MARC LEVINE

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MODULES OF FINITE LENGTH AND K-GROUPS OF SURFACE SINGULARITIES

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Introduction

Let X be a variety over an algebraically closed field k. The study of vector bundles on X can be divided into local and global aspects, the local data coming from the singularities of X. For example, if X is a curve, and $f: Z \to X$ is the normalization of X, then the long exact cohomology sequence

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(Z, \mathcal{O}_Z^*) \to \mathcal{O}_Z^*/\mathcal{O}_X^* \to \operatorname{Pic}(X) \xrightarrow{f^*} \operatorname{Pic}(Z) \to 0$$

associated to the short exact sheaf sequence

$$0 \to \mathcal{O}_X^* \to f_*(\mathcal{O}_Z^*) \to f_*(\mathcal{O}_Z^*)/\mathcal{O}_X^* \to 0$$

shows how the kernel of the surjection $f^*: \operatorname{Pic}(X) \to \operatorname{Pic}(Z)$ can be computed from the local invariant $\mathcal{O}_Z^*/\mathcal{O}_X^*$ and the global invariant $H^0(Z, \mathcal{O}_Z^*)$. If X is complete, then the above sequence reduces to

$$0 \to \mathcal{O}_Z^*/\mathcal{O}_X^* \to \operatorname{Jac}(X) \to \operatorname{Jac}(Z) \to 0,$$

which describes the difference between the Jacobian of Z and the generalized Jacobian of X in terms of the local data $\mathcal{O}_Z^*/\mathcal{O}_X^*$.

Using higher K-theory, one can generalize the above to the case of normal surfaces. If X is a normal surface with singular locus S, and $f: Z \to X$ a resolution of singularities, we let X^* denote $\operatorname{Spec}(\mathcal{O}_{X,S})$, and Z^* the inverse image $f^{-1}(X^*)$. The Leray spectral sequence for the sheaf $\mathcal{K}_2$ on Z has the five term exact sequence

$$0 \to H^1(X, f_*(\mathcal{K}_2)) \to H^1(Z, \mathcal{K}_2) \to H^1(Z^*, \mathcal{K}_2)$$

$$\to H^2(X, \mathcal{K}_2) \xrightarrow{f^*} H^2(Z, \mathcal{K}_2).$$

Letting $F_0K_0(X)$ denote the subgroup of $K_0(X)$ generated by the residue fields of smooth closed points of X, and similarly defining $F_0K_0(Z)$, there are natural maps

$$F_0K_0(Z) \to H^2(Z, \mathcal{K}_2)$$

$$F_0K_0(X) \to H^2(X, \mathcal{K}_2),$$

shown to be isomorphisms by Bloch [B] in the smooth case, and by Collino [C] in the singular case. Since X is normal, the kernel of $f^* : K_0(X) \to K_0(Z)$ is a subgroup of $F_0K_0(X)$, hence the above gives a description of $\ker(f^*)$ as

$$\ker\big(f^* : K_0(X) \to K_0(Z)\big) = H^1(Z^*, \mathcal{K}_2)/\mathrm{Im}\big(H^1(Z, \mathcal{K}_2)\big).$$

In contrast with the case of curves, however, the local invariant $H^1(Z^*, \mathcal{K}_2)$ is very difficult to compute. The local invariant $\mathcal{O}_Z^*/\mathcal{O}_X^*$ is easy to compute because it depends only on the *analytic* type of the singularity; the group $H^1(Z^*, \mathcal{K}_2)$ is not a priori an analytic invariant, and depends on the more subtle algebraic nature of the semi-local ring $\mathcal{O}_{X,S}$.

M.P. Murthy and N. Mohan Kumar ([MK] and [MM]) have employed a different approach to the problem of computing $K_0(X)$. They consider the algebraic local ring of a normal singular point on a rational surface, and attempt to classify all such rings in a given analytic isomorphism class. From this analysis, they are able to show that $F_0K_0(X) = 0$ if X is an affine rational surface with a rational double point of type A_n $(n \neq 7, 8)$ or D_n $(n \neq 8)$. Using deformation theory and K-theory, Bloch (unpublished) has shown that $F_0K_0(X) = \mathbf{Z}$ by length if X is a projective rational surface with only rational double points. In a joint work with Srinivas [LS], we have analyzed the effect of a special type of quotient singularity on singular elliptic surfaces, and have shown that the map $f^* : K_0(X) \to K_0(Z)$ is injective for these surfaces.

Another attack on the problem of computing $K_0(X)$ for singular X comes from considering the category of sheaves of modules of finite projective dimenson, supported in the singular locus S. We denote this category by $\mathcal{C}_{X,S}$. It turns out that the image of a portion of the Grothendieck group $K_0(\mathcal{C}_{X,S})$ in $K_0(X)$ is exactly the kernel of $f^* : K_0(X) \to K_0(Z)$, where Z is a resolution of singularities of X. In addition, $K_0(\mathcal{C}_{X,S})$ depends only on the analytic neighborhood of S in X, hence this gives a description of the effect of the singularity S on $K_0(X)$ in terms of the rough analytic nature of the singularity, rather than the more subtle algebraic structure of the Zariski neighborhood of S in X. For example, if X is the cone $x^2 + y^2 = z^2$, then the knowledge of $K_0(\mathcal{C}_{X,S})$ would give information of the effect of ordinary double points on $K_0(Y)$ for any surface Y with only this type of singularity.

Except for curves, the group $K_0(\mathscr{C}_{X,S})$ has been computed for only a very few examples. Coombes and Srinivas [CS] have used results from algebraic K-theory to show that length: $K_0(\mathscr{C}_{X,S}) \to \mathbf{Z}$ is an isomorphism if X is a Zariski surface (inseparable degree p cover of the plane) with a normal singular point. Srinivas [Sr 2] has computed $K_0(C_{X,S})$ if $\mathcal{O}_{X,S}$ is a UFD; in particular, he shows that $K_0(\mathscr{C}_{X,S})$ is isomorphic to $\mathbf{Z}$ by length if (X, S) is an E_8 singularity.

In this paper, we consider the problem of computing $K_0(\mathscr{C}_{X,S})$ when (X, S) is a normal surface singularity. We refine the technique employed by Coombes and Srinivas to relate $K_0(\mathscr{C}_{X,S})$ to certain K-theoretic invariants of a resolution of singularities of X. More precisely, let X be a semi-local normal surface with singular locus S, and let $f: Z \to X$ be a resolution of singularities of X with exceptional divisor $E = \bigcup_i E_i$. Let $SK_0'(E)$ be the kernel of the map rank: $K_0'(E) \to \mathbf{Z}$, and let N be the subgroup of $H^1(Z, K_2)$ coming from $SK_1'(E)$. Then we have an exact sequence (Theorem 2.1)

$$0 \to H^1(Z, \mathscr{K}_2)/N \to K_0(\mathscr{C}_{X,S}) \xrightarrow{f^*} SK_0'(E) \to 0.$$

As an application, we show that the map length: $K_0(\mathscr{C}_{X,S}) \to \mathbf{Z}$ is an isomorphism when (X, S) is a quotient singularity.

We also consider the relationship between $K_0(\mathscr{C}_{X,S})$ and $K_0(X)$. Let $SK_0(\mathscr{C}_{X,S})$ be the kernel of the map $f^*: K_0(\mathscr{C}_{X,S}) \to SK_0'(E)$ above. We show (Proposition 4.1) that the image of $SK_0(\mathscr{C}_{X,S})$ in $K_0(X)$ is the kernel of $f^*: K_0(X) \to K_0(Z)$, if X is a normal quasi-projective surface with resolution $f: Z \to X$. We also generalize the exact sequence described above for curves to yield an exact sequence

$$H^1(Z, \mathscr{K}_2) \to SK_0(\mathscr{C}_{X,S}) \to K_0(X) \xrightarrow{f^*} K_0(Z)$$

describing the kernel of f^* in terms of the local analytic invariant $SK_0(\mathscr{C}_{X,S})$, and the global invariant $H^1(Z, \mathscr{K}_2)$. This shows, for example, that f^* is injective if X has only quotient singularities.

The main technical tool is a new localization sequence in algebraic K-theory which generalizes the localization sequence for projective modules [G]. The construction of this sequence occupies the first part of the paper. We apply this to the computation of $K_0(\mathscr{C}_{X,S})$ in section two, and compute $K_0(\mathscr{C}_{X,S})$ for a number of examples in section three. In the fourth section, we relate $K_0(X)$ and $K_0(\mathscr{C}_{X,S})$, and we use this machinery to compute $K_0(X)$ for several types of singular rational surfaces.

We fix at the outset an algebraically closed field k. Except for section one, we assume that all schemes and morphisms are over k.

Upon completion of this work, the author received a manuscript from V. Srinivas, in which he also computes $K_0(\mathscr{C}_{X,S})$, for quotient singularities (X, S). The method is essentially that of showing that every such singularity has an algebraic model that is a UFD, and then applying the methods of [Sr 2]. Finally, I would like to thank Srinivas for discussing his work in K-theory with me, and suggesting how one should link up the K-theory of the resolution of (X, S) with the K-theory of $\mathscr{C}_{X,S}$.

Section 1

Let X be a noetherian scheme, such that every coherent $\mathcal{O}_X$ module admits a surjection from a locally free $\mathcal{O}_X$ module. Let Y be a closed subscheme of X of pure codimension d. We assume that, for each y in Y,

$$\text{proj dim}_{\mathcal{O}_{X,y}}(\mathcal{O}_{Y,y}) = d.$$

We refer to the above property by saying that Y *has pure projective dimension d over X*. We fix an affine open subset U of Y, and let C denote the (set-theoretic) complement. We suppose that C is *locally set-theoretically principal* on Y, i.e., there is a closed subscheme Z of Y with $\text{supp}(Z) = C$, such that the sheaf of ideals $\mathscr{I}_z \subset \mathcal{O}_Y$ is locally principal, and locally generated by a non-zero divisor of $\mathcal{O}_Y$.Let $i: Y \to X$, $j: U \to Y$ be the inclusions.

We now define some subcategories of $\mathscr{M}^*_X$ (quasi-coherent sheaves on X). We describe only the objects; the categories will be full subcategories of $\mathscr{M}^*_X$, and will be given the admissible monomorphisms and epimorphisms to make them into exact categories in the usual way. That these actually form exact categories is an easy exercise in homological algebra:

	$\mathscr{P}_Y$:	the category of coherent, locally free $\mathcal{O}_Y$ module
	$\mathscr{P}_U$:	the category of coherent, locally free $\mathcal{O}_U$ modules
$(r \geqslant d)$	$\mathscr{P}^r(Y)$:	the category of $\mathcal{O}_Y$ modules of projective dimension at most r over $\mathcal{O}_X$.
	$\mathscr{P}^r(Y, U)$:	the subcategory of $\mathscr{P}^r(Y)$ of modules M such that $j^*(M)$ is in $\mathscr{P}_U$, and M has no associated primes supported in C
	$\mathscr{P}^r_C(Y)$:	the subcategory of $\mathscr{P}^r(Y)$ consisting of modules M with support contained in C.

The object of this section is to prove the following theorem:

THEOREM 1.1: *There is a natural long exact localization sequence* $(i \geqslant 0)$:

$$\to K_{i+1}(\mathscr{P}_U) \xrightarrow{\partial Y} K_i(\mathscr{P}^{d+1}_C(Y)) \to K_i(\mathscr{P}^d(Y, U)) \to K_i(\mathscr{P}_U). \tag{*}$$

In addition, the sequence (*) *is compatible with the localization sequence for $j: U \to Y$, i.e., the inclusions $\mathscr{P}_Y \hookrightarrow \mathscr{P}^d(Y, U)$, $\mathscr{H}_C \hookrightarrow \mathscr{P}^{d+1}_C(Y)$ (here*

$\mathscr{H}_C$ is the category of hereditary $\mathcal{O}_Y$ modules supported on C) induce a commutative diagram:

$$\begin{array}{ccccccc}
\to K_{i+1}(\mathscr{P}_U) \to & K_i(\mathscr{P}_C^{d+1}(Y)) & \to & K_i(\mathscr{P}^d(Y, U)) & \to & K_i(\mathscr{P}_U) \\
\| & \uparrow & & \uparrow & & \| \\
\to K_{i+1}(\mathscr{P}_U) \to & K_i(\mathscr{H}_C) & \to & K_i(\mathscr{P}_Y) & \to & K_i(\mathscr{P}_U).
\end{array}$$

The formal details of the proof of Theorem 1.1 are essentially the same as in Grayson's article [G], and we will indicate here only the necessary modifications. We will use freely the notations and constructions developed in that paper.

Let V be the exact subcategory of $\mathscr{P}_U$ consisting of coherent sheaves of the form $j^*(M)$ for M in $\mathscr{P}^d(Y, U)$. Let $\mathscr{T} = \mathrm{Iso}(\mathscr{V})$, $\mathscr{S} = \mathrm{Iso}(\mathscr{P}^d(Y, U))$. To spare the notation, we will write $\mathscr{P}^d$ for $\mathscr{P}^d(Y, U)$, $\mathscr{P}_C^{d+1}$ for $\mathscr{P}_C^{d+1}(Y)$.

Let $\mathscr{E}$ be the extension construction over $Q\mathscr{V}$:

$$\mathrm{Obj}(\mathscr{E}) = \{L \twoheadrightarrow M, \text{ with } L, M \text{ in } V\}.$$

Over an injective arrow $M' \rightarrowtail M$ in $Q\mathscr{V}$, we allow the pull-back diagram

$$\begin{array}{ccc}
L' = L \times_{M'} M & \rightarrowtail & L \\
\downarrow & & \downarrow \\
M' & \rightarrowtail & M
\end{array}$$

Over a surjective arrow $M' \twoheadleftarrow M$ in $Q\mathscr{V}$, we allow the diagram

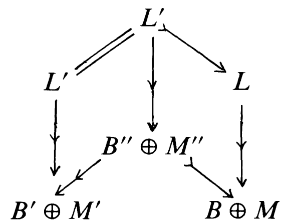
$$\begin{array}{ccc}
L & = & L \\
\downarrow & & \downarrow \\
M' & \twoheadleftarrow & M.
\end{array}$$

We also allow isomorphisms

$$\begin{array}{ccc}
L & \xrightarrow{\sim} & L \\
\downarrow & & \downarrow \\
M & = & M
\end{array}$$

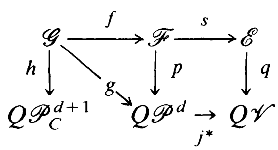
We define the category $\mathscr{F}$ to be the pull-back of $\mathscr{E}$ over $j^*: Q\mathscr{P}^d \to Q\mathscr{V}$, i.e., an object of $\mathscr{F}$ is a pair $(B, Z \twoheadrightarrow j^*B)$, with B in $\mathscr{P}^d$, Z in $\mathscr{V}$, and arrows are component-wise. Let $\mathscr{G}$ be the category whose objects are surjections $L \twoheadrightarrow B \oplus M$, with L and B in $\mathscr{P}^d$, and M in $\mathscr{P}_C^{d+1}$. Arrows are defined by giving an arrow from say B' to B in $Q\mathscr{P}^d$, an arrow from

M' to M in $Q\mathscr{P}_C^{d+1}$, and completing to a diagram of the form



with the right-hand square a pull-back diagram. We identify two such diagrams if they differ by an isomorphism of the middle column.

Putting all these categories together, we obtain a diagram



where

$$q(L \twoheadrightarrow M) = M$$

$$f(L \twoheadrightarrow B \oplus M) = (B,\ j^*L \twoheadrightarrow j^*B)$$

$$g(L \twoheadrightarrow B \oplus M) = B$$

$$h(L \twoheadrightarrow B \oplus M) = M$$

$$s = p_2,\ p = p_1.$$

The maps h, g, p, and q are fibered. We also have the following

LEMMA 1.2: *Let M be in $\mathscr{P}^d$. Then $M \to j_* j^*(M)$ is injective, and $j_* j^*(M) = \bigcup_n \mathscr{I}_Z^{-n} M$, where Z is the locally principal subscheme of Y with* $\operatorname{supp}(Z) = C$.

PROOF: Since M has no associated primes along C, sections of $\mathscr{I}_Z$ act as non-zero divisors on M. The lemma is an easy consequence of this. □

The monoidal category $\mathscr{S}$ acts on $\mathscr{E}$ by

$$A + (L \twoheadrightarrow M) = L \oplus j^*(A) \twoheadrightarrow M,$$

and similarly on $\mathscr{F}$ and $\mathscr{G}$. We note that the map $j^* : \mathscr{S} \to \mathscr{T}$ is co-final, so $\mathscr{S}^{-1}\mathscr{E}$ is homotopy equivalent to $\mathscr{T}^{-1}\mathscr{E}$. The following facts are proven exactly as in Grayson, and we omit the proof:

(1) $\mathscr{S}^{-1}\mathscr{E}$ is contractible, and

$$\begin{array}{ccc} \mathscr{S}^{-1}\mathscr{T} & \to & \mathscr{S}^{-1}\mathscr{E} \\ \downarrow & & \downarrow \\ \mathrm{pt} & \to & Q\mathscr{V} \end{array}$$

is homotopy cartesian. $\mathscr{S}^{-1}\mathscr{E} \to Q\mathscr{V}$ is a fibration.

(2) $h: \mathscr{G} \to Q\mathscr{P}_C^{d+1}$ is a homotopy equivalence.

(3) $f: \mathscr{G} \to \mathscr{F}$ is a homotopy equivalence. Since $\mathscr{S}$ acts trivially on $Q\mathscr{P}_C^{d+1}$, $\mathscr{S}$ acts invertibly on $\mathscr{G}$ and $\mathscr{F}$, so $\mathscr{S}^{-1}f: \mathscr{S}^{-1}\mathscr{G} \to \mathscr{S}^{-1}\mathscr{F}$ is a homotopy equivalence.

(4)

$$\begin{array}{ccc} \mathscr{S}^{-1}\mathscr{F} & \to & \mathscr{S}^{-1}\mathscr{E} \\ \downarrow & & \downarrow \\ Q\mathscr{P}^d & \to & Q\mathscr{V} \end{array} \qquad \text{is homotopy cartesian.}$$

(5) $BQ\mathscr{P}_C^{d+1} \to BQ\mathscr{P}^d \to BQ\mathscr{V}$ has the homotopy type of a fibration. As $\mathscr{V}$ is cofinal in $\mathscr{P}_U$, and exact sequences in $\mathscr{V}$ and $\mathscr{P}_U$ split, we have $K_i(Q\mathscr{V}) \to K_i(Q\mathscr{P}_U)$ is an isomorphism for $i \geqslant 1$, and injective for $i = 0$. This gives the desired long exact sequence (*).

The compatibility of the above construction with the original construction in [G] shows that the localization sequence (*) is compatible with the localization sequence

$$K_{i+1}(\mathscr{P}_U) \to K_i(\mathscr{H}_C) \to K_i(\mathscr{P}_Y) \to K_i(\mathscr{P}_U)$$

as claimed. This completes the proof of Theorem 1.1. □

Note: Let Y' be a closed subscheme of X, containing Y, and let C' be a closed subset of Y'. We assume that Y' is of pure codimension d, and pure projective dimension d over X, and that C' is locally set-theoretically principal on Y'. Let U' be the complement $Y' - C'$. We also assume that U' is affine, that C' contains C, and that $U' \cap U$ is both open and closed in U'. Let $i: Y \to Y'$, $j: U' \cap U \to U$, and $h: U' \cap U \to U'$ denote the inclusions.

The maps i, j, and h induce exact functors

$$i_*: \mathscr{P}^d(Y, U) \to \mathscr{P}^d(Y', U')$$

$$i_*: \mathscr{P}_C^{d+1}(Y) \to \mathscr{P}_{C'}^{d+1}(Y')$$

$$h_* \circ j_*: \mathscr{P}_U \to \mathscr{P}_{U'}.$$

We claim that these induce a commutative diagram

$$\begin{array}{cccccccc} \to K_{\iota+1}(\mathscr{P}_U) & \to & K_\iota(\mathscr{P}_C^{d+1}(Y)) & \to & K_\iota(\mathscr{P}^d(Y, U)) & \to & K_\iota(\mathscr{P}_U) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \to K_{\iota+1}(\mathscr{P}_{U'}) & \to & K_\iota(\mathscr{P}_{C'}^{d+1}(Y')) & \to & K_\iota(\mathscr{P}^d(Y', U')) & \to & K_\iota(\mathscr{P}_{U'}). \end{array}$$

Indeed, let $\mathscr{E}'$, $\mathscr{F}'$, $\mathscr{G}'$, $\mathscr{V}'$, and $\mathscr{S}'$ be categories constructed as above, only for the primed subschemes Y', U'. One easily checks that the maps i, j, and h map the diagram on the left to that on the right, in a commutative fashion:

$$\begin{array}{ccccc} \mathscr{S}^{-1}\mathscr{G} & \to & \mathscr{S}^{-1}\mathscr{F} & \to & \mathscr{S}^{-1}\mathscr{E} \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathscr{P}_C^{d+1}(Y) & \to & Q\mathscr{P}^d(Y, U) & \to & Q\mathscr{P}_U \end{array} \quad \begin{array}{ccccc} \mathscr{S}'^{-1}\mathscr{G}' & \to & \mathscr{S}'^{-1}\mathscr{F}' & \to & \mathscr{S}'^{-1}\mathscr{E}' \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathscr{P}_{C'}^{d+1}(Y') & \to & Q\mathscr{P}^d(Y', U') & \to & Q\mathscr{P}_{U'} \end{array}$$

This proves our claim.

Section 2

We now give an application of the localization sequence (*) to the study of the category of modules of finite length, and finite projective dimension on a normal, two dimensional semi-local ring. We first recall some basic fact about the K-theory of smooth k-schemes; we refer the reader the Quillen's article [Q] for details.

Let Z be a smooth scheme, essentially of finite type over k. Let $\mathscr{K}_p$ denote the sheaf (for the Zariski topology) defined by

$$\mathscr{K}_{p,z} = K_p(\mathcal{O}_{Z,z}),$$

where $\mathrm{K}_p(\mathcal{O}_{Z,z})$ is the p^{th} (Quillen) K-group of $\mathcal{O}_{Z,z}$. The sheaf $\mathscr{K}_p$ has an acyclic resolution (Z^i = set of codimension i points of Z)

$$0 \to \mathscr{K}_p \to i_*\big(K_p(k(Z))\big) \to \coprod_{z \in Z^1} i_{z*}\big(K_{p-1}(k(z))\big)$$

$$\to \dots \to \coprod_{z \in Z^p} i_{z*}\big(K_0(k(z))\big) \to 0$$

called the Gersten resolution. From this follows Bloch's formula

$$H^p(Z, \mathscr{K}_p) = CH^p(Z),$$

where $CH^p(Z)$ is the free abelian group of codimension p cycles on Z, modulo divisors of functions on codimension $p-1$ subvarieties of Z.

One can give similar descriptions of the other cohomology groups $H^p(Z, \mathscr{K}_q)$ as well; for instance, if Z is a surface, $q = 2$, then the above sequence shows that $H^p(Z, \mathscr{K}_2)$ is the p^{th} cohomology group of the complex

$$0 \to K_2(k(Z)) \xrightarrow{T} \bigoplus_{\substack{C \text{ a curve} \\ \text{on } Z}} k(C)^* \xrightarrow{\text{div}} \bigoplus_{\substack{p \text{ a point} \\ \text{of } Z}} \mathbf{Z} \to 0$$

where T is the tame symbol map, and div is the divisor map. In particular, each element of $H^1(Z, \mathscr{K}_2)$ is represented by a collection $\{(f_i, C_i)\}$, where the C_i are curves on Z, f_i is in $k(C_i)^*$, and $\sum_i \text{div}(f_i) = 0$ as a zero-cycle on Z.

Let R be the semi-local ring of a finite set S of normal points on a surface X over k. For brevity, we denote the category $\mathscr{C}_{X,S}$ by $\mathscr{C}_R$. Let $X^* = \text{Spec}(R)$, let $f: Z^* \to X^*$ be a resolution of singularities of X^*, and let E be the reduced exceptional divisor of f. We write E as a union of irreducible components, $E = \bigcup_i E_i$. Let N be the subgroup of $H^1(Z^*, \mathscr{K}_2)$ generated by collections $\{(f_i, E_i)\}$, $f_i \in k(E_i)^*$, with $\sum_i \text{div}(f_i) = 0$. Let $SK_0'(E)$ denote the kernel of the map rank : $K_0'(E) \to \bigoplus_i \mathbf{Z}E_i$. We will prove the following theorem.

THEOREM 2.1: *There is an exact sequence*

$$0 \to H^1(Z^*, \mathscr{K}_2)/N \xrightarrow{\beta} K_0(\mathscr{C}_R) \xrightarrow{f^*} SK_0'(E) \to 0.$$

In the next section, we will use this result to compute $K_0(\mathscr{C}_R)$ for several types of singular local rings. Before we proceed to the proof of the Theorem, we first prove the following lemma.

LEMMA 2.2: *Let A be a normal domain containing k, with fraction field F. Then $K_2(F)$ is generated by symbols $\{a, b\}$ with a, b in A, and with* div(a), div(b) *reduced and having no common components.*

PROOF: In [M], Milnor gives an argument of Tate which shows that $K_2(F)$ is generated by symbols $\{a, b\}$, with a, b in A, and with div(a) and div(b) having no components in common, in case A is a Dedekind domain. The same proof works for any domain A which is regular in codimension one, as the reader can easily verify. Let now a be an arbitrary element of A, and T a finite set of height one primes of A prime to div(a). Let $p_1, \ldots, p_s$ be the primes in div(a). By the Chinese remainder theorem, we can find an element c of A which vanishes with multiplicity one at each p_i and prime to T. By Bertini's theorem, we can find such a c such that $\mathrm{div}(c)$ is reduced. Let $q_1, \ldots, q_r$ be the primes of $\mathrm{div}(c)$ not amoung the p_i. Arguing as above, we can find an element d of A with reduced divisor, vanishing at each q_j, and prime to T and the p_i. Since A is normal, c divides $a \cdot d$, $a \cdot d/c = e$, and multiplicity of e at each prime of $\mathrm{div}(e)$ is strictly less than $\max(v_{p_i}(a), 2)$. By induction, we have proved the following fact:

Let T be a finite set of height one primes of A, a an element of A with $\mathrm{div}(a)$ prime to T. Then we can express a as

$$a = \frac{\prod c_i}{\prod d_i}$$

where c_i, d_i are in A, $\mathrm{div}(c_i)$, $\mathrm{div}(d_i)$ are prime to T, and $\mathrm{div}(c_i)$, $\mathrm{div}(d_i)$ are reduced.

The lemma is now immediate from the bilinearity of $\{a, b\}$. □

Let now Y be a reduced, principal subscheme of X^*, and let U be the affine open subset $Y - S$. Since S is set-theoretically principal on Y, we may apply the results of section one to obtain the localization sequences:

$$K_1(\mathscr{P}^1(Y, U)) \to K_1(\mathscr{P}_U) \overset{\partial Y}{\to} K_0(\mathscr{P}_S^2(Y)). \tag{2.1}$$

We note that $K_i(\mathscr{P}_U) = \bigoplus_{x \text{ in } U^0} K_i(k(x))$.

Let $\mathscr{P}^1$ be the direct limit of the $\mathscr{P}^1(Y, U)$ over reduced, principal Y, and let $\mathscr{P}^2$ be the direct limit of the $\mathscr{P}_S^2(Y)$, We claim that $\mathscr{P}^2$ is just $\mathscr{C}_R$. Indeed, $\mathscr{P}^2$ is clearly a subcategory of $\mathscr{C}_R$. On the other hand, if M is a module of finite length, and finite projective dimension over R, then M is supported on S, and $\operatorname{proj\,dim}_R(M) = 2$. In addition, we may find an element t of R, with reduced divisor, such that $tM = 0$. Letting Y be subscheme of X^* defined by t, we see that M is in $\mathscr{P}_S^2(Y)$, are desired. As K-theory commutes with direct limits, we obtain the exact sequence

$$K_1(\mathscr{P}^1) \overset{\alpha}{\to} \bigoplus_{x \text{ in } X^{*1}} k(x)^* \overset{\partial}{\to} K_0(\mathscr{C}_R) \tag{2.2}$$

We recall that the sequence (2.1) is compatible with the localization sequence

$$K_1(\mathscr{P}_Y) \to K_1(\mathscr{P}_U) \overset{\delta_Y}{\to} K_0(\mathscr{H}_S(Y)). \tag{2.3}$$

If f is an element of R with reduced divisor, and g an element of R such that (f, g) is a regular sequence, then $\delta_Y(\bar{g}, R/(f))$ is the class of

$R/(f, g)$ in $K_0(\mathscr{H}_S(Y))$, where we take $Y = \mathrm{Spec}(R/(f))$. Thus $\partial(\bar{g}, R/(f))$ is the class of $R/(f, g)$ in $K_0(\mathscr{C}_R)$. As $K_0(\mathscr{C}_R)$ is generated by such modules (see [CS] for an argument by Mohan Kumar. The result is originally due to Hochster), this implies that ∂ is surjective. Similarly, using Lemma 2.2, we see that ∂ is zero on the subgroup of $\bigoplus_{x \text{ in } X^{*1}} k(x)^*$ generated by tame symbols from $K_2(k(X^*))$.

Let $f: Z^* \to X^*$ be a resolution of singularities of X^*, with exceptional divisor $E = \bigcup_i E_i$. If M is a module in P^1, then we have an exact sequence

$$0 \to R^n \overset{T}{\to} R^n \to M \to 0$$

hence we have

$$0 \to \mathrm{Tor}^1_R(\mathscr{O}_{Z^*}, M) \to \mathscr{O}^n_{Z^*} \overset{f^*(T)}{\to} \mathscr{O}^n_{Z^*} \to f^*(M) \to 0.$$

As M is a torsion module, T is generically an isomorphism, hence $\ker(f^*(T))$ is a torsion module, hence zero. Thus $f^*: \mathscr{P}^1 \to \mathscr{M}^1_{Z^*}$ is an exact functor, where $\mathscr{M}^1_{Z^*}$ is the category of torsion $\mathscr{O}_{Z^*}$ modules. We therefore have the commutative diagram

$$\begin{array}{ccccccc} K_1(\mathscr{C}_R) \to & K_1(\mathscr{P}^1) & \overset{\alpha}{\to} & \bigoplus_{x \text{ in } X^{*1}} k(x)^* & \to & K_0(\mathscr{C}_R) \\ & f^* \downarrow & & \| & & \\ K_1'(E) \to & K_1(\mathscr{M}^1_{Z^*}) & \to & \bigoplus_{x \text{ in } X^{*1}} k(x)^* & \underset{\mathrm{div}}{\to} & K_0'(E) \end{array}$$

where the bottom row is part of the localization sequence

$$\to K_i'(E) \to K_i(\mathscr{M}^1_{Z^*}) \to K_i(\mathscr{M}^1_{Z^*-E}) \to K_{i-1}'(E) \to .$$

The map $\mathrm{div}: \bigoplus_{x \text{ in } X^{*1}} k(x)^* \to K_0'(E)$ therefore induces a homomorphism $f^*: K_0(\mathscr{C}_R) \to K_0'(E)$. As the image of div is $SK_0'(E)$, we get a surjection $f^*: K_0(\mathscr{C}_R) \to SK_0'(E)$. In addition, from the localization sequence

$$\bigoplus_{\substack{\text{components } E_i \\ \text{of } E}} k(E_i)^* \underset{\mathrm{div}}{\to} \bigoplus_{p \text{ in } E^1} \mathbf{Z} \to K_0'(E) \to \bigoplus_{E_i} \mathbf{Z} \to 0$$

we see that, if z is in $\bigoplus_{x \text{ in } X^{*1}} k(x)^*$, then

$$f^* \circ \partial(z) = [\mathrm{div}(z)] = 0 \quad \text{in } K_0'(E)$$

if and only if there are f_i in $k(E_i)^*$ such that $\text{div}(z)+\sum_i \text{div}(f_i)=0$ as a cycle on E. Thus there is an element w

$$\text{w in} \bigoplus_{x \text{ in } Z^{*1}} k(x^*), \quad \text{div}(w)=0,$$

restricting to z on X^*. As ∂ (tame symbols) = 0, the projection

$$\bigoplus_{x \text{ in } Z^{*1}} k(x)^* \to \bigoplus_{x \text{ in } X^{*1}} k(x)^*$$

induces a surjection $H^1(Z^*, \mathcal{K}_2) \to \ker(f^*)$. The subgroup N of $H^1(Z^*, \mathcal{K}_2)$ generated by $\{(f_i, E_i)\}$, with f_i in $k(E_i)^*$, and $\sum_i \text{div}(f_i) = 0$ clearly goes to zero under this map.

To conclude the proof, it suffices to show that $\alpha(K_1(\mathcal{P}^1))$ is contained in the group of tame symbols from $K_2(k(X^*))$. We have the localization sequence

$$K_2(k(X^*)) \xrightarrow{T} K_1(\mathcal{M}^1_{Z^*}) \to K_1(Z^*)$$

and a commutative triangle

$$\begin{array}{ccc} K_1(\mathcal{P}^1) & \xrightarrow{f^*} & K_1(\mathcal{M}^1_{Z^*}) \\ & {\scriptstyle\alpha}\searrow \quad \swarrow & \\ & \bigoplus_{x\in X^{*1}} k(x)^* & \end{array}$$

so it suffices to show that $f^*(K_1(\mathcal{P}^1))$ goes to zero in $K_1(Z^*)$. As $K_1(\mathcal{P}^1) \to K_1(Z^*)$ factors through $K_1(R)$, it suffices to show that $K_1(\mathcal{P}^1) \to K_1(R)$ is zero. Let $\mathcal{H}$ be the category of torsion R modules of finite projection dimension. Then the map $K_1(\mathcal{P}^1) \to K_1(R)$ factors through $K_1(\mathcal{H})$. We have the localization sequence

$$\begin{array}{ccccc} K_1(\mathcal{H}) & \to & K_1(R) & \to & K_1(k(X^*)) \\ & & \| \wr & & \| \wr \\ & & R^* & \hookrightarrow & k(X^*)^* \end{array}$$

hence $K_1(\mathcal{H}) \to K_1(R)$ is zero. This completes the proof of Theorem 2.1. □

For later use, we will denote the kernel of f^* by $SK_0(\mathcal{C}_R)$.

Note: In an earlier version of this work, a weaker form of Theorem 2.1 was proved, which did not show that $H^1(Z^*, \mathscr{K}_2)/N \to K_0(\mathscr{C}_R)$ is injective. The argument above showing injectivity was communicated to the auther by V. Srinivas.

Section 3

We now use Theorem 2.1 to compute $K_0(\mathscr{C}_R)$ for a number of examples. The main trick is the following:

As every M in $\mathscr{C}_R$ is killed by a power of the maximal ideal m or R, we have $\mathscr{C}_R = \mathscr{C}_{\hat{R}}$ ($\hat{R} = m$-adic completion of R), so $\mathscr{C}_R$ depends only on the analytic type of R. On the other hand, for particular choices of R with a given analytic type, $H^1(Z^*, \mathscr{K}_2)$ is relatively easy to compute. For example, we have

PROPOSITION 3.1: *Let X be a projective ruled surface, smooth outside a single normal singularity* 0. *Let $f: Z \to X$ be a resolution of singularities of X, $X^* = Spec(\mathscr{O}_{X,0})$, $Z^* = f^{-1}(X^*)$. Suppose $f^*: K_0(X) \to K_0(Z)$ is injective. Then*

$$\mathrm{Pic}(Z^*) \otimes_{\mathbf{Z}} k^* \to H^1(Z^*, \mathscr{K}_2)$$

is surjective.

PROOF: For a scheme Y, we let $F_0K_0(Y)$ denote the subgroup of $K_0(Y)$ generated by the residue fields of smooth closed points of Y. We have isomorphisms (proved by Collino [C] for X, Bloch [B] for Z):

$$F_0K_0(X) \cong H^2(X, \mathscr{K}_2)$$

$$F_0K_0(Z) \cong H^2(Z, \mathscr{K}_2).$$

The Leray spectral sequence $H^p(X, R^qf_*(\mathscr{K}_2)) \Rightarrow H^{p+q}(Z, \mathscr{K}_2)$ has the five term exact sequence

$$0 \to H^1(X, f_*\mathscr{K}_2) \to H^1(Z, \mathscr{K}_2) \to H^1(Z^*, \mathscr{K}_2)$$

$$\to H^2(X, f_*\mathscr{K}_2) \xrightarrow{f^*} H^2(Z, \mathscr{K}_2).$$

As f is an isomorphism away from 0, $H^2(X, f_*\mathscr{K}_2) = H^2(X, \mathscr{K}_2)$. Since Z is ruled, $H^1(Z, \mathscr{K}_2)$ is generated by $\mathrm{Pic}(Z) \otimes k^*$. As f^* is injective, $H^1(Z^*, \mathscr{K}_2)$ is therefore generated by $\mathrm{Pic}(Z) \otimes k^*$. Finally, the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Pic}(Z)\otimes k^* & \overset{\mathrm{res}}{\to} & \mathrm{Pic}(Z^*)\otimes k^* \\ \downarrow & & \downarrow \\ H^1(Z,\mathscr{K}_2) & \overset{\mathrm{res}}{\to} & H^1(Z^*,\mathscr{K}_2), \end{array}$$

completes the proof. □

We say that a two dimensional local ring R is a *quotient singularity* if there is a finite group G acting linearly on $\mathbf{A}^2_k$ such that the local ring of the origin 0 of $\mathbf{A}^2/G$ has completion isomorphic to the completion of R (both completions taken with respect to the maximal ideals), and in addition, R is a rational singularity. In characteristic zero, the second condition is superfluous. We now compute $K_0(\mathscr{C}_R)$ for quotient singularities.

THEOREM 3.2: *Let R be the local ring of a surface singularity. Suppose that R is a quotient singularity. Then the map length: $K_0(\mathscr{C}_R)\to\mathbf{Z}$ is an isomorphism.*

PROOF: Let G be a finite group acting on $\mathbf{A}^2$ as above. Since $K_0(\mathscr{C}_R)=K_0(\mathscr{C}_{\hat{R}})$, we may assume that R is the local ring of 0 in $\mathbf{A}^2/G=X$. Let p be a point of X, with q a point of $\mathbf{A}^2$ lying over p. If p is not 0, there is a line L in $\mathbf{A}^2$ passing through q and avoiding (0, 0), hence the image of L in X is a rational curve passing through p and missing 0. From this one sees easily that $F_0K_0(X)=0$. In particular, if $f: Z\to X$ is a resolution of singularities, then $f^*: K_0(X)\to K_0(Z)$ is injective. Since the map $\mathbf{A}^2\to X$ is generically etalé, X is a rational suface by the criterion of Castelnuovo. Using the notation of proposition 3.1, we see that $H^1(Z^*,\mathscr{K}_2)$ is generated by $\mathrm{Pic}(Z^*)\otimes k^*$. As the class group of $\mathscr{O}_{X,0}$ is finite, and k^* is divisible, we have that $\mathrm{Pic}(Z^*)\otimes k^*$ is generated by $\sum_i E_i\otimes k^*$, where the E_i are the irreducible components of $E=f^{-1}(0)$. From Theorem 2.1, this implies that $f^*: K_0(\mathscr{C}_R)\to SK'_0(E)$ is an isomorphism. The group $SK'_0(E)$ is just the group of zero-cycles on E modulo divisors of rational functions, hence, as 0 is a rational singularity, $SK'_0(E)$ is isomorphic to $\mathbf{Z}$ by degree. The commutativity of the diagram,

$$\begin{array}{ccc} \bigoplus_{x\text{ in }X^{*1}} k(x)^* & \overset{\partial}{\to} & K_0(\mathscr{C}_R) \\ \mathrm{div}\downarrow & \swarrow f^* & \\ SK'_0(E) & & \\ \deg\downarrow\wr & & \\ \mathbf{Z} & & \end{array}$$

and the fact that $\partial(R/f, g) = R/(f, g)$, shows that $\deg \circ f^* = \text{length}$, which proves the theorem. □

A similar argument shows that $K_0(\mathscr{C}_R) \cong \mathbf{Z}$ by length if R is a rational singularity, and if there exists a rational surface X with sole singularity 0, such that $\hat{\mathscr{O}}_{X,0} = \hat{R}$, and with $f^* : K_0(X) \to K_0(Z)$ injective for a resolution of singularities $f: Z \to X$. As an illustration, we prove the following complement to the previous theorem.

THEOREM 3.3: *Suppose R is the local ring of a rational surface singularity in characteristic zero. Suppose further that the fundamental cycle on a minimal resolution of* Spec(R) *is reduced. Then length*: $K_0(C_R) \to \mathbf{Z}$ *is an isomorphism.*

PROOF: As in Theorem 3.2, we will exhibit a surface X with isolated singularity analytically isomorphic to R, such that X is covered by rational curves which do not pass through the singular point.

Let X_0 be a surface having Spec(R) as local ring at a point 0. We assume that $k = \mathbf{C}$, and let X^* be a small neighborhood of 0 on X_0 in the complex topology. Let $f: Z^* \to X^*$ be a minimal resolution, and W the fundamental cycle. Write the exceptional divisor E of f as a sum of irreducible divisors, $E = \sum_1^s E_i$, and set $n_i = -\deg(W \cdot E_i)$. Let U be the blow up of a small disk about (0, 0) in $\mathbf{C}^2$, and let F be the exceptional curve. For each i, choose n_i distinct points, p_{ij}, on $E_i - \bigcup_{k \neq i} E_k$, and for each p_{ij} glue a copy of U onto Z^* so that F intersects E_i transversely at p_{ij} and at no other point of E. Call the resulting surface Y, and let E' be the divisor on Y gotten by adding all the new F's to E. Then E' is reduced, has arithmetic genus zero, and satisfies $E' \cdot E_i' = 0$ for each irreducible component E_i' of E'.

By deformation theory, there is a one parameter family of deformations of E' in Y, with generic member a smooth rational curve disjoint from E'. Since E' has only nodes, the versal deformation space of E' is smooth, hence by Artin approximation, there is a smooth algebraic surface Y' containing E', in which E' smooths to a rational curve disjoint from E'. Blow down E in Y' to yield a singular surface X. If Y' approximates Y to a sufficiently high infinite smal neighborhood of E, then a neighborhood of the singular point p on X is isomorphic to X^* (as complex analytic spaces) so the local ring of p on X is analytically isomorphic to R. In addition, the deformations of E' in Y' give rise to a family of rational curves on x, the generic member of which misses p. Arguing as in Theorem 3.2 completes the proof. □

We now consider the computation of $K_0(\mathscr{C}_R)$ where R is the local ring of certain non-rational singularities. Let C be a smooth complete curve, projectively normal in a $\mathbf{P}^N$. Let g be the genus of C, and d the degree of C in $\mathbf{P}^N$. We assume that either

(a) $\mathrm{char}(k) = p > 0$
(b) $d \leqslant 2N - 1$
(c) $d > 2g + 1$.

Let X be the affine cone over C, with vertex 0, and let $R = \mathcal{O}_{X,0}$. Let $f: Z \to X$ be the blowup of 0. Then Z is smooth, and is a line bundle over C, $p: Z \to C$, with zero section the exceptional locus E of f. This implies that $p^*: H^1(C, \mathscr{K}_2) \to H^1(Z, \mathscr{K}_2)$ is an isomorphism. Let $X^* = \mathrm{Spec}(\mathcal{O}_{X,0})$, $Z^* = f^{-1}(X^*)$. Let $s: C \to Z$ be the zero section. As s factors through Z^*, $p^*: H^1(C, \mathscr{K}_2) \to H^1(Z^*, \mathscr{K}_2)$ is injective. On the other hand, Srinivas [Sr] has shown that any one of the conditions (a), (b), (c) implies that $F_0K_0(X) = 0$, hence the map $f^*: K_0(X) \to K_0(Z)$ is injective. Arguing as in Proposition 3.1, this shows that $H^1(Z, \mathscr{K}_2) \to H^1(Z^*, \mathscr{K}_2)$ is surjective, hence

$$p^*: H^1(C, \mathscr{K}_2) \to H^1(Z^*, \mathscr{K}_2)$$

is an isomorphism.

From Theorem 2.1, there is an exact sequence

$$p^{*-1}(N) \to H^1(C, \mathscr{K}_2) \xrightarrow{\beta} K_0(\mathscr{C}_R) \xrightarrow{f^*} \mathrm{Pic}(C) \to 0.$$

On the other hand, letting h be the class of $\mathcal{O}_C(1)$ in $\mathrm{Pic}(C)$, it is clear that the subgroup $\mathrm{Im}(p^{*-1}(N))$ is just the subgroup of $H^1(C, \mathscr{K}_2)$ generated by $h \otimes k^*$. This gives the exact sequence

$$0 \to \left(H^1(C, \mathscr{K}_2)/\langle h \otimes k^* \rangle\right) \xrightarrow{\beta} K_0(\mathscr{C}_R) \xrightarrow{f^*} \mathrm{Pic}(C) \to 0,$$

describing $K_0(\mathscr{C}_R)$. In the case $k = \overline{\mathbf{F}}_p$, Srinivas has pointed out to me that $K_0(\mathscr{C}_R)$ is isomorphic to $\mathrm{Pic}(C)$. Indeed, we have a surjection

$$\mathrm{Pic}(C) \otimes_{\mathbf{Z}} k^* \to H^1(C, \mathscr{K}_2).$$

As $\mathrm{Pic}^0(C)$ is torsion, and k^* is divisible, $\mathrm{Pic}^0(C) \otimes_{\mathbf{Z}} k^* = 0$, and hence $h \otimes k^*$ generates $H^1(C, \mathscr{K}_2)$. This and the above exact sequence proves the result.

Section 4

We now give applications to the study of vector bundles on normal surfaces. We have already seen a close connection between the kernel of

$f^*: K_0(X) \to K_0(Z)$ for a resolution of singularities $f: Z \to X$ of a normal surface X, and the kernel of the map $f^*: K_0(\mathscr{C}_R) \to K_0'(E)$, where R is the semi-local ring of Sing(X) in X, and E is the exceptional divisor of f.

We have the following proposition which clarifies this relationship.

PROPOSITION 4.1: *Let X be a normal, quasi-projective surface with singular locus S. Let $R = \mathscr{O}_{X,S}$, $f: Z \to X$ a resolution of singularities of X, $X^* =$ Spec(R). Let $i_*: K_0(\mathscr{C}_R) \to K_0(X)$ be the map induced by the inclusion $i: X^* \to X$. Then there is an exact sequence*

$$H^1(Z, \mathscr{K}_2) \xrightarrow{\gamma} SK_0(\mathscr{C}_R) \xrightarrow{i_*} K_0(X) \xrightarrow{f^*} K_0(Z)$$

PROOF: We first note the following. Let f be a rational function on X, g a section of a line bundle L on X so that div(f) and div(g) have no common components. We also assume that f is in R. Let z be the class of $R/(f, g)$ in $K_0(\mathscr{C}_R)$. Let w be the portion of the intersection cycle div(f)$\cdot$div(g) supported in the smooth locus of X. The cycle w determined a class cl(w) in $K_0(X)$, and we have

$$i_*(z) = -\mathrm{cl}(w) \tag{*}$$

$$\partial((\bar{f}, R/(g)) = z,$$

where we consider $(\bar{f}, R/(g))$ as an element of $\coprod_{x \, in \, X^{*1}} k(x)^*$ in the usual way.

Since X is normal, the kernel of f^* is a subgroup of $F_0K_0(X)$. In addition, it has been shown in [L] and in [PW] that $F_0K_0(X)$ and $H^2(X, \mathscr{K}_2)$ are naturally isomorphic. Letting Z^* denote the subscheme $f^{-1}(X^*)$ of Z, we have the exact sequence

$$\begin{array}{ccccccc} H^1(Z, \mathscr{K}_2) & \xrightarrow{\mathrm{res}} & H^1(Z^*, \mathscr{K}_2) & \xrightarrow{j} & H^2(X, \mathscr{K}_2) & \xrightarrow{f^*} & H^2(Z, \mathscr{K}_2) \\ & & & & \alpha \downarrow \wr & & \| \wr \\ & & & & F_0K_0(X) & \underset{f^*}{\to} & F_0K_0(Z) \end{array}$$

Let (C^*, h^*) represent an element of $H^1(Z^*, \mathscr{K}_2)$, let C be the closure of C^* in Z and let h be the extension of h^* to a function on C. Then one easily checks that

$$\alpha \circ j((C^*, h^*)) = \mathrm{cl}(f_*(\mathrm{div}(h))) \quad \text{in } F_0K_0(X).$$

By the computation (*) of $i_* \circ \beta$, this implies that the following diagram commutes up to sign

$$\begin{array}{ccc} H^1(Z^*, \mathcal{K}_2) & \xrightarrow{\alpha \circ j} & F_0K_0(X) \\ & {}_{\beta}\searrow \quad \nearrow_{i_*} & \\ & SK_0(\mathcal{C}_R) & \end{array}$$

As the image of $\alpha \circ j$ is $\ker(f^*)$, and β is surjective, the proof is complete, where we take $\gamma : H^1(Z, \mathcal{K}_2) \to SK_0(\mathcal{C}_R)$ to be $\beta \circ \text{res}$. □

COROLLARY 4.2: *Let X be a normal quasi-projective surface having only quotient singularities. Let $f : Z \to X$ be a resolution of singularities of X. Then $f^* : K_0(X) \to K_0(Z)$ is injective.*

PROOF: Let $S = \text{Sing}(X)$. By Theorem 3.3, $SK_0(C_{X,S}) = 0$. The result is then immediate from Proposition 4.1. □

COROLLARY 4.3: *Let X be a normal quasi-projective surface, birationally isomorphic to $C \times \mathbf{P}^1$ for a smooth complete curve C. Suppose that X has only quotient singularities. If X is projective, then $F_0K_0(X)$ is isomorphic to* $\text{Pic}(C)$, *if X is affine, then $F_0K_0(X) = 0$, and every vector bundle on X is a direct sum of line bundles. If $X =$* $\text{Spec}(A)$, *and I is an ideal of A that is purely of height two and locally a complete intersection, then I is a complete intersection, $I = (f, g)$ for suitable f, g in A. In particular, the maximal ideal of every smooth point of X is a complete intersection.*

PROOF: The computations of $F_0K_0(X)$ follow from Corollary 4.2 and the known results for smooth ruled surfaces. The statement about vector bundles on affine X follows from $F_0K_0(X) = 0$, and the cancellation theorem of Murthy-Swan [MS]. The remaining assertions are then a consequence of the"Ext trick" of Serre [S], namely, given such an ideal I, there is a rank two projective P, and an exact sequence

$$0 \to A \to P \to I \to 0.$$

Since $F_0K_0(X) = 0$, P goes to zero in $K_0(X)$. By Murthy-Swan, P is free, hence I is two-generated. □

As a final example, we consider a quartic surface X in $\mathbf{P}^3$ with a triple point $S = (0, 0, 0, 1)$ as sole singularity. Let $F(X, Y, Z, W)$ be the homogeneous polynomial defining X, and let $f(x, y, z)$ be the dehomogenized polynomial defining X away from $W = 0$. We write f as

$$f = f_3 + f_4,$$

where f_i is homogeneous of degree i. We assume that the cubic curve C in $\mathbf{P}^2$ defined by $f_3 = 0$ is smooth, and that the quartic curve defined by $f_4 = 0$ intersects C at twelve distinct points $q_1, \ldots, q_{12}$. Then it is easily seen that the points q_i correspond to twelve lines $l_1, \ldots, l_{12}$ lying on X and passing through S. Also, projection from S defines a birational map $p: X \to \mathbf{P}^2$, which shows that X is gotten by blowing up the twelve points $q_1, \ldots, q_{12}$, and then blowing the proper transform of C down to S. Let $u: Z \to X$ be the blow up of X at S, E the exceptional curve, and let

$$\begin{array}{ccc} & Z & \\ {}^{u}\swarrow & & \searrow^{v} \\ X & & \mathbf{P}^2 \end{array}$$

be the factorization of p described above. Then $E = v^{-1}$ [C], $H^1(Z, \mathscr{K}_2) \cong \operatorname{Pic}(Z) \otimes k^*$, and $\operatorname{Im}(H^1(Z, \mathscr{K}_2) \to H^1(C, \mathscr{K}_2))$ is the subgroup of $H^1(C, \mathscr{K}_2)$ generated by $\oplus_i\, k(q_i)^*$. In addition, (X, S) is analytically isomorphic to the singularity of the vertex on the cone over C (assume $\operatorname{char}(k) > 3$), so by Theorem 2.1 and the discussion at the end of Section 3,

$$SK_0(\mathscr{C}_{X,S}) \cong H^1(C, \mathscr{K}_2)/\operatorname{Im}(k(p)^*),$$

where p is a flex on C. By proposition 4.1, we have the exact sequence

$$0 \to H^1(C, \mathscr{K}_2)/\langle \bigoplus_i k(q_i)^* \rangle \to F_0K_0(X) \to \underset{\displaystyle \mathbf{Z}}{\overset{}{F_0K_0(Z)}} \to 0, \qquad F_0K_0(Z) \cong \mathbf{Z}$$

describing $F_0K_0(X)$.

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Department of Mathematics
Northeastern University
360 Huntington Avenue
Boston, MA 02115
USA