V. AVERBUCH

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# REMARK ON THE DEFINITION OF AN AUTOMORPHIC FORM

V. Averbuch

## Introduction

Let $k$ be a global field, $G$ a reductive algebraic group over $k$. Let $K$ be a maximal compact subgroup of $G_A$, and let $K^f$ be it's finite component.

In |1|, an autmorphic form a $G$ is defined as a smooth $C$-valued function on $G_A$, which satisfied the following conditions.

(1) $f$ is $K^f$-finite and Hecke-finite.

(2) $f$ is a function on $G_k \backslash G_A$

(3) $f$ transforms according to a character under the action of the centre.

(4) $f$ satisfies a moderate growth condition in the Siegel domain. (Precise definition is given in the beginning of Part 1 for $GL_n$ over $Q$ and in the beginning of Part 2 for the general case).

These conditions are not independent. If char $k > 0$, then (4) can be deduced from (1)–(3). If char $k = 0$, then Hecke-finiteness at finite places follows from other conditions and Hecke-finiteness at Archemedian places. Here we prove the following theorem:

THEOREM: *If* char $k = 0$, *then* (4) *can be deduced from* (2), (3), $K^f$*-finiteness and Hecke-finiteness at some big finite place* $-p$, *s.t.* $f$ *is invariant under* $K_p$.

*Big finite place is any place* $p$ *at which* $N_p = \#O_p/m_p \geqslant C$ *where* $C$ *depends on G nd on* $k$.

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## Part 1. Case of $G = GL_n$, $k = Q$

We prove the main theorem first in this case because of two reasons. First, the proof in the general case is essentially the same, but includes some technical details. Second, the intermediate results of this case are needed in the general case.

NOTATION: $B$ is the standard Borel subgroup, $\alpha_1, \ldots, \alpha_{n-1}$ simple roots, $\chi_1, \ldots, \chi_{n-1}$ the fundamental weights. For each $\chi_i$, there is a character $\mu_i : G_A \to R^*$, trivial on $G_k$, s.t. $|\chi_i \mu_i|$ is trivial on the centre.

Let $K$ be the standard maximal compact subgroup. We have the Iwasawa decomposition $G_A = B_A \cdot K_A$.

If $\chi$ is an algebraic character on $B$, we can consider $|\chi|$ as a function on $G_A$, using the Iwasaw decomposition, $|\chi(bk)| = |\chi(b)|$.

Recall our definition of the Hecke ring $H(G_p)$. It consists of all compactly supported smooth functions on $G_p$. Multiplication is given by convolution. If $K'$ is an open compact subgroup of $G_p$, we can define $H^{K'}(G_p)$ as the subring consisting of all two-sided $K'$-invariant functions. If $g \in G_p$, we can define an element $f^g \in H^{K'}(G_p)$, by

$$f^g(g') = \begin{cases} 1/\mu(K') & g' \in K'gK' \\ 0 & \text{otherwise} \end{cases}$$

Here $\mu$ is the standard Haar measure. (The measure of maximal compact subgroup is 1).

We have a decomposition

$$K'gK' = \cup g_i K' \quad \text{(disjoint union)}$$

Since $K'$ is compact and open, the union is finite. If $f$ is any smooth $K'$-invariant function on $G_p$ or $G_A$, the action of $f^g$ on $f$ is given by

$$(\pi(f^g)f)(g') = \sum f(g'g_i)$$

We recall some facts about fundamental domains. For $c > 0$, set

$$\sigma(c) = \{ b \in B_A \mid |\alpha_i(b)| \geqslant c \quad \forall_i \}$$

These sets are called Siegel domains.

THEOREM 1: *There is $c_0 > 0$, s.t. $G_A = G_Q \cdot \sigma(c_0)K$. Any $\sigma(c)$ which satisfies this condition is called a fundamental domain. We fix $c_0$ to be s.t. $\sigma(c_0)$ is a fundamental domain.*

*For each $C > 0$, let*

$$M^C = \left\{ b \cdot k \mid b \in \sigma(c_0),\ k \in K,\ \prod_i |\chi_i \mu_i(b)| \leqslant C \right\}.$$

*$B_Q$ acts on $M^C$ from the left, and the quotient is compact, modulo the centre.*

Now we can state the main theorem.

MAIN THEOREM: *Let $f$ be a function on $G_Q \backslash G_A$ s.t.*

(1) *$f$ is smooth and $K_p$-invariant, and transforms by a character under action of the centre.*
(2) *$f$ is Hecke finite at place $p$.*

*Let* $L(c) = \max_{g \in M^c} |f(g)|$.

*If* $p \geqslant C$, *where* $C$ *is some constant, which depends only on* $G$, *then* $L(c)$ *has a moderate growth as* $c \to \infty$, *t.i. there are constants* $A_1$, $A_2$, $A_3$, *s.t.* $L(c) \leqslant A_1 \cdot c^{A2}$ *for* $c \geqslant A_3$.

*We need one simple property of the decomposition* $G_A = G_Q \cdot \sigma(c_0) \cdot K$.

LEMMA 1: *Let* $b$, $b' \in B_A$, $b' = q \cdot b \cdot k$, *where* $q \in G_Q$, $k \in K$. *Let* $\chi$ *be a dominant weight. Then,*

$$|\chi(b')| \leqslant \max_{w \in W} |(w\chi)(b)|$$

*Here* $W$ *is the Weyl group*

PROOF: We shall first prove the case of $\chi = \chi_{n-1}$. Since $|\det b'| = |\det b|$, it is sufficient to show, that $|b'_n| \geqslant \min |b_i|$, where $b_i$, $b'_i$ are the diagonal entries of $b$, $b'$. Let $\vec{b}_1, \dots, \vec{b}_n$ be the row vectors of $b$. It is easy to see, that

$$|b'_n| = \prod_p |b'_n|_p = \prod_p |\left(\sum q_{ni}\vec{b}_i\right)_p|_p$$

Let $j$ be minimal, s.t. $q_{nj} \neq 0$. Since $b$ is upper triangular,

$$|\left(\sum q_n \vec{b}_i\right)_p|_p \geqslant |q_{nj} b_j| p$$

So

$$|b'_n| \geqslant \prod_p |q_{nj} b_j|_p = |b_j|$$

So, the case of $\chi = \chi_{n-1}$ is proved.

Now, let us use the representation $g \to \Lambda^{n-1}g$. Since

$$\chi_{n-1}(\Lambda^{n-1}b) = \chi_1(b) \cdot (\det b)^{n-2}$$

the case $\chi = \chi_1$, be follows from the previous case. Let now $\chi$ be any dominant weight. Consider the representation $V(\chi)$ with highest weight $\chi$. Using this representation we see, that

$$|\chi(b')| \leqslant |\chi_1(b)|,$$

where $\chi_1$ is some weight in the representiation $V(\chi)$. It can be easily seen, that there exists $w \in W$, s.t.

$$|\chi_1(b)| \leqslant |w\chi(b)|.$$

We need some properties of Hecke operators.

THEOREM 2:

(1) *Let* $a_i = \text{diag}(p^{-1}, \ldots, p^{-1}, 1, \ldots, 1)$ (*i times* $p^{-1}$). *Let* $Ka_iK = \cup b_jK$ *be a disjoint uniton with* $b_j \in B_p$. *Then each* $b_j$ *has i eigenvalues with absolute value p and n − i eigenvalues with absolute value* 1. *Among them, there is only one whose first i eigenvalues have absolute value p. This element lies in aK.*

(2) *If* $g,\ g' \epsilon G_p$, *then* $g \in Kg'K \Leftrightarrow$ *for all j, the maximal absolute value of* $j \cdot j$ *minors of g is equal to that of* $g'$.

*This theorem is well-known and we shall not prove it. We use this theorem in the following formulation. Let us denote by* $A^+$ *the set of all diagonal matrixes, absolute values of whose eigenvalues are not increasing.*

THEOREM 3:

*Let* $a \in A^+$, *let* $KaK = aK \cup (\cup b_jK)$ *be a disjoint decomposition with* $b_j \in B$.

(1) *Let* $\chi$ *be any dominant weight, and* $w \in W$ *any element of the Weyl group. Then,*

$$|\chi(a)| \geqslant |w\chi(b_j)|.$$

(2) *Suppose that* $\forall_i$ *s.t.* $|\alpha_i(a)| > 1$, *we have* $\chi/\chi_i$ *is a dominant weight. Then,*

$$|\chi(a)| \geqslant p|\chi(b_l)|.$$

PROOF 1: (1) We can write $\chi$ as a product of fundamental weights, and since both sides of the inequality ar multiplicative in $\chi$, we can assume that $\chi = \chi_i$ for some $i$. Then $|\chi(a)|$ is the maximal absolute value of a $i \times i$ minor of $a$, and $|w\chi(b_j)|$ is the absolute value of some $i \times i$ minor of $b_j$. Since $b_j \in KaK$, (1) now follows from Theorem 2, (2).

(2) We can write $a$ as a product of $a_0 \cdot c \cdot \prod a_i^{n_i}$, where $a_0 \in K \cap A$, $c$ lies in the centre, $a_i$ are as in Theorem 2 and $n_i \geqslant 0$. For $a = a_0 \cdot c$ the theorem is trivial because $KaK = aK$. So, suppose that $a = a' \cdot a_i$, $a' \in A^+$ and the theorey is true for $a'$, and let us prove the theorem for $a$. Let

$$Ka'K = a'K \cup \left(\bigcup_j b_j'K\right), \quad Ka_iK = a_iK \cup \left(\bigcup_j b_j^iK\right)$$

Since $KaK \subset Ka'K \cdot Ka_iK$,

$$\begin{aligned} KaK &\subset \left(a'K \cup \left(\bigcup_j b_j'K\right)\cdot\right) \cdot Ka_iK \\ &= a'Ka_iK \cup \left(\cup b_j'Ka_iK\right) \\ &= a'a_iK \cup \left(\bigcup_j b_j'a_iK\right) \cup \left(\bigcup_j a'b_j^iK\right) \cup \left(\bigcup_{j_1, j_2} b_{j_1}'b_{j_2}^iK\right). \end{aligned}$$

Since $b_l \notin aK$, $b_l$ lies in one of the classes $b'_j a_l K$ or $a' b^i_j k$ or $b'_j b^i_j K$. It is easy to see, that if $b_l \in b \cdot K$, then $|\chi(b_l)| = |\chi(b)|$. Since $|\chi|$ is multiplicative on $B$, it is enough to prove, that

(1) $|\chi(a')| > |\chi(b'_i)|$

(2) $|\chi(a_i)| > |\chi(b^i_j)|$.

since $\chi$ and a satisfy second condition of the theorem, $\chi$ and $a'$ also satisfy this condition. So, (1) is checked. Also, from second condition we have: $\chi = \chi_i \cdot \chi'$, where $\chi'$ is a dominant weight. From part (1) of the theorem, $|\chi'(a_i)| \geqslant |\chi'(b^i_j)|$. From part (1) of theorem 2 we have that

$$|\chi_i(a_i)| > |\chi_i(b^i_j)|$$

Now, we can prove the main theorem.

We assumed, that $f$ is Hecke-finite at $p$. Let $f_1 = f, \ldots, f_N$ be the basis of the finite-dimensional space that $f$ spans under the action of the Hecke ring $H^{K_p}(G_p)$. They all are $K_p$-invariant from the right.

Recall our sets $M^c$, which were defined at beginning of the part. Let us define

$$L'(c) = \max_{i,\, g \in M^c} |f_i(g)|.$$

$L'(c)$ exists, since $B_q \backslash M^c$ is compact, modulo the centre. We shall show, that there are constants $c_1$, $c_2$, $c_3$ s.t. for $c \geqslant c_3$.,

$L'(c) \leqslant \max(c_1 L'(c/c_2),\ L'(c/2))$. Here $c_1$, $c_2$ depend on $p$, and this will prove the theorem. Furthermore, we can take $c_2 = p/A'$, where $A'$ is a constant. Let us estimate $L'(c)$, with $c \gg 1$.

Let us take $g \in M^c$, $g \notin M^{c/2}$, $g = b \cdot k$. Since $f_i$ are $K_p$ invariant, we can assume $k_p = 1$.

Since $\prod |\chi_i(b)| \gg 1$, there is some simple root $\alpha_i$ s.t. $|\alpha_i(b)| \gg 1$.

We use the Hecke operator $T_i = K a_i K$. If $K a_i K = a_i K \cup (\cup b_l K)$, we have

$$f_j(g a_i) + \sum_l f_j(g b_l) = T_i f_j(g),$$

so

$$f_j(g) = T_i f_j(g a_i^{-1}) - \sum_l f_j(g a_i^{-1} b_l).$$

In order to prove our estimate, it is sufficient to check, that every argument in the right side can be written as $q \cdot g'$ with $q \in G_Q$ and $g' \in M^{c/c_2}$:

It is easy to see, that $g a_i^{-1} \in M^{c/p}$.

Let us write $ga_\iota^{-1}b_l = q \cdot g'$, $g' \in \sigma(c_0) \cdot K$.
From Lemma 1, $|\chi_K(g')| \leqslant \max |(w\chi_K)(ga_\iota^{-1}b_l)|$. (although $g'$, $ga_i^{-1}b_l$ do not lie in $B$, we already defined $|\chi|$ on the group $G$).

LEMMA 2: *(i)* $|w\chi_K(ga_i^{-1}b_l)| \leqslant A|\chi_K(g)|$, *A doesn't depend on p.*
*(ii)* $|w\chi_i(ga_\iota^{-1}b_l)| \leqslant |\chi_i(g)|/p$.

PROOF: (i) Since $g_p \in B$,

$$|w\chi_K(ga_\iota^{-1}b_l)| = |w\chi_K(g)| \cdot \left|\frac{w\chi_K(b_l)}{w\chi_K(a_\iota)}\right|.$$

We can write

$$\chi_K/w\chi_K = \prod \alpha_m^{n_m}, \quad n_m \geqslant 0.$$

*Case 1*: $n_\iota > 0$. In this case, $|w\chi_K(g)| \ll |\chi_K(g)|$, and $|w\chi_K(a_\iota^{-1}b_l)|$ is bounded, so the assertion is proved.

*Case 2*: $n_\iota = 0$. Then, $|w\chi_K(g)/\chi_K(g)| = \Pi|\alpha_m(g)|^{-n_m}$. Since $n_m \geqslant 0$ and $|\alpha_K(g)|$ are bounded from below, since $g \in \sigma(c_0) \cdot K$, this expression is bounded from above.

Since $n_\iota = 0$, $|w\chi_K(a_\iota)| = |\chi_K(a_\iota)|$. By Theorem 3,

$$\left|\frac{w\chi_K(b_l)}{w\chi_K(a_\iota)}\right| = \left|\frac{w\chi_K(b_l)}{\chi_K(a_\iota)}\right| \leqslant 1.$$

Q.E.D.

(ii) Again, we write $\chi_\iota/w\chi_\iota = \Pi\alpha_m^{n_m}$, $n_m \geqslant 0$.
We have two cases:

*Case 1*: $n_\iota > 0$. Then, as we saw before,

$$|w\chi_\iota(ga_\iota^{-1}b_l)| \ll |\chi_\iota(g)|.$$

*Case 2*: $n_\iota = 0$. Then $w\chi_\iota = \chi_\iota$. So, we need only to check that

$$|\chi_\iota(a_\iota)| \geqslant p|\chi_\iota(b_l)|.$$

Q.E.D.

This is proved in Theorem 3, (2)

So, we see that $|\chi_K(g')| \leqslant \max_w |w\chi_K(ga_i^{-1}b_l)| \leqslant A|\chi_K(g)|$, and $|\chi_i(g')| \leqslant P^{-1}|\chi_\iota(g)|$.
So, $\Pi|\chi_i(g')| \leqslant A' \cdot p^{-1} \cdot \Pi|\chi_\iota(g)|$.
Hence, we can take $c_2 = p/A'$. If $p \gg 1$, $c_2 > 1$.

## Part 2. Notation and basic facts

If $k$ is a number field, we always normalise it's absolute values by the condition, that the product formula should be satisfied, and at any non-archimedian place the value group must be generatd by the number of elements in the residue field.

Recall the following facts ( |2| ).

Let $S$ be a maximal split torus. Let $\phi$ be the root system (possibly non-reduced) relative to $S$, and $\alpha_1, \dots, \alpha_l$ a base. We shall call $\alpha_i$ the simple roots. Let $B$ be the minimal parabolic subgroup relative to the base $(\theta_1, \dots, \alpha_l)$. Let $Z$ be the centralizer of $S$ in $G$, and let $K$ be a maximal compact subgroup. We have the Iwasawa decomposition $G_A = B_A K_A$.

Let $\tilde{k}$ be a finite extension of $k$, in which $G$ is split, and let $S_1$ be a maximal torus of $G(\tilde{k})$ which contains $S(\tilde{k})$.

If $\alpha$ is a root of $G(\tilde{k})$ relative to $S_1$, then the restriction of $\alpha$ to $S$ is a character from $S$ to $k$. There is a base $\tilde{\alpha}_1, \dots, \tilde{\alpha}_K$ in $G(\tilde{k})$ relative to $S_1$, s.t. restriction of each $\tilde{\alpha}_i$ to $S$ is either a positive root or 1.

Let $\tilde{B}$ be the Borel subgroup of $G(\tilde{k})$ relative to the base $\tilde{\alpha}_1, \dots \tilde{\alpha}_K$. Let $\tilde{K}$ be a maximal compact subgroup of $G(\tilde{k})$ which contains $K$. We have $G(\tilde{k}_A = \tilde{B}_A \cdot \tilde{K}_A$.

For each simple root $\tilde{\alpha}_i$, there is a copy of $SL_2(\tilde{k})$ imbeded in $G(\tilde{k})$, whose standard unipotent subgroup coincides with the root space, corresponding to $\tilde{\alpha}_i$. Let $H_i \subset G(\tilde{k})$ be the standard maximal compact subgroup of this copy.

We can find a finite set $\tilde{\Sigma}$ of places $p$, s.t. $|H_i|_p \subset \tilde{K}_p$ for $p \notin \tilde{\Sigma}$.

Let $\Sigma$ be the set of restrictions of $\tilde{\Sigma}$ to $k$.We define Siegel domains

$$\sigma(c) = \{ s \cdot z \cdot u \mid |\alpha_i(s)| \geqslant c,\ z \in Z_0,\ u \in U \}$$

where $Z_0$ is a compact subset in $Z_A$ s.t. $Z_A = Z_Q \cdot Z_0 \cdot S_A$, and $U$ is the unipotent radical of $B$.

THEOREM 1*: *There is a constant* $c_0 > 0$, *s.t.*

$$G_A = G_k \cdot \sigma(c) \cdot K$$

*for* $0 < c \leqslant c_0$. *For the proof, see* [3].
*Any such* $\sigma(c)$ *will be called the fundamental domain. Let* $\overline{U}$ *be the Lie algebra of* $U$, *and* $V_i \subset \overline{U}$ *be the subspace generated by root spaces with respect to* $S$, *whose roots are* $\geqslant \alpha_i$, *and let* $\chi_i$ *denote the determinant of the adjoint action of* $B$ *in* $V_i$.

*If $\chi$ is any algebraic character of $B$, it gives a character of $\tilde{B}$, and a function $|\tilde{\chi}|$ as $G(\tilde{k})_A$. Restriction of this function to $G_A$ satisfies*

$$|\tilde{\chi}|_{|G_A} = |\chi|^n, \qquad \text{where} \quad n = [\tilde{k}:k].$$

*Finally, if $p$ is non-archimedian place of $k$, we denote by $|p|$ the number of elements in the residue field of $p$.*

*We now state the main theorem.*

*Again, we introduce the sets*

$$M^c = \{b \cdot k \mid b \in \sigma(c_0),\ k \in K : \Pi |\chi_i(b)| \leqslant c\}$$

*$S_k \cdot U_k$ acts on $M^c$ from the left, and the quotient is compact, modulo the centre.*

Main theorem (for general case):

*Let $f$ be a function on $G_k \backslash G_A$ s.t.*

(1) *$f$ is a smooth function on $G_A$ and $K_p$-invariant for some place $p$.*

(2) *$f$ transforms by a character under the action of the centre.*

(3) *$f$ is Hecke-finite at $p$.*

*If $|p| \geqslant C_1$, where $C_1$ depends only on $G$ and $k$, then*

$$\max_{g \in M^c} |f(g)| \leqslant C_2 c^{C_3}$$

*Of course, $\max_{g \in M^c} |f(g)|$ exists, since $S_k U_k \backslash M^c$ is compact modulo the centre.*

*We shall now generalize the proof of case $GL_n(Q)$.*

First we prove the generalizations of Lemma 1 and Theorem 3.

Lemma 1*: *If $b$, $b' \in B_A$, $b' = q \cdot b \cdot k$, $q \in G_k$, $k \in K_A$, $\chi$ is a dominant weight over $\tilde{k}$, $W$ the Weyl group over $\tilde{k}$, then*

$$|\chi(b')| \leqslant C \max_{w \in W} |w\chi(b)|$$

*Here $b'$, $b$ are considered as elements of $G(\tilde{k})$, an $|\chi|$, $|w\chi|$ are defined on $G(\tilde{k})$ as was explained before.*

Proof: In fact, the assertion is about $G(\tilde{k})$, so we can assume, that $G$ is split and $k = \tilde{k}$. Let $V(\chi)$ be the irreducible representation with highest weight $\chi$. If $g \in G$, we shall denote by $\bar{g}$ its image in $GL(V(\chi))$. Also, for each weight $\chi_1$ of representation $V(\chi)$, there is defined the standard weight $\bar{\chi}_1$ of $GL(V(\chi))$

$$\chi_1(g) = \bar{\chi}_1(\bar{g})$$

Let $K_1$ be a maximal compact subgroup in $GL(V(\chi))_A$ which contains the image of $K$. Let $K_0 \subset GL(V(\chi))_A$ be a standard maximal compact subgroup relative to a basis in which $S$ acts diagonality. Let $B_0$ be the standard Borel subgroup containing $B$.

Let $g = q \cdot b$. Then, there is $k' \in K_0$, s.t. $\bar{g}k' \in B_0$. Also, $\bar{g}k \in B_0$. We know from Lemma 1, that

$$|\bar{\chi}(\bar{g} \cdot k')| \leqslant \max_{\chi' \in M} |\bar{\chi}'(\bar{b})|$$

where $M$ is the set of all weights of $V(\chi)$. It is easy to see, that

$$\max_{\chi' \in M} |\bar{\chi}'(\bar{b})| = \max_{w} |w\bar{\chi}(\bar{b})| = \max_{w} |w(b)|$$

Now, $|\bar{\chi}(\bar{g}^k)/\bar{\chi}(\bar{g}k')| = |\bar{\chi}(k'^{-1} \cdot \bar{k})| \leqslant C$, where $C = \max_{K_0 \cdot K_1} |\bar{\chi}|$.So,

$$|\chi(b')| = |\bar{\chi}(\bar{g}k)| \leqslant C |\bar{\chi}(\bar{g} \cdot \bar{k}')| \leqslant C \max |w\chi(b)|.$$

Q.E.D.

THEOREM 3*: *Let $p \notin \Sigma$.*
*Let $s \in S^+ = \{s \in S_p \mid |\alpha_i(s)| \geqslant 1\}$.*
*Let $K_p s K_p = sK_p \cup (\cup b_i K_p)$, $b_i \in B$ be a disjoint union.*
*Let $w$ be an element of the Weyl group over $\tilde{k}$, and $\chi$ a dominant weight over $\tilde{k}$.*

(1) $|\chi(s)| \geqslant |w\chi(b_i)|$.

(2) *Let now, $\chi_0$ be the determinant of the action of $B$ on the span of those root spaces $\bar{V}_\alpha$ or which $|\alpha(s)| > 1$. Suppose that $\chi/\chi_0$ is a dominant weight. Then*

$$|\chi(s)| \geqslant |p| \, |\chi(b_i)|.$$

PROOF: Again, the theorem is about $G(\tilde{k})$, so we can assume that $G$ is split.

(1) Use the representation with highest weight $\chi$.

(2) We need a lemma.

LEMMA: *Let $b \in B$, and suppose that $|\alpha_i(b)| < 1$.*
*Then, there is $b' \in KbK \cap B$ s.t.*

(1) $|\alpha_i(b')| > 1$, $|\chi_i(b')| \geqslant |\chi_i(b)| \cdot |p|$.

(2) $|\chi_j(b')| = |\chi_j(b)|$, $j \neq i$.

PROOF: We shall find such $b'$ in $H_i b H_i \cap B$. First we seek for $b'$ which satisfies (1). Then this problem reduces to the case $G = SL_2$, an this is easy. Now, $H_i$ is contained in the parabolic subgroup on which $\chi_j$ is defined as a character, if $j \neq i$. Since $|\chi_j|_{|H_i} = 1$ (since $H_i$ is compact)

$$|\chi_j(b')| = |\chi_j(b)|, \quad j \neq i.$$

Q.E.D.

Now, take $b_i$ and suppose that $|\chi(b_i)| \geqslant |\chi(s)|$. Using the Lemma, we get a sequence $b^1 = b_i$, $b^2, \ldots,$ which can stop at some $b^n$ if $|\alpha_j(b_n)| \geqslant 1 \forall j$.

Since one of the characters $|\chi_j|$ always grows at $|p|$ from $b^n$ to $b^{n+1}$, and on the other side $|\chi_j(b^k)| \leqslant |\chi_j(s)|$ by part 1 of the theorem, this sequence must stop. So, there is some $b^n$ s.t. $|\alpha_j(b^n)| \geqslant 1 \forall j$.

If $|\chi(b_i)| \geqslant |\chi(s)|$ then $|\chi(b_i)| = |\chi(s)|$ by part 1. If $|\alpha;(s)| > 1$, then $|\chi/\chi_j|$ is dominant, by assumption that $|\chi/\chi_0|$ is dominant. Sine $|\chi_j(b^n)| \leqslant |\chi_j(s)|$ for all $j$, $|\chi_j(b^n)| = |\chi_j(s)|$ if $|\alpha_j(s)| > 1$. Weight $\chi_j^2/\alpha_j$ is always dominant or 1. So,

$$|\chi_j^2/\alpha_j(b^n)| \leqslant |\chi_j^2/\alpha_j(s)|.$$

So, if $|\alpha_j(s)| > 1$, we have $|\chi_j(b^n)| = |\chi_j(s)|$, so $|\alpha_j(b^n)| \geqslant |\alpha_j(s)|$. If $|\alpha_j(s)| = 1$, then obviously $|\alpha_j(b^n)| \geqslant |\alpha_j(s)|$.

Let $\chi'$ be a dominant weight s.t. every simple root enters $\chi'$ with positive coefficients. Since

$$|\chi'(b^n)| \leqslant |\chi'(s)|, \quad |\alpha_j(b^n)| \geqslant |\alpha_j(s)|$$

we have

$$|\alpha_j(b^n)| = |\alpha_j(s)|.$$

We use now any faithful representation of $G$ in some $GL_N$ an Theorem 3 to show that $b^n \in Ks$.

Suppose that $b_i \neq b^n$. Then we used some simple root $\alpha_j$ s.t. $|\alpha_j(b^{n-1})| < 1$ and the lemma to get $b^n$. There are two possibilities.

1) $|\alpha_j(s)| = 1$. Then, by the first part of the lemma, $|\alpha_j(b^n)| > 1 \Rightarrow |\alpha_j(b^n)| > |\alpha_j(s)|$ contradiction.

2) $|\alpha_j(s)| > 1$. Then,

$$|\chi_j(b^n)| \geqslant |p|\,|\chi_j(b^{n-1})| \geqslant |p|\,|\chi_j(b_i)| = |p|\,|\chi_j(s)|$$

again contradiction. Q.E.D.

Now, we can prove the main theorem. The results which we used in case of $GL_n(Q)$, that is Lemma 1 and Theorem 3 were already generalized.

We assume that $p \notin \Sigma$, and we choose a basis $f_1 = f, f_2, \ldots, f_n$ of the space which $f$ spans under the action of the Hecke ring. We estimate

$$L(c) = \max_{g \in M^c,\, i} |f_i(g)|$$

We prove that $L(c) \leqslant \max(CL(c/c_2), L(c/2))$, where $c_2$ can be chosen greater than 1 for $p \gg 1$.

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Department of Mathematics
Tel Aviv University
Ramat-Aviv
Israel