup_{p+q=2d-2} H^{pq} \]

where \( H^{pq} = H^{q}(E^{an}, \Omega_X^{p-an}) \) and \( H^{pq} = \overline{H^{pq}} \). Since \( E^{an} \) is real \((2d-2)\)-dimensional with components \( E_1, \ldots, E_k \) the group \( H^{2d-2}(E^{an}, \mathbb{C}) \) is a \( k \)-dimensional \( \mathbb{C} \)-vectorspace. Since \( \overline{H^{d,d-2}} \cong \overline{H^{d-2,d}} \cong H^{d}(E^{an}, \Omega_X^{d-2}) = 0 \) we get \( H^{d-1,d-1} \cong H^{2d-2}(E^{an}, \mathbb{C}) \cong \mathbb{C}^k \) as desired.

**Lemma 1.4:** Situation as in (1.3). Assume moreover that \( H^1(X', \mathcal{O}_{X'}) = 0 \).

Then the canonical map induced by the logarithmic derivation

\[ \text{Pic}(X') \otimes \mathbb{Z} \mathbb{C} \to H^1(X', \Omega^1_{X'}) \]

is injective.

**Proof:** From \( H^1(X', \mathcal{O}_{X'}) = 0 \) we get that \( \text{Pic}(X') \to \text{Pic}(E)/\text{Pic}^0(E) \) is injective, see e.g. the arguments in [3], Appendix or in [24]. From this fact together with (1.2) the assertion easily follows.

We will now prove (1.1): As remarked above we may assume \( k = \mathbb{C} \). By Artin [1] \( A \) is the completion of a convergent analytic \( \mathbb{C} \)-algebra and by Bingener [2] the divisor class group of a normal analytic algebra with isolated singularity does not change under completion. Hence we may as well assume that \( A \) is a convergent analytic \( \mathbb{C} \)-algebra. With the notation of (1.3) we consider the following diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & H^1_E(X', \mathcal{O}_{X'}) \otimes \mathbb{Z} \mathbb{C} & \to & \text{Pic}(X') \otimes \mathbb{Z} \mathbb{C} & \to & \text{Pic}(U) \otimes \mathbb{Z} \mathbb{C} & \to & 0 \\
& & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\
& & \cdots & & H^1_E(\Omega^1_{X'}) & & \to \to H^1(X', \Omega^1_{X'}) & \to & H^1(U, \Omega^1_{X'}) & \to & \cdots
\end{array}
\]

Here \( \alpha, \beta, \gamma \) are induced by the logarithmic derivation, and \( H^1_E(\Omega^1_{X'}) \) is easily seen to be the free subgroup of \( \text{Pic}(X') \) generated by \( E_1, \ldots, E_k \).

By (1.4) \( \beta \) is injective, hence \( \alpha \) is injective, and since by (1.3) \( H^1_E(\Omega^1_{X'}) \) is of rank \( k \) the map \( \alpha \) is even bijective. Hence we obtain by a simple diagram chasing that \( \gamma \) is injective as desired.

**Remark 1.5:** For a normal isolated singularity of dimension \( d \geq 3 \) \( \text{Cl}(A) \) has a natural structure of a Lie-group, see [4], [5]. More generally as in (1.1) the proof given above shows that

\[ \text{Cl}(A)/\text{Cl}^+(A) \to H^1(U, \Omega^1_U) \]

is injective (without the assumption depth \( A \geq 3 \)).
COROLLARY 1.6: Let $A = k[X]/\alpha$ be a Cohen-Macaulay ring of dimension $d$ such that $A$ is regular in codimension $\leq 2$ (i.e. $A$ satisfies $R_2$). Set $X = \text{Spec } A$, $U = \text{Reg } X$ and let $\xi : \text{Cl}(A) \to H^1(U, \Omega^1_U)$ be the mapping induced by the logarithmic derivation. Then $\ker \xi$ is a torsion group.

PROOF: If $d = 3$ then (1.6) is contained in (1.1). If $d > 3$ let $t \in A$ be a nonzero divisor such that $B = A/tA$ has property $R_2$ too; set $V := V(t) \cap U \subseteq U$. In the diagram

$$
\begin{array}{ccc}
\text{Pic}(U) & \xrightarrow{\xi} & H^1(U, \Omega^1_U) \\
\rho \downarrow & & \downarrow \\
\text{Pic}(V) & \xrightarrow{\xi} & H^1(V, \Omega^1_V)
\end{array}
$$

the restriction map $\rho$ is injective by [23] or [15], Exp. XI. Now the result follows by induction on $d$.

REMARK 1.7: If $A = \bigcup_{i > 0} A_i$ is quasihomogeneous, $A_0 = \mathbb{C}$, then the results above can be shown under much weaker assumptions: Set $X := \text{Spec } (A)$, $U := X \setminus \{m_A\}$, $m_A$ denoting the maximal homogeneous ideal. By $\text{Pic}_h(U)$ we denote the subgroup of $\text{Pic}(U)$ generated by those invertible $\mathcal{O}_U$-modules $\mathcal{L}$ such that $\Gamma(U, \mathcal{L})$ has a grading. Then

$$
\text{Pic}_h(U)/\text{Pic}_h^0(U) \xrightarrow{\xi} H^1(U, \Omega^1_U)
$$

is injective, if depth $A \geq 3$. Here we do not assume that $A$ has isolated singularity or even that $A$ is reduced. If $A$ is in addition normal then the same holds also for the completion of $A$ since in this case $\text{Pic}_h(U) = \text{Pic}(U) = \text{Pic}(\hat{U})$ by [9], (1.5) and its proof, where $\hat{U} := \text{Spec } (A) \setminus \{m_{\hat{A}}\}$. We shortly sketch the proof in the homogeneous case: $H^1(U, \Omega^1_U)$ has a natural grading and $\xi(\text{Pic}_h(U))$ is easily seen to be contained in $H^1(U, \Omega^1_U)_0$. If $Y = \text{Proj } (A)$ the natural mapping $\text{Pic}(Y)/\mathbb{Z} \cdot [\mathcal{O}_Y(1)] \to \text{Pic}_h(U)$ given by $\mathcal{L} \mapsto \bigcup_{i \geq 0} H^0(Y, \mathcal{L}(i))$ is bijective. In the diagram

$$
\begin{array}{ccc}
\text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{C} & \to & \text{Pic}_h(U) \otimes_{\mathbb{Z}} \mathbb{C} \\
\beta \downarrow & & \downarrow \xi \\
\mathbb{C} & \xrightarrow{\alpha} & H^1(Y, \Omega^1_Y) \to H^1(U, \Omega^1_U)_0
\end{array}
$$

where the last exact sequence is induced by the Euler-sequence, $\alpha(\mathbb{C}) = \mathbb{C} \cdot \beta([\mathcal{O}_Y(1)])$. Since $\beta$ is injective by (1.2) this implies the injectivity of $\xi$. We remark that these arguments can be carried over to the quasihom-
mogeneous case. One may ask if $\xi$ is also injective under these weaker assumptions if $A$ is not quasihomogeneous.

§2. Applications

Let $k$ be always a field of characteristic 0. In the following we will formulate our results for complete local $k$-algebras $A = k[\mathbb{X}]_n/\mathfrak{a}$. We remark that they are also valid in the corresponding analytic or algebraic situation.

THEOREM 2.1: Let $A = k[\mathbb{X}]_n/\mathfrak{a}$ be an isolated Gorenstein singularity of dimension $d \geq 3$ satisfying $T_{d-2}^2(A) = 0$. Then $A$ is almost factorial.

PROOF: It is well known and follows easily from the spectral sequence

$$E_2^{p,q} = \text{Ext}^p_A(A, A) \Rightarrow T_A^q(A),$$

that $T_A^{d-2}(A) = \text{Ext}^{d-2}_A(\Omega^1_A, A)$ in this case. By Grothendieck-duality $\text{Ext}^{d-2}_A(\Omega^1_A, A)$ is dual to $H^2_m(\Omega^1_A) \cong H^1(U, \Omega^1_U)$, where $U = \text{Spec}(A) \setminus \{m_A\}$ as usual. By (1.1) our result follows.

In particular (2.1) implies, that a 3-dimensional rigid isolated Gorenstein singularity is almost factorial. We remark that the condition $T_{d-2}^2(A) = 0$ in (2.1) is necessary: If $A$ is the completion of the local ring at the vertex of the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with respect to $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}(2)$, then $A$ is an isolated Gorenstein singularity, which is even rigid, but $\text{Cl}(A) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/(2)$.

From (2.1) it is easily to deduce a similar result for non isolated singularities.

COROLLARY 2.2: Let $A = k[\mathbb{X}]_n/\mathfrak{a}$ be a $d$-dimensional Gorenstein singularity which is regular in codimension $\leq k$, where $3 \leq k < d$. Suppose $T_A^{d-2}(A) = \ldots = T_A^{k-1}(A) = 0$. Then $A$ is almost factorial.

PROOF: In the case $d = k + 1$ this is just (2.1). In the case $d > k + 1$ choose $t \in A$ such that $B = A/tA$ is also regular in codimension $\leq k$. By the Lefschetz theorem of [23] or [15], Exp. XI, $\text{Cl}(A) \rightarrow \text{Cl}(B)$ is injective. From the exact cohomology sequence of tangent functors

$$\ldots \rightarrow T_A^i(A) \rightarrow T_A^i(A) \rightarrow T_A^i(B) \rightarrow T_A^{i+1}(A) \rightarrow \ldots$$

$$\ldots \rightarrow T_{A/B}^i(B) \rightarrow T_B^i(B) \rightarrow T_A^i(B) \rightarrow T_{A/B}^{i+1}(B) \rightarrow \ldots$$

and the vanishing of $T_{A/B}^i(B)$, $i \geq 2$, we obtain $T_B^{d-3}(B) = \ldots = T_B^{k-1}(B) = 0$. Now the assertion follows by induction on $d$. [5] Infinitesimal M. Noether theorem for singularities 45
In the quasihomogeneous case (2.2) has been shown by Buchweitz (unpublished). By [6], [20], the assumptions on the vanishing of the tangent functors are satisfied, if \( A \) is linked to a complete intersection. For other results in this direction see also [21], [28].

In the case \( d = 3 \) we now show a refined version of (2.1), which is an analogue of the infinitesimal M. Noether theorem in [7]. Let \( A = k[X]^n/\alpha \) be an isolated Gorenstein singularity of dimension 3 and \( U := \text{Spec}(A) \setminus \{ m_A \} \). Suppose \( L \) is a reflexive \( A \)-module of rank 1 and denote by \( \mathcal{L} \) the associated invertible sheaf on \( U \). If \( k[\epsilon] \rightarrow A'(\epsilon^2 = 0) \) is a first order deformation of \( A \), we set \( U' = \text{Spec}(A') \setminus \{ m_{A'} \} \).

**Theorem 2.3:** Suppose that for each first order deformation \( k[\epsilon] \rightarrow A' \) of \( A \) \( \mathcal{L} \) can be extended to a locally free sheaf \( \mathcal{L}' \) on \( U' \). Then \( L \) is a torsion element in \( \text{Cl}(A) \).

**Proof:** Let \( \xi_L \in H^1(U, \Omega_U^1) \) be the class associated to \( L \) under the map \( \text{Cl}(A) \rightarrow H^1(U, \Omega_U^1) \). It is well known that the group \( \text{Ext}^1_A(\Omega^1_A, A) \) describes the first order deformations of \( A \). Denote by \([A']\) the cohomology class in \( \text{Ext}^1_A(\Omega^1_A, A) \) associated to \( A' \). Then it is not difficult to see that in the canonical pairing

\[
\text{Ext}^1_A(\Omega^1_A, A) \times H^1(U, \Omega_U^1) \rightarrow H^2(U, \mathcal{O}_U)
\]

\( \langle [A'], \xi_L \rangle \) is just the obstruction for extending \( \mathcal{L} \) to a \( \mathcal{L}' \). But by Grothendieck duality this pairing is nondegenerated, and so by our assumption \( \xi_L = 0 \), which implies by (1.1) that \( L \) is a torsion element in \( \text{Cl}(A) \).

For the case of complete intersections it is possible to strengthen (2.3):

**Proposition 2.4:** Let \( A \) be as in (2.3) and suppose moreover that \( A \) is a complete intersection. Then \( A' = k[X]^n/\alpha \) is parafactorial.

**Proof:** Let \( \mathcal{L}' \) be a locally free module on \( U' := \text{Spec}(A') \setminus \{ m_{A'} \} \). If \( A = k[X]^n/(f_1, \ldots, f_{n-3}) \), denote by \( A_i \) the first order deformation

\[
A_i := k[X]^n/(f_1, \ldots, f_{i-1}, f_i^2, f_{i+1}, \ldots, f_{n-3}), \quad \epsilon \mapsto \overline{f_i}
\]

of \( A \) and \( U_i := \text{Spec}(A_i) \setminus \{ m_{A_i} \} \). Moreover let \( \mathcal{L} \) resp. \( \mathcal{L}_i \) be the sheaf on \( U \) resp. \( U_i \) induced by \( \mathcal{L}' \). By assumption \( \mathcal{L} \) can be extended to the locally free sheaf \( \mathcal{L}_i \) on \( U_i \), hence with the notations in the proof of the last result

\[
\langle [A_i], \xi_{\mathcal{L}_i} \rangle = 0, \quad i = 1, \ldots, n-3.
\]
But the $A_{i}$ generate $\text{Ext}^{1}_{A}(\Omega^{1}_{A}, A)$ as an $A$-module and so $\xi_{\mathcal{L}} = 0$, and $\mathcal{L} \in \text{Pic}(U)$ is a torsion element. Since $\text{Pic}(U) = \text{Cl}(A)$ is known to have no torsion, see [4] (3.2), we get $\mathcal{L} \cong \mathcal{O}_{U}$ and hence $\mathcal{L}' \cong \mathcal{O}_{U'}$, as desired.

We will now apply these results to normal bundles of Gorenstein singularities.

**Theorem 2.5:** Let $A = k\llbracket X \rrbracket_{\alpha}$ be a $d$-dimensional isolated Gorenstein singularity, $d \geq 3$, $W := \text{Spec}(k\llbracket X \rrbracket_{\alpha}) \setminus \{m\}$, $X := \text{Spec}(A)$, $U := X \setminus \{m_{A}\}$, $Y \subseteq X$ a divisor and $V := Y \setminus \{m_{A}\}$. If the sequence of normal bundles

$$0 \to N_{V/U} \to N_{V/W} \to N_{U/W} \otimes \mathcal{O}_{V} \to 0$$

splits on $V$ then $Y$ represents a torsion element in $\text{Cl}(A)$, i.e. $Y$ is given set-theoretically by one equation.

**Proof:** First we will assume $d = 3$. Let $R$ denote the ring $k\llbracket X \rrbracket_{\alpha}$ and $\tilde{B} := H^{0}(V, \mathcal{O}_{V})$, which by Grothendieck's finiteness theorem is finite over $B := H^{0}(Y, \mathcal{O}_{Y})$. Then $H^{0}(V, N_{V/U}) = T_{R/B}^{1}(\tilde{B})$, $H^{0}(V, N_{V/W}) = T_{B}^{1}(\tilde{B})$, $H^{0}(V, N_{U/W} \otimes \mathcal{O}_{V}) = T_{R/A}^{1}(\tilde{B})$, and our assumption implies that

$$0 \to T_{A/B}^{1}(\tilde{B}) \to T_{R/B}^{1}(\tilde{B}) \to T_{R/A}^{1}(\tilde{B}) \to 0$$

is exact. In particular in the diagram

$$
\begin{array}{ccc}
T_{R/B}^{1}(\tilde{B}) & \to & T_{R/A}^{1}(\tilde{B}) \\
\downarrow \alpha & & \downarrow \beta \\
T_{B}^{1}(\tilde{B}) & \to & T_{A}^{1}(\tilde{B})
\end{array}
$$

$\gamma$ is onto, and since $R$ is regular, $\alpha$, $\beta$ are surjective too, from which we obtain the surjectivity of $\delta$. Consider the diagram

$$
\begin{array}{ccc}
T_{A}^{1}(A) \\
\downarrow \\
T_{B}^{1}(\tilde{B}) \\
\delta \\
T_{A}^{1}(\tilde{B})
\end{array}
$$

That $\delta$ is surjective means: If $k[\epsilon] \to A'$ is a first order deformation of $A$ then there exists an extension $[B'] \in T_{B}^{1}(\tilde{B})$ and a commutative diagram

$$
\begin{array}{ccc}
0 \to A & \to A' & \to A \\
\downarrow & \downarrow & \downarrow \\
0 \to \tilde{B} & \to B' & \to \tilde{B} \\
\end{array}
$$
see [8], §1. In particular $V \subseteq U$ can be extended to a first order deformation $V' := \text{Spec}(B') \setminus \{m_{B'}\} \subseteq U' := \text{Spec}(A') \setminus \{m_{A'}\}$ or, equivalently, $\mathcal{L} = \mathcal{O}_U(V)$ can be extended to a locally free sheaf $\mathcal{L}' = \mathcal{O}_{U'}(V')$. By (2.3) $\mathcal{L}$ is a torsion element in $\text{Cl}(A)$ and so $Y$ can be described set-theoretically by one equation.

Now suppose $d > 3$. Let $t \in R$ be a generic linear combination of $X_1, \ldots, X_n$ with coefficients in $k$. Set $\overline{W} := V(t) \subseteq W$, $\overline{U} := V(t) \cap U$, $\overline{V} := V(t) \cap V$. Then $\overline{W}$, $\overline{U}$ are smooth, and $\overline{A} := A/tA$ is an isolated Gorenstein singularity of dimension $d - 1$. Once more by [15,23] $\text{Cl}(A)$ $\to \text{Cl}(\overline{A})$ is injective, and moreover the normal bundle sequence

$$0 \to \mathcal{N}_{\overline{V}/\overline{U}} \to \mathcal{N}_{\overline{V}/W} \to \mathcal{N}_{\overline{U}/W} \otimes \mathcal{O}_V \to 0$$

splits, since it is the restriction of our original normal bundle sequence to $\overline{V}$. Now the assertion follows by induction on $d$.

Applying this result to the cone over a projective variety we immediately obtain a generalization of the results [11] Chap. IV, (f), and [16] mentioned in the introduction.

**COROLLARY 2.6:** Suppose $X \subseteq \mathbb{P}^n = \mathbb{P}^n_k$ is an arithmetically Cohen-Macaulay submanifold of dimension $d \geq 2$ such that $\omega_X = \mathcal{O}_X(\ell)$ for some $\ell$. If $Y \subseteq X$ is a 1-codimensional Cartier-divisor and if the sequence of normal bundles

$$0 \to \mathcal{N}_{Y/X} \to \mathcal{N}_{Y/P^*} \to \mathcal{N}_{X/P^*} \otimes \mathcal{O}_Y \to 0$$

splits, then there is a hypersurface $H \subseteq \mathbb{P}^n$ such that $Y = H \cap X$ set-theoretically.

**REMARKS:** (1) In the case $d = 3$ in (2.5) it is obviously sufficient to require that $H^0(V, \mathcal{N}_{Y/W}) \to H^0(V, \mathcal{N}_{U/W} \otimes \mathcal{O}_V)$ is surjective. Similarly in (2.6) it suffices that $H^0(\mathcal{N}_{Y/P^*}(\ell)) \to H^0(\mathcal{N}_{X/P^*} \otimes \mathcal{O}_Y(\ell))$ is surjective if $d = 2$.

(2) If in (2.5) resp. (2.6) $\text{Cl}(A)$ resp. Pic($X$) has no torsion then $Y$ is even scheme-theoretically given by one equation. This is e.g. satisfied if $A$ resp. $X$ is a complete intersection, see [4], (3.2).

(3) I do not know whether these results continue to be true without the assumption char($A) = 0$. At least the proofs given here do not apply since we have heavily used the Hodge-decomposition theorem of Oshawa.

(4) If the problem mentioned at the end of section 1 would be true, then (2.3) would be valid in the case of any 3-dimensional Gorenstein singularity (not necessarily isolated) if $\mathcal{L}$ is assumed to be locally free. In order to show this let $\text{Ext}^1_{\mathcal{A}}(\Omega^1_{\mathcal{A}}, A) \to T^1_{\mathcal{A}}(A)$ be the map induced by
the canonical projection $L^*_{A} \to \Omega^1_{A}$, where $L^*_{A}$ is the cotangent complex of $A$. Then in the diagram

$$
\begin{array}{c}
\text{Ext}^1(A, \Omega^1_{A}) \times H^1(U, \Omega^1_U) \\
\downarrow \alpha \\
T^1(A) \times \text{Pic}(U)
\end{array}
\begin{array}{c}
\n
\n\end{array}

\begin{array}{c}
\uparrow \xi \\
H^2(U, \Omega^1_U)
\end{array}

$$

is commutative in the sense, that $\{\alpha(x), [L]\} = \langle x, \xi_L \rangle$. Here $\{[A'], L\}$ denotes the obstruction of extending $L$ to $A'$. Now the proof of (2.3) applies. In a similar way, then it would be possible to generalize (2.4), (2.5). In (2.5) we could replace the condition “isolated singularity” by “$U$ is locally a complete intersection in $W$”. By the last remark in section 1 this is at least true for quasihomogeneous singularities, and so we obtain:

**Corollary 2.7**: (2.6) remains true if the condition “submanifold” is replaced by “locally a complete intersection”.

**References**


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