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ON THE DEMAZURE CHARACTER FORMULA II GENERIC HOMOLOGIES

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1. Introduction

1.1. This paper is a continuation of [1] referred to hereafter as I and whose notation and conventions we adopt. In particular all modules are assumed to admit a locally finite semisimple action of $\mathfrak{h}$ with weights in $P(R)$ (I, notation 1.1).

1.2. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ (notation I, 1.1). For each $\lambda \in P(R)^+$, $w \in W$ we define the U($\mathfrak{b}$) module $F(w\lambda) := U(\mathfrak{n})e_{w\lambda}$ (notation I, 1.1, 1.2). Now $F(w\lambda)$ admits a weight space decomposition with respect to $\mathfrak{h}$ and hence a formal character. In I we showed that for λ sufficiently large (in short $\lambda \gg 0$) that this formal character is given by the Demazure expression. Our method applies in principle to arbitrary $\lambda \in P(R)^+$; but for λ small one runs into a delicate combinational question involving Weyl chambers. To understand this better one should determine the character formulae for the homology spaces $H_i(\mathfrak{n}, F(w\lambda))$ for $\lambda \gg 0$ and study their degeneration as λ becomes small.

1.2. In section 2 we describe the general form of the weights of $\mathfrak{n}$ homology for $\lambda \gg 0$. In section 3 we derive a weak translation principle which shows the essential independence of the weight spaces on λ. We further give an algorithm for computing these weight spaces. For the present this seems rather unwieldly; yet we show that it simplifies considerable if a rather natural condition on the weight spaces is satisfied. At present it appears to establish this condition is a question of considerable combinatorial difficulty.

1.4. Some general features of the $H_*(\mathfrak{n}, F(w\lambda))$ emerge from our analysis. First we determine $H_1(\mathfrak{n}, F(w\lambda))$ completely (for $\lambda \gg 0$). Secondly, it is noted that weight spaces can have multiplicity > 1 (in types G_2, A_3 for example). Thirdly, it is noted that the degeneration for λ small can occur before λ is non-regular. This is in distinction to the Verma module situation and although there is an obvious conjecture for the way in which this degeneration should take place it would appear very hard to prove. Finally we note that (for $\lambda \gg 0$) any non-zero weight space has weight μ satisfying $(\mu, \mu) \geqslant (\lambda, \lambda)$ with equality if and only if $i = 0$. Conversely this easily implies the Demazure character formula and so it would be of interest to give a direct proof of this inequality valid for all $\lambda \in P(R)^+$.

2. General form of generic $\mathfrak{n}$ homology

2.1. If $S \subset R^+$ (I, notation 1.1) we set

$$\langle S \rangle = \sum_{\alpha \in S} \alpha.$$

The result below depends heavily on (I, 2.21) to which we refer the reader. In this $\leqslant$ denotes the Bruhat order on W and w_0 its unique maximal element.

Proposition: *Assume $\lambda \gg 0$. Then the weights of $H_i(n, F(w\lambda))$ are amongst those of the form $y\lambda + \langle S \rangle$ where*

(*i*) $y \leqslant w$.

(*ii*) $S \subset R^+ \cap yR^+$.

(*iii*) $l(w) - l(y) \leqslant i \leqslant l(w_0) - l(y)$.

(*iv*) *card* $S \leqslant i \leqslant$ *card* $S + l(w) - l(y)$.

(*v*) $\langle S \rangle = 0 \Leftrightarrow y = w$ *and* $i = 0$.

The proof is by backward induction on length $l(w)$ and by forward induction on i. Suppose first that $w = w_0$. Then $F(w_0\lambda)$ is just the simple $\mathfrak{g}$ module $E(\lambda)$ with highest weight λ. Then by the Bott-Kostant theorem a weight of $H_i(\mathfrak{n}, F(w_0\lambda))$ takes the form $y(w_0\lambda - \rho) + \rho$ where $l(y) = i$. Now $l(yw_0) = l(w_0) - l(y)$, so $l(w_0) - l(yw_0) = l(y) = i$. Again $\rho - y\rho = \langle S(y^{-1}) \rangle$ where $S(y^{-1}) := R^+ \cap yR^- = R^+ \cap yw_0R^+$. Finally card $S(y^{-1}) = l(y^{-1}) = l(y) = i$. This case is rather exceptional as most inequalities become equalities.

Assume $\lambda \gg 0$ and fixed. Given $z \in W$, let $\Omega_i(z)$ denote the set of weights of $H_i(\mathfrak{n}, F(z\lambda))$. Fix $w \in W$ and assume our assertion on $\Omega_i(z)$ proved for all z such that $l(z) > l(w)$. Take $\mu \in \Omega_i(w)$. If $i = 0$ then $\mu = w\lambda$ which has the required form. If $i > 0$ then the hypothesis $\lambda \gg 0$ ensures via the analysis of (I, 2.21) that μ cannot be antidominant. Otherwise the conclusion of (I, 2.21) shows that the situation following (iii) in the proof of (I, 2.21) always holds and this eventually implies that $H_{i+\text{card}\, R^+}(\mathfrak{n}, F(w\lambda)) \neq 0$ which is impossible. Consequently there exists $\alpha \in B$ such that $s_\alpha\mu < \mu$.

Set $B' = \{\beta \in B \mid w^{-1}\beta \in R^-\}$. Our proof breaks up into the study of the following four cases:

(i) *One has* $\alpha \in B'$.

When (i) holds, $F(w\lambda)$ is a $U(\mathfrak{p}_\alpha)$ module (notation I, 2.1). Then by (I, 2.19) it follows that $\mu' := s_\alpha\mu + \alpha \in \Omega_{i-1}(w)$. By the induction hypothesis we can write $\mu' = y'\lambda + \langle S' \rangle$ where y', S' satisfy (i)–(v).Now $s_\alpha\mu' > \mu'$ and since $\lambda \gg 0$ this implies that $s_\alpha y' < y'$ and hence $\alpha \notin R^+ \cap y'R^+$. We can therefore write $\mu = y\lambda + \langle S \rangle$ with $y = s_\alpha y'$, $S = s_\alpha S' \cup \{\alpha\}$. Now $y' \leqslant w$ and $s_\alpha y' < y'$, so $y \leqslant w$ which is (i). Again $S' \subset R^+ \cap y'R^+$ and since $\alpha \notin R^+ \cap y'R^+$ we obtain $S \subset s_\alpha(R^+ \cap y'R^+) \cup \{\alpha\} = R^+ \cap yR^+$ which is (ii). From the inequality $l(w) - l(y') \leqslant (i-1) \leqslant l(w_0) - l(y')$ and $l(y) = l(y') - 1$ we obtain (iii). Again card $S =$ card $S' + 1$ gives (iv) and (v) is obvious.

Suppose $\alpha \notin B'$. Then the analysis of (I, 2.21) shows that one of three following possibilities hold:

(ii) *One has* $\mu' := s_\alpha\mu \in \Omega_{i-1}(w)$.

By the induction hypothesis we can write $\mu' = y'\lambda + \langle S' \rangle$ where y', S' satisfy (i)–(v). As in case (i), we obtain $s_\alpha y' < y'$ and so $\alpha \notin R^+ \cap y'R^+$. Thus we can write $\mu = y\lambda + \langle S \rangle$ where $y = s_\alpha y'$, $S = s_\alpha S'$. Then everything goes through as in (i) except that card $s_\alpha S =$ card S. This means that the first inequality of (iv) can be strict. Finally if $\langle s_\alpha S \rangle = 0$, then $\langle S \rangle = 0$ so by (v) we must have $y' = w$. Yet $s_\alpha y' < y'$ whereas $\alpha \in B'$ and so $s_\alpha w < w$ which is a contradiction.

(iii) *One has* $\mu \in \Omega_i(s_\alpha w)$.

Since $s_\alpha w < w$ we can write $\mu = y\lambda + \langle S \rangle$ by the induction hypothesis where (i)–(v) are satisfied with respect to the triple y, $s_\alpha w$, S. Now as in case (i) we have $s_\alpha y > y$ and then because $w \leqslant s_\alpha w$, $y \leqslant s_\alpha w$ we obtain $y \leqslant w$ (by what can be considered as a defining property of the Bruhat order). We have $l(s_\alpha w) - l(y) \leqslant i \leqslant l(w_0) - l(y)$ and since $l(s_\alpha w) = l(w) + 1$ we obtain (iii) except that now the first inequality can be strict. Finally S remains unchanged so (ii), (iv), (v) result.

(iv) *One has* $\mu' := s_\alpha\mu \in \Omega_i(s_\alpha w)$.

Again $l(s_\alpha w) > l(w)$ and so by the induction hypothesis we can write $\mu' = y'\lambda + \langle S' \rangle$ where (i)–(v) are satisfied with respect to the triple y', $s_\alpha w$, S'. As before $s_\alpha y' < y'$ and so $\alpha \notin R^+ \cap y'R^+$ and we can write $\mu = y\lambda + \langle S \rangle$ where $s_\alpha y' = y$, $s_\alpha S' = S$. We have $l(s_\alpha w) - l(y') = l(w) - l(y)$, $s = s_\alpha S' \subset R^+ \cap s_\alpha y'R^+ = R^+ \cap yR^+$ and card $S' =$ card S from which (ii)–(iv) are easily checked. Since $y' \leqslant s_\alpha w$, $y = s_\alpha y' \leqslant y'$ and $w \leqslant s_\alpha w$ we obtain $y \leqslant w$ by again a defining property of the Bruhat order. Finally if $\langle S \rangle = 0$, then $\langle s_\alpha S \rangle = 0$, so $i = 0$ and $y = s_\alpha w$. This would imply that $s_\alpha w\lambda$ is a weight of $H_0(\mathfrak{n}, F(w\lambda))$ which is obviously false. This proves (v) and completes the proof of the proposition.

2.2. Eventually we should like to show that 2.1 holds without restriction on $\lambda \in P(R)^+$. It is therefore interesting to point out the following two properties of this general solution either of which would imply the Demazure character formula through the subsequent use of (I, 2.18).

LEMMA: *Fix* $\lambda \in P(R)^+$, $w \in W$. *Suppose* $\mu = y\lambda + \langle S \rangle$ *where* $y \in W$, $S \subset R^+$ *satisfy* (i)–(ii) *of* 2.1. *Then*

(i) $(\mu, \mu) \geqslant (\lambda, \lambda)$ *with equality if and only if* $S = \emptyset$, $y = w$, $i = 0$.

(ii) μ *is antidominant if and only if* $S = \emptyset$, $i = 0$ *and* $y\lambda = w\lambda = w_0\lambda$.

(i) Set $\sigma = \langle S \rangle$. Then $(\mu, \mu) = (\lambda, \lambda) + (\sigma, \sigma) + 2(\sigma, y\lambda)$. Since $y^{-1}S \subset R^+$ and $\lambda \in P(R)^+$ we have $(\sigma, y\lambda) \geqslant 0$ and so $(\mu, \mu) \geqslant (\lambda, \lambda)$ with equality if and only if $\sigma = 0$. Thus implies $S = \emptyset$ and $y = w$, $i = 0$ by 2.1(v). Hence (i).

(ii) Suppose $-\mu \in P(R)^+$. Since $S \subset R^+$ we obtain $(\sigma, \mu) \leqslant 0$ and so $(\sigma, \sigma) + 2(\sigma, y\lambda) \leqslant 0$. As before this implies $S = \emptyset$ and then $y = w$. Hence (ii).

2.3. We view 2.1 as a first step in an inductive procedure for computing $H_*(\mathfrak{n}, F(w\lambda))$. It is convenient to summarize its conclusions as below. Assume $\lambda \gg 0$ and fixed. Let us use $\Omega_i(w)$ to denote the set of weights of $H_i(\mathfrak{n}, F(w\lambda))$ counted with their multiplicities.

Suppose $\mu \in \Omega_i(w)$. Then we can write $\mu = y\lambda + \langle S \rangle$ and either $i = 0$ in which case $\mu = w\lambda$ or $i > 0$ and we can choose $\alpha \in B$ such that $s_\alpha y > y$. Furthermore in the latter case either

(i) $s_\alpha w < w$ *and then* $s_\alpha \mu + \alpha \in \Omega_{i-1}(w)$

or $s_\alpha w > w$ *and then one of the following hold*

(ii) $s_\alpha \mu \in \Omega_{i-1}(w)$.

(iii) $\mu \in \Omega_i(s_\alpha w)$ *and then* $s_\alpha \mu + \alpha \in \Omega_{i-1}(s_\alpha w)$.

(iv) $s_\alpha \mu \in \Omega_i(s_\alpha w)$.

2.4. The above result shows that we should try to determine which weights of the form $y\lambda + \langle S \rangle$ occur in $\Omega_i(w)$ by induction on $l(w) - l(y)$. For this purpose we note a partial converse to 2.3. Here we fix $\lambda \in P(R)^+$, $w \in W$ and assume that the Demazure character formula holds for all $z \in W$ with $l(z) > l(w)$ – equivalently that the surjective map (I, 2.18) φ_z: $\mathscr{D}_z\mathbb{C}_\lambda \to F(z\lambda)$ is bijective. This is weaker than assuming $\lambda \gg 0$.

LEMMA: *Fix* $\alpha \in B$ *such that* $s_\alpha w > w$. *Assume that* $\mu \in P(R)$ *satisfies* $(\mu, \alpha) < 0$.

(i) *Suppose* $\mu \in \Omega_i(s_\alpha w)$. *If* $\mu \notin \Omega_i(w)$ *and* $\mu + \alpha \notin \Omega_i(s_\alpha w)$, *then* $s_\alpha \mu \in \Omega_i(w)$.

(ii) *Suppose* $\mu \in \Omega_i(s_\alpha w)$. *If* $\mu - \alpha \notin \Omega_i(s_\alpha w)$ *and* $\mu - \alpha \notin \Omega_{i-1}(w)$ *then* $s_\alpha \mu + \alpha \in \Omega_{i+1}(w)$.

(iii) *Suppose* $\mu \in \Omega_i(w)$. *If* $\mu \notin \Omega_i(s_\alpha w)$ *and* $s_\alpha \mu \notin \Omega_{i+2}(s_\alpha w)$ *then* $s_\alpha \mu \in \Omega_{i+1}(w)$.

This result is obtained by following through the analysis of (I, 2.21) and we leave the details to the reader. It is obviously not quite enough as an inductive procedure and in particular gives no insight into possible multiplicities. We remark that the fact that $w\lambda \in \Omega_0(w)$ can be conveniently thought of as a consequence of (i) taking $\mu = s_\alpha w\lambda$ and $i = 0$.

2.5. We shall compute $H_1(\mathfrak{n}, \mathrm{F}(w\lambda))$ when $\lambda \gg 0$. First we obtain a result under the slightly weaker hypothesis of 2.4.

LEMMA: (Notation I, 3.6).

(*i*) *Every non-antidominant weight of* $H_1(\mathfrak{n}, F(w\lambda))$ *is amongst those of the form* $y\lambda + \gamma$ *with* $y \xrightarrow{\gamma} w$ *and those of the form* $w\lambda + \gamma$ *with* $\gamma \in R^+ \cap yR^+$.

(*ii*) *Suppose* $\lambda \gg 0$. *If* $y \xrightarrow{\gamma} w$, *then* $y\lambda + \gamma \in \Omega_1(w)$ *and has multiplicity one.*

(i) When $\lambda \gg 0$, then (i) is immediate from 2.1 taking $i = 1$. However for the non-antidominant weights one easily checks that the argument of 2.3 still applies. Here one should observe that in applying 2.3(iv) we can assume that $\Omega_1(s_\alpha w)$ has no antidominant elements for $s_\alpha w > w$. This is because $\ker \varphi_{s_\alpha w} = 0$ by hypothesis and so (I, 3.4, 3.5) apply.

(ii) The proof is by backward induction on length. For $w = w_0$ the assertion follows from the Bott-Kostant theorem. Set $B' = \{\alpha \in B \mid w^{-1}\alpha \in R^-\}$. If $\gamma \in B'$ then $F(w\lambda)$ has a $\mathfrak{p}_\gamma$ module structure and so the assertion follows from (I, 2.19) and the fact that $w\lambda \in \Omega_0(w)$. If $\gamma \in B \setminus B'$ then $l(s_\gamma w) > l(w)$. Since $s_\gamma w\lambda \in \Omega_0(s_\gamma w)$ and obviously $s_\gamma w\lambda - \gamma \notin \Omega_0(s_\gamma w) \cap \Omega_{-1}(w)$ we obtain $w\lambda + \gamma \in \Omega_1(w)$ by 2.4(ii). Finally assume $\gamma \in R^+ \setminus B$. Write $w_0 = z'w = zy$. Then we can write $z' = xx'$, $z = xs_\beta x'$ where $x, x' \in W$, $\beta \in B$ and lengths add. One has $x'\gamma = \beta$, so $x' \neq 1d$. We can therefore pick $\alpha \in B$ such that $x'\alpha \in R^-$ and then $s_\alpha y > y$, $s_\alpha w > w$ and $s_\alpha y \xrightarrow{s_\alpha\gamma} s_\alpha w$. By the induction hypothesis $s_\alpha y\lambda + s_\alpha\gamma \in \Omega_1(s_\alpha w)$. Now for $\lambda \gg 0$ we have $s_\alpha y\lambda + s_\alpha\gamma \notin \Omega_1(w)$ because $s_\alpha y \not< w$. Again $s_\alpha y\lambda + s_\alpha\gamma + \alpha \notin \Omega_1(s_\alpha w)$ because by (i) we should then have $s_\alpha\gamma + \alpha = s_\alpha\gamma$ which is absurd. Hence $y\lambda + \gamma \in \Omega_1(s_\alpha w)$ by 2.4(i) as required. Finally under the hypothesis $\lambda \gg 0$ one has $\ker \varphi_w = 0$ and then the multiplicity one assertion is an obvious consequence of (I, 3.8).

REMARK: One can analyse the degeneration of $H_1(\mathfrak{n}, F(w\lambda))$ for λ small by this method; but the computation obviously gets somewhat messy. Again in view of (I, 3.9) one can hope to show that the natural map $H_1(\mathfrak{n}, \mathscr{D}_w\mathbb{C}_\lambda) \to H_1(\mathfrak{n}, F(w\lambda))$ is injective. Such a result would strongly suggest that $\ker \varphi_w = 0$ which in turn would establish the Demazure character formula for all λ.

2.6. Assume $\lambda \gg 0$. It is false that $w\lambda + \gamma \in \Omega_1(w)$ implies $w \xrightarrow{\gamma} s_\gamma w$ though by (I, 3.12) all such weights do occur and with multiplicity one.

Actually we can solve more generally the question of when $w\lambda + \langle S \rangle \in \Omega_i(w)$. Set $\mathfrak{n}_w = \{\mathbb{C}X_\gamma \mid \gamma \in R^+ \cap wR^+\}$ which is a subalgebra of $\mathfrak{n}$ complemented by the subalgebra $\mathfrak{n}_{ww_0}$. Consider $\mathbb{C}_{w\lambda}$ as an $\mathfrak{h} \oplus \mathfrak{n}_w$ module by restriction (notation I, 2.3) and set $M = U(\mathfrak{n}) \oplus_{U(\mathfrak{n}_w)} \mathbb{C}_{w\lambda}$. One can describe (though somewhat implicitly) the space $H_*(\mathfrak{n}, M)$. Take the standard free resolution $U(\mathfrak{n}_w) \otimes \Lambda^*\mathfrak{n}_w$ of the $U(\mathfrak{n}_w)$ module $\mathbb{C}_w$. Then $U(\mathfrak{n}) \otimes \Lambda^*\mathfrak{n}_w$ is a free resolution of M and so we have an isomorphism $H_*(\mathfrak{n}, M) \cong H_*(\mathfrak{n}_w, \mathbb{C}_{w\lambda})$ of $\mathfrak{h}$ modules. In particular $H_1(\mathfrak{n}, M) \cong \mathbb{C}_{w\lambda} \otimes \mathfrak{m}_w$ where $\mathfrak{m}_w$ is an ad $\mathfrak{h}$ stable complement to $[\mathfrak{n}_w, \mathfrak{n}_w]$ in $\mathfrak{n}_w$ (the former being the image of the differential map $\mathfrak{n}_w \Lambda \mathfrak{n}_w \to \mathfrak{n}_w$).

Lemma: *Assume $\lambda \gg 0$. If $\mu \in \Omega_i(w)$ takes the form $\mu = w\lambda + \langle S \rangle$ then* $\dim H_i(\mathfrak{n}, F(w\lambda))_\mu = \dim H_i(\mathfrak{n}, M)_\mu$. *In particular if $w\lambda + \langle S \rangle \in \Omega_1(w)$, then $\langle S \rangle$ is a root vector in $\mathfrak{m}_w$.*

Recall that $F(w\lambda)$ is a cyclic $U(\mathfrak{n})$ module with cyclic generator $e_{w\lambda}$ of weight $w\lambda$ satisfying $X_\gamma \mathbb{C}_{w\lambda} = 0$, $\forall \gamma \in R^+ \cap wR^+$ and hence $\mathbb{C}e_{w\lambda} \cong \mathbb{C}_{w\lambda}$ as $\mathfrak{h} \oplus \mathfrak{n}_w$ modules. Of course $\operatorname{Ann}_{U(\mathfrak{n}_{ww_0})} e_{w\lambda} \neq 0$, so $F(w\lambda)$ is only an image of M. Yet if we fix $i \in \mathbb{N}$ then $\operatorname{Ann}_{U^i(\mathfrak{n}_{ww_0})} e_{w\lambda} = 0$ for $\lambda \gg 0$. This follows from the corresponding fact that $\operatorname{Ann}_{U^i(\mathfrak{n}^-)} e_\lambda = 0$ for $\lambda \gg 0$ which in turn follows from Verma module theory (see for example the argument in I, 2.14). Set $K = \operatorname{Ker}(M \to F(w\lambda))$. For each weight of the form $\mu = w\lambda + \langle S \rangle$: $S \subset R^+$ we have shown that $(K \otimes \Lambda^j\mathfrak{n})_\mu = 0$, $\forall j = 1, 2, \ldots, \operatorname{card} R^+$ (taking $\lambda \gg 0$) and so the natural map $H_i(\mathfrak{n}, M)_\mu \to H_i(\mathfrak{n}, F(w\lambda))_\mu$ is an isomorphism for each i. This proves the lemma.

Remarks: Lemmas 2.5 and 2.6 determine $H_1(\mathfrak{n}, F(w\lambda))$ completely for $\lambda \gg 0$. Representatives of each weight space are given by (I, 3.8). Unlike the Bott-Kostant case these representatives are not uniquely determined by their weights.

2.7. A further consequence of 2.6 is that the occurrence in $\Omega_i(w)$ of weights of the form $y\lambda + \langle S \rangle$ with $y = w$ is independent of λ for $\lambda \gg 0$. We shall see (3.7) that this holds for arbitrary y. Thus to each triple $(y, w, S) \in W \times W \times p(R^+)$ satisfying 2.1 and for each $i \in \mathbb{N}$ we can assign an integer $\geqslant 0$ given by the multiplicity of the weight $y\lambda + \langle S \rangle$ in $\Omega_i(w)$. Our ultimate aim is to determine these integers. A key difficulty is that $\langle S \rangle$ does not determine S. Indeed set $\Sigma = \{\sigma \in \mathbb{N}B \mid \sigma = \langle S \rangle$ for some $S \subset R^+\}$ and define for each $\sigma \in \Sigma$ the positive integer $m_\sigma = \operatorname{card}\{S \subset R^+ \mid \sigma = \langle S \rangle\}$. It is well-known that $m_\sigma = 1$ if and only if $\sigma = \rho - z^{-1}\rho$ for some $z \in W$ (and then $\sigma = \langle S(z) \rangle$) and $m_\sigma = 0$ (mod 2) otherwise.

We define $(y, w, \sigma) \in W \times W \times \Sigma$ to be an *acceptable triple* if $y\lambda + \sigma \in \Omega_i(w)$ for some i and we let Y (resp. Y_i) denote the set of acceptable triples (resp. occurring in dimension i). We can give a fairly simple algorithm for determining Y and the corresponding weight multiplicities if it is known that $Y_i \cap Y_{i+1} = \emptyset$ for all i. The analogous question for simple infinite dimensional highest weight modules is known to have a positive answer; but this is a deep consequence of a property of Deligne-Goretsky-Macpherson sheaves. It is probable that the present question is much simpler and it even seems likely that $Y_i \cap Y_j = \emptyset$ for all $i \neq j$ which fails for simple highest weight modules. We have checked it for rank $\mathfrak{g} \leqslant 2$ and in $sl(4)$.

2.8. We can translate 2.3 to rule on acceptable triples. This takes the form of the

LEMMA: *Suppose* $(y, w, \sigma) \in Y_i$. *If* $y = w_0$, *then* $w = w_0$, $\sigma = 0$ *and* $i = 0$. *Otherwise for each* $\alpha \in B$ *such that* $s_\alpha y > y$ *there exists* $S \subset R^+$ *such that* $\sigma = \langle S \rangle$ *and one of the following hold*

(*i*) $s_\alpha w < w$; *then* $\alpha \in S$ *and* $(s_\alpha y, w, \langle s_\alpha(S \setminus \{\alpha\})\rangle) \in Y_{i-1}$.
(*ii*) $s_\alpha w > w$, $\alpha \notin S$ *and* $(s_\alpha y, w, \langle s_\alpha S\rangle) \in Y_{i-1}$.
(*iii*) $s_\alpha w > w$, $\alpha \in S$ *and* $(s_\alpha y, s_\alpha w, \langle s_\alpha(S \setminus \{\alpha\})\rangle) \in Y_{i-1}$.
(*iv*) $s_\alpha w > w$, $\alpha \notin S$ *and* $(s_\alpha y, s_\alpha w, \langle s_\alpha S\rangle) \in Y_i$.

REMARKS: Here the notation $\langle s_\alpha S\rangle$ means that we can continue to use $s_\alpha S$ as a representative. The snag in using this result to prove the conjectures in 2.7 is that the choice of representative may depend on α. Indeed if $\sigma \neq 0$ we can always pick a representative such that $S \cap B \neq \emptyset$. If this is the "correct representative" then given $\alpha \in S \cap B$, we must have $s_\alpha y > y$ by 2.1(ii) and then either (i) or (iii) holds in the above. This argument would eventually give $i = \text{card } S$ which is generally false.

2.9. There is one nice class of acceptable triples (y, w, σ) and as one might guess from the previous discussion these are characterized by the condition $m_\sigma = 1$. This yields a generalization of the Bott-Kostant theorem to any $F(w\lambda)$ which is natural but not a priori obvious.

Given $m_\sigma = 1$ we can write $\sigma = \langle S(z)\rangle$ for some $z \in W$. Let $z = s_t s_{t-1} \cdots s_1$: $s_i = s_{\alpha_i}$ be a reduced decomposition. Define w_i: $i = 0, 1, 2, \dots, t$ inductively through $w_0 = w$,

$$w_i = \begin{cases} s_i w_{i-1} & \text{if } s_i w_{i-1} > w_{i-1} \\ w_{i-1} & \text{otherwise.} \end{cases}$$

LEMMA: *Fix* $w \in W$. *Suppose* $\sigma = \langle S(z)\rangle$ *for some* $z \in W$. *Then there is exactly one value of* $y \in W$ (*namely* $z^{-1} w_t$) *such that* $(y, w, \sigma) \in Y_j$ *and furthermore* $j = l(z)$. *Finally* $y\lambda + \langle S(z)\rangle$ *occurs in* $\Omega_{l(z)}(w)$ *with multiplicity one* (*for* $\lambda \gg 0$).

Uniqueness. This is essentially the argument given in the remarks following 2.8 and goes by induction on $l(z)$. If $l(z)=0$, then $\sigma=0$, so $y=w$ and $i=0$ by 2.1(v). Otherwise we can choose $\alpha \in S(z) \cap B$ and then $s_\alpha y > y$ by 2.1(ii). If $s_\alpha w < w$ then (i) of 2.8 holds, otherwise (iii) holds. Noting that $s_\alpha(S(z)\setminus\{\alpha\}) = S(zs_\alpha)$ the desired conclusion is obtained.

Existence. The proof is again by induction on $l(z)$ the case $l(z)=0$ being trivial. Choose $\alpha \in S(z) \cap B$. If $s_\alpha w < w$, then $(s_\alpha y, w, \langle S(zs_\alpha)\rangle) \in Y_{i-1}$ implies $(y, w, \langle S(z)\rangle) \in Y_i$ by (I, 2.19). Finally we must show that if $s_\alpha w > w$ then $(s_\alpha y, s_\alpha w, \langle S(zs_\alpha)\rangle) \in Y_{i-1}$ implies $(y, w, \langle S(z)\rangle) \in Y_i$. Setting $\tau = \langle S(zs_\alpha)\rangle$, this will follow from 2.4(ii) if we can show that $(s_\alpha y, s_\alpha w, \tau-\alpha)$ and $(s_\alpha y, w, \tau-\alpha)$ are not acceptable triples. This latter fact is simply a consequence of the fact that $\tau-\alpha$ cannot be written in the form $\langle S\rangle$ with $S \subset R^+$. Otherwise $\rho-(\tau-\alpha)$ is a weight of $E(\rho)$. Now $\rho-\tau=(zs_\alpha)^{-1}\rho$ and so $\|\rho-(\tau-\alpha)\|^2 = \|s_\alpha z^{-1}\rho+\alpha\|^2 > \|\rho\|^2 - 2(z\alpha, \rho) > \|\rho\|^2$ since $z\alpha \in R^-$ by hypothesis. This contradiction proves our assertion. Finally refining 2.4 in the obvious manner shows that multiplicity of the corresponding weight spaces can only be increased in the above process. Since $H_i(\mathfrak{n}, \mathbb{C}_\lambda) \cong H_i(\mathfrak{n}, \mathbb{C}_0) \otimes \mathbb{C}_\lambda$ as an $\mathfrak{h}$ module, the latter can be computed by Bott–Kostant (giving $\mathbb{C}_0$ the trivial $\mathfrak{g}$ module structure) and is found to have no multiplicity. Hence neither does $H_i(\mathfrak{n}, F(w\lambda))$ for these set of weights.

3. An induction formula for $\mathfrak{m}$ homology

3.1. The analysis of section 2 is based on backward induction on length. This is a necessary feature of our analysis since we use the nonexistence of antidominant weights in one-homology to deduce the Demazure character formula. More specifically we deduce the bijectivity of the maps φ_w (I, notation 2.18) and this via (I, 2.8(ii)) establishes the structure of the successive quotients $F(s_\alpha w\lambda)/F(w\lambda)$: $s_\alpha w > w$. However having obtained this result it is more natural to apply forward induction on length to deduce an algorithm for the $\mathfrak{n}$-homologies. As we shall see this is not so straightforward as the corresponding result (c.f. I, 2.22) for characters. Our analysis must go via $\mathfrak{m}$ homologies where $\mathfrak{m}=\mathfrak{m}_\alpha$: $\alpha \in B$ and this is studied below.

3.2. Fix $\alpha \in B$ and (notation I, 2.1, 2.19) set $\mathfrak{p}=\mathfrak{p}_\alpha$, $\mathfrak{m}=\mathfrak{m}_\alpha$, $\mathfrak{r} = \mathfrak{r}_\alpha := \mathfrak{h} \oplus \mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha}$, $\mathfrak{c} = \mathfrak{h} \oplus \mathbb{C}X_\alpha$. If E (resp. F) is a $\mathfrak{p}$ (resp. $\mathfrak{b}$) module then $H_*(\mathfrak{m}, E)$ (resp. $H_*(\mathfrak{m}, F)$) inherits a natural $\mathfrak{r}$ (resp. $\mathfrak{c}$) module structure. Furthermore defining the $\mathscr{D}_\alpha^i$ with respect to the pair $\mathfrak{r}$, $\mathfrak{c}$ (notation I, 2.1, 5.3) one has

$$\begin{aligned} &H_j(\mathfrak{m}, E) \xrightarrow{\sim} \mathscr{D}_\alpha^0 H_j(\mathfrak{m}, E), \ \forall j \in \mathbb{N}, \\ &\mathscr{D}_\alpha^i H_j(\mathfrak{m}, E) = 0, \ \forall j \in \mathbb{N}, \ \forall i > 0. \end{aligned} \qquad (*)$$

Again let Q be a $\mathfrak{b}$ module with the property that the $\mathfrak{b}$ module structure on $Q \otimes \mathbb{C}_\rho$ extends (necessarily uniquely I, 2.7) to a $\mathfrak{p}$ module structure. Then since $\mathbb{C}_\rho$ is trivial as an $\mathfrak{m}$ module we obtain an isomorphism $H_j(\mathfrak{m}, Q \otimes \mathbb{C}_\rho) \xrightarrow{\sim} H_j(\mathfrak{m}, Q) \otimes \mathbb{C}_\rho$ of $\mathfrak{r}$ modules and hence from (I, 5.4(i)) that

$$\mathscr{D}^i_\alpha H_j(\mathfrak{m}, Q) = 0, \ \forall i, \ j \in \mathbb{N}. \tag{$**$}$$

LEMMA: *Let $0 \to F \to E \to Q \to 0$ be an exact sequence of finite dimensional $\mathfrak{b}$ modules. Assume that $\mathfrak{b}$ module structure on E and on $Q \otimes \mathbb{C}_\rho$ extends to a $\mathfrak{p}$ module structure. Then for all $j \in \mathbb{N}$ one has a direct sum of $\mathfrak{r}$ modules*

$$H_j(\mathfrak{m}, E) \cong \mathscr{D}^0_\alpha H_j(\mathfrak{m}, F) \oplus \mathscr{D}^1_\alpha H_{j-1}(\mathfrak{m}, F).$$

Split the long exact sequence for $\mathfrak{m}$ homology into short exact sequences:

$$0 \to Y_j \to H_j(\mathfrak{m}, Q) \to X_j \to 0. \tag{1}$$

$$0 \to Z_j \to H_j(\mathfrak{m}, E) \to Y_j \to 0. \tag{2}$$

$$0 \to X_{j+1} \to H_j(\mathfrak{m}, F) \to Z_j \to 0. \tag{3}$$

From (1), $(**)$ and (I, 5.4(iii)) we obtain

$$\mathscr{D}^1_\alpha Y_j = 0, \quad \mathscr{D}^1_\alpha X_j \xrightarrow{\sim} \mathscr{D}^0_\alpha Y_j, \quad \mathscr{D}^0_\alpha X_j = 0. \tag{4}$$

Then from (2), $(*)$ and (4) we obtain

$$\mathscr{D}^1_\alpha Z_j = 0, \quad 0 \to \mathscr{D}^0_\alpha Z_j \to H_j(\mathfrak{m}, E) \to \mathscr{D}^0_\alpha Y_j \to 0. \tag{5}$$

Combining (3), (4) and (5) we obtain

$$\mathscr{D}^1_\alpha X_{j+1} \xrightarrow{\sim} \mathscr{D}^1_\alpha H_j(\mathfrak{m}, F), \quad \mathscr{D}^0_\alpha H_j(\mathfrak{m}, F) \xrightarrow{\sim} \mathscr{D}^0_\alpha Z_j. \tag{6}$$

Finally from (4), (5) and (6) we obtain

$$0 \to \mathscr{D}^0_\alpha H_j(\mathfrak{m}, F) \to H_j(\mathfrak{m}, E) \to \mathscr{D}^1_\alpha H_{j-1}(\mathfrak{m}, F) \to 0$$

and hence the assertion of the lemma.

3.3. The above formula shows that $H_*(\mathfrak{m}, F)$ determines $H_*(\mathfrak{m}, E)$ (but not conversely). Indeed by (I, 2.3) any $\mathfrak{c}$ module is a direct sum of its string submodules $F(\mu, \nu)$. Here we recall (I, 2.3) that a string module is by definition a $\mathfrak{c}$ module which is cyclic as an $\mathbb{C}X_\alpha$ module and its isomorphism class is determined by its highest weight μ and lowest weight ν. From the string module decomposition of $H_*(\mathfrak{m}, F)$ we can then use (I, 2.4(*)) to deduce from 3.2 the $\mathfrak{r}$ module structure of $H_*(\mathfrak{m}, E)$. We shall see that the knowledge of the $\mathfrak{r}$ module structure of $H_*(\mathfrak{m}, E)$ is equivalent to a knowledge of the $\mathfrak{h}$ module structure of $H_*(\mathfrak{n}, E)$. Conversely if $H_*(\mathfrak{m}, F)$ is a string module then we shall see that its highest and lowest weights are determined by the $\mathfrak{h}$ module structure of $H_*(\mathfrak{n}, F)$. Unfortunately if many strings occur in $H_*(\mathfrak{m}, F)$ the knowledge of the $\mathfrak{h}$ module structure of $H_*(\mathfrak{n}, F)$ is insufficient as the highest and lowest weights of the different strings cannot be distinguished. We shall examine this question in the subsequent sections.

3.4. Fix a pair $\mathfrak{n}$, $\mathfrak{m}$. For what follows it is only important that $\mathfrak{m}$ is an ideal of codimension one in $\mathfrak{n}$ (and eventually that both are ideals in $\mathfrak{b}$ and that $\mathfrak{m}$ is an ideal in $\mathfrak{p}$). Let F be a $\mathfrak{b}$ module and take $H_{-1}(\mathfrak{m}, F) = 0$.

LEMMA: *For each $j \in \mathbb{N}$ one has an exact sequence*

$$0 \to H_0\big(\mathfrak{n}/\mathfrak{m}, H_j(\mathfrak{m}, F)\big) \xrightarrow{\bar{\varphi}} H_j(\mathfrak{n}, F)$$

$$\xrightarrow{\bar{\psi}} H_1\big(\mathfrak{n}/\mathfrak{m}, H_{j-1}(\mathfrak{m}, F)\big) \to 0$$

of $\mathfrak{h}$ modules.

This result was noted in (I, 2.19). However for what follows it is useful to see quite explicitly how this result obtains.

Since $\mathfrak{m}$ is an ideal of $\mathfrak{n}$ we have an action of $\mathfrak{n}$ on each $\Lambda^j\mathfrak{m} \otimes F$ defined by

$$Y(\xi \otimes f) = (\operatorname{ad} Y)\xi \otimes f + \xi \otimes Yf, \ \forall Y \in \mathfrak{n},$$

$\xi \in \Lambda^j\mathfrak{m}$, $f \in F$. Fix $X \in \mathfrak{n}$, $X \notin \mathfrak{m}$. (It will eventually be convenient to take $X = X_\alpha$). Identify $\mathbb{C}X \otimes \Lambda^j\mathfrak{m}$ with (an $\mathfrak{h}$ stable) complement to $\Lambda^{j+1}\mathfrak{m}$ in $\Lambda^{j+1}\mathfrak{n}$. One easily checks that

$$d(X \otimes b) = Xb - X \otimes db, \ \forall b \in \Lambda^j\mathfrak{m} \otimes F. \tag{1}$$

Since $Xb \in \Lambda^j\mathfrak{m} \otimes F$ we conclude that

$$d(X \otimes b) = 0 \Leftrightarrow Xb = 0 \quad \text{and } db = 0. \tag{2}$$

Given $a \in \Lambda^j \mathfrak{n} \otimes F$ we may write $a = X \otimes b + b'$, where $b \in \Lambda^{j-1}\mathfrak{m} \otimes F$, $b' \in \Lambda^j \mathfrak{m} \otimes F$ are uniquely determined. We define a linear map ψ_j: $\Lambda^j \mathfrak{n} \otimes F \to \mathfrak{n}/\mathfrak{m} \otimes (\Lambda^{j-1}\mathfrak{m} \otimes F)$ by setting $\psi_j(a) = X \otimes b$. From (1) we obtain $da = Xb - X \otimes db + db'$ and so

$$da = 0 \Leftrightarrow db = 0 \quad \textit{and } Xb + db' = 0. \tag{3}$$

$$\psi_j(db) = -X \otimes db. \tag{4}$$

From (3) and (4) one deduces that ψ_j factors to a linear map $\bar{\psi}_j$: $H_j(\mathfrak{n}, F) \to \mathfrak{n}/\mathfrak{m} \otimes H_{j-1}(\mathfrak{m}, F)$. Furthermore from the relation in the extreme right hand side of (3) we deduce that $\operatorname{Im} \bar{\psi}_j \in H_1(\mathfrak{n}/\mathfrak{m}, H_{j-1}(\mathfrak{m}, F))$. Conversely given $\bar{b} \in H_{j-1}(\mathfrak{m}, F)$ satisfying $X\bar{b} = 0$ we can choose a representative $b \in \Lambda^{j-1}\mathfrak{m} \otimes F$ satisfying $db = 0$ and $Xb = db'$ for some $b' \in \Lambda^j \mathfrak{m} \otimes F$. Then by (1) we have $d(X \otimes b - b') = 0$ whereas $\psi_j(X \otimes b - b') = X \otimes b$ and we conclude that $\bar{\psi}_j$ is surjective.

Let φ_j: $\Lambda^j \mathfrak{m} \otimes F \to \Lambda^j \mathfrak{n} \otimes F$ be defined through the embedding $\mathfrak{m} \hookrightarrow \mathfrak{n}$. One obtains a commuting diagram

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \Lambda^j \mathfrak{m} \otimes F & \xrightarrow{\varphi_j} & \Lambda^j \mathfrak{n} \otimes F & \xrightarrow{\psi_j} & \mathfrak{n}/\mathfrak{m} \otimes \Lambda^{j-1}\mathfrak{m} \otimes F & \to 0 \\
 & & \downarrow d_j & & \downarrow d_j' & & \downarrow & & \\
0 \to & & \Lambda^{j+1}\mathfrak{m} \otimes F & \xrightarrow{\varphi_{j+1}} & \Lambda^{j+1}\mathfrak{n} \otimes F & \xrightarrow{\psi_{j+1}} & \mathfrak{n}/\mathfrak{m} \otimes \Lambda^j \mathfrak{m} \otimes F & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

with exact rows. One deduces that φ_j factors to a linear map $\tilde{\varphi}_j$: $H_j(\mathfrak{n}, F) \to H_j(\mathfrak{m}, F)$ with $\ker \tilde{\varphi}_j = \{b \in \ker d_j \mid \varphi_j(b) \in \operatorname{Im} d_{j-1}'\}/\operatorname{Im} d_{j-1}$. Now suppose $\tilde{b} = X\tilde{c}$ for some $\tilde{c} \in H_j(\mathfrak{m}, F)$. Choosing representatives we can write $b = Xc + c'$ where $c, c' \in \Lambda^j \mathfrak{m} \otimes F$, $dc = 0$ and $c' \in \operatorname{Im} d_{j-1}'$. Then by (1) we obtain $\varphi(b) = Xc + c' = d(X \otimes c) + c' \in \operatorname{Im} d_{j-1}'$. We conclude that $\tilde{\varphi}_j$ further factors to a linear map $\bar{\varphi}_j$ of $H_0(\mathfrak{n}/\mathfrak{m}, H_j(\mathfrak{m}, F))$ into $H_j(\mathfrak{n}, F)$. The remaining assertions namely $\ker \bar{\varphi}_j = 0$ and $\operatorname{Im} \bar{\varphi}_j = \ker \bar{\psi}_j$ are left to the reader.

3.5. (Notation, 2.1, 2.7). For each $y \in W$ we define $\mathfrak{n}^y = \sum_{\gamma \in R^-} \mathbb{C} X_{y\gamma}$. This is a subalgebra of $\mathfrak{g}$ conjugate to $\mathfrak{n}^-$.

LEMMA: *Assume $\lambda \gg 0$. If $\mu \in \Omega_i(w)$ takes the form $\mu = y\lambda + \sigma$: $\sigma \in \Sigma$ then a representative of the corresponding homology class can be given the form*

$$\sum_{\sigma' \in \Sigma} a_{\sigma'} \otimes f_{y\lambda + \sigma - \sigma'}$$

where $f_{y\lambda+\sigma-\sigma'} \in F(w\lambda)$ *has weight* $y\lambda + \sigma - \sigma'$ *and* $a_{\sigma'} \in \Lambda^*\mathfrak{n}$ *has weight* σ'. *Moreover one can write* $f_{y\lambda+\sigma-\sigma'} = b_{\sigma-\sigma'}e_{y\lambda}$ *for some uniquely determined* $b_{\sigma-\sigma'} \in U(\mathfrak{n}^y)$.

Only the last statement is not obvious. It results from the embedding $F(w\lambda) \hookrightarrow E(\lambda)$, the fact that $e_{y\lambda}$ is a cyclic vector for $E(\lambda)$ considered as a $U(\mathfrak{n}^y)$ module and (for uniqueness) the observations made in the proof of 2.6.

3.6. We should like to show that the a_σ, $b_{\sigma-\sigma'}$, defined above can be chosen in a manner independent of λ (for $\lambda \gg 0$). The situation is analogous to 2.6 and except that one still has to specify which elements of $U^i(\mathfrak{n}^y)e_{y\lambda}$: i small, belong to $F(w\lambda)$. For each $\alpha \in R$, let G_α denote the subalgebra of $U(\mathfrak{g})$ generated by X_α. For each pair w, $y \in W$ with $w \geqslant y$ we define a subspace $H_{w,y}$ of $U(\mathfrak{g})$ inductively as follows. Set $H_{1,1} = \mathbb{C}$. If $w \in W\setminus\{1\}$ choose $\alpha \in B$ such that $w' := s_\alpha w < w$ and assume that $H_{w',y'}$ is defined for all $y' \leqslant w'$. Consider $H_{w,y}$. If $s_\alpha y > y$, then $y \leqslant s_\alpha w = w'$ and we set $H_{w,y} = G_{-\alpha}H_{w',y}$. If $s_\alpha y < y$, then $s_\alpha y \leqslant s_\alpha w = w'$ and we set $H_{w,y} = G_\alpha s_\alpha(H_{w',s_\alpha y})$. The resulting expression can in principle depend on the above choice of α, in particular how w is presented as a reduced decomposition; but we show that this is not so. First one checks that $H_{w,y} \subset U(\mathfrak{n}^y)$. For each $i \in \mathbb{N}$, set $H^i_{w,y} = H_{w,y} \cap U^i(\mathfrak{n}^y)$. Recall that $F(w\lambda)$ identifies in a unique fashion with a $U(\mathfrak{b})$ submodule of $E(\lambda)$.

LEMMA: *Fix* y, $w \in W$ *with* $y \leqslant w$.

$$(i)\ H_{w,y}e_{y\lambda} = F(w\lambda).$$

Suppose $i \ll (\lambda, \alpha^\vee)$, $\forall \alpha \in B$. *Then*

$$(ii)\ xe_{y\lambda} \in F(w\lambda)\colon x \in U^i(\mathfrak{n}^y) \Leftrightarrow x \in H^i_{w,y}.$$

(i) The proof is by induction on $l(w)$. It is obvious for $l(w) = 0$. Assume that the assertion has been proven for $w' = s_\alpha w < w$ with $\alpha \in B$. Then $F(w\lambda)$ is generated by $X_{-\alpha}$ over $F(w'\lambda)$, that is $F(w\lambda) = G_{-\alpha}F(w'\lambda)$. Assume $s_\alpha y > y$. Then $y \leqslant s_\alpha w = w'$ and so $F(w\lambda) = G_{-\alpha}H_{w',y}e_{y\lambda} = H_{w,y}e_{y\lambda}$, as required. Furthermore $F(w\lambda)$ is s_α stable and so $F(w\lambda) = G_\alpha s_\alpha(H_{w',y})e_{s_\alpha y\lambda} = H_{w,s_\alpha y}e_{s_\alpha y\lambda}$, which proves the assertion in the case $s_\alpha y < y$. Hence (i). (ii) follows from (i) and the uniqueness property discussed in 3.5.

REMARK: It follows from (ii) that $H_{w,y}$ is well defined and in particular independent of the reduced decomposition of w implicitly chosen. Combinatorially this is a quite non-trivial fact. For example, in type A_2

it means for example that $G_{-\alpha}G_{-\beta}G_{-\alpha} = G_{-\beta}G_{-\alpha}G_{-\beta}$ where $\{\alpha, \beta\} = B$. This last result was alluded to and can be derived from (I.2.9).

3.7. Combining 3.5, 3.6 obviously proves the essential independence of homology on λ for $\lambda \gg 0$ alluded to in 2.7. More precisely we have the

PROPOSITION: *Assume* $\lambda \gg 0$. *Given* $(y, w, \sigma) \in W \times W \times \Sigma$ *one can choose* $a_{\sigma'}$, $b_{\sigma-\sigma'}$ *in the conclusion of* 3.5 *in a manner independent of* λ *and in particular* $\dim H_i(\mathfrak{n}, F(w\lambda))_{y\lambda+\sigma}$ *is independent of* λ.

For completion we sketch a geometric argument which can also give this result. Let G be the connected, algebraic group with Lie algebra $\mathfrak{g}$ and B a Borel subgroup of G. It is known that each $\nu \in P(R)$ lifts to character on B and we let $\mathscr{L}_\nu$ denote the associated invertible sheaf over G/B. For each $w \in W$, let $\mathscr{L}_\nu(w)$ denote the restriction of $\mathscr{L}_\nu$ to the Schubert variety $\overline{Bn_wB/B}$. Since G/B is projective it follows from Serre's theorem that for $\nu \gg 0$, the space of global sections on $\mathscr{L}_\nu(w)$ coincides with the dual of $F(w\nu)$. Furthermore each $\mathscr{L}_\nu(w)$ admits a finite B equivariant resolution in terms the standard sheaves $\mathscr{L}_\mu$: $\mu \in P(R)$.This resolution may be translated by tensoring it with $\mathscr{L}_\lambda$ over the structure sheaf $\mathscr{L}_0$ and using the relations $\mathscr{L}_\nu(w) \otimes_{\mathscr{L}_0} \mathscr{L}_\lambda \cong \mathscr{L}_{\lambda+\nu}(w)$. This gives a translation principle valid for $\lambda \gg 0$ on the $\mathfrak{h}$ weights of the homology spaces $H_*(\mathfrak{n}, F(w\nu))$.

3.8. We can give an algorithm for computing the $a_{\sigma'}$, $b_{\sigma-\sigma'}$ in the conclusion of 3.5. Assume that these elements have been determined for $H_*(\mathfrak{n}, F(w\lambda))$ and choose $\alpha \in B$ such that $s_\alpha w > w$. Let us show how to compute $H_*(\mathfrak{n}, F(s_\alpha w\lambda))$ using 3.2, 3.3, 3.5 and 3.6. In what follows we can obviously group the elements of $\Omega_*(w)$ into subsets whose members differ by integer multiples of α. Since we assume $\lambda \gg 0$, this means that we can fix $\sigma \in \Sigma$ and $y \in W$ with $s_\alpha y > y$ and restrict our attention to weights of the form $\mu = y\lambda + \sigma + \mathbb{Z}\alpha$.

Now let $a_\mu \in \Lambda^i\mathfrak{n} \otimes F(w\lambda)$ be a representative of a weight vector in $H_j(\mathfrak{n}, F(w\lambda))_\mu$. We can write $a_\mu = X_\alpha \otimes b_{\mu-\alpha} + b'_\mu$ for some unique $b_{\mu-\alpha} \in (\Lambda^{i-1}\mathfrak{m} \otimes F(w\lambda))_{\mu-\alpha}$, $b'_\mu \in (\Lambda^i\mathfrak{m} \otimes F(w\lambda))_\mu$. In the notation of 3.3 we have $b'_\mu \in \ker \bar{\psi}_i = \operatorname{Im} \bar{\varphi}_i$ and $X_\alpha \otimes b_{\mu-\alpha} \in \operatorname{Im} \bar{\psi}_i$. We use 3.6 to determine whether these elements represent zero in $\mathfrak{m}$ homology. In practice it is nearly always unnecessary to do any computation. For example if μ occurs with multiplicity one in $H_i(\mathfrak{n}, F(w\lambda))$ then exactly one of these elements is homologous to zero and it is usually obvious which it must be. (We discuss the possible complete elimination of these ambiguities below). Following through this procedure we obtain according to 3.3 a complete set of lowest and highest weight vectors in the string module decomposition of $H_*(\mathfrak{m}, F(w\lambda))$. To determine the $\mathfrak{c}$

module structure of $H_*(\mathfrak{m}, F(w\lambda))$ it remains to analyse the action of X_α on the lowest weight vectors. If only small powers of X_α are involved then by the uniqueness in 3.6(ii) there is no difficulty of principle in doing this. If large powers of X_α are involved and this corresponds to passing from a weight of the form $s_\alpha y\lambda + \sigma$: $s_\alpha y > y$, $\sigma \in \Sigma$ to a weight of the form $y\lambda + \sigma'$: $\sigma' \in \Sigma$, $\sigma - \sigma' = t\alpha$: $t \in \mathbb{Z}$ (and small), then we observe that $s_\alpha(U(\mathfrak{n}^{s_\alpha y})) = U(\mathfrak{n}^y)$ and use the reflection s_α. In this fashion we can (in principle) decompose $H_*(\mathfrak{m}, F(w\lambda))$ into a sum of string module $F(\mu, \nu)$ and find for each such module highest b_μ and lowest b_ν weight vectors with the property that one of the following hold (in homology):

$$(i)\ \ X_\alpha^t b_\nu = b_\mu$$

$$(ii)\ \ X_\alpha^t s_\alpha(b_\nu) = b_\mu$$

$$(iii)\ \ s_\alpha\left(X_\alpha^t b_\nu\right) = b_\mu$$

where $t \in \mathbb{N}$ is small. (The exact value of t depends on $\mathfrak{g}$. It is bounded by the longest length of an α string in $E(\rho)$). Now from 3.2 we may compute $H_*(\mathfrak{m}, F(s_\alpha w\lambda))$ and hence by 3.3 this gives $H_*(\mathfrak{n}, F(s_\alpha w\lambda))$. We remark that in order to compute representatives in $\Lambda^*\mathfrak{n} \otimes F(s_\alpha w\lambda)$ via the functors $\mathscr{D}_\alpha^0$, $\mathscr{D}_\alpha^1$ we first consider the given representations in $\Lambda^*\mathfrak{m} \otimes F(w\lambda)$ as elements of $\Lambda^*\mathfrak{m} \otimes F(s_\alpha w\lambda)$ described through 3.5, 3.6. We can now use the fact that $F(s_\alpha w\lambda)$ is s_α stable and that small powers of $X_{-\alpha}$ can be applied to any $b \in U(\mathfrak{n}^y)$ if $s_\alpha y > y$. For example suppose (i) above holds with $(\mu, \alpha^\vee) \geqslant 0$. Then we must have $\mu = y\lambda + \sigma$ with $s_\alpha y > y$ (because $\lambda \gg 0$). Then the highest weight vectors in $H_*(\mathfrak{m}, F(s_\alpha w\lambda))$ of the simple $\mathfrak{r}$ submodules arising through $\mathscr{D}_\alpha^0$ from the pair b_μ, b_ν take the form $X_\alpha^t b_\nu = b_\mu$, a suitable linear combination of $X_\alpha^{t-1} b_\nu$, $X_{-\alpha} X_\alpha^t b_\nu$, a suitable linear combination of $X_\alpha^{t-2} b_\nu$, $X_{-\alpha} X_\alpha^{t-1} b_\nu$, $X_{-\alpha}^2 X_\alpha^t b_\nu$ etc. Again if $(\mu, \alpha^\vee) \leqslant 0$, then we must have $\mu = s_\alpha y\lambda + \sigma$ with $s_\alpha y > y$. In this case we replace the pair b_μ, b_ν by the pair $s_\alpha(X_\alpha b_\nu)$, $s_\alpha(X_\alpha b_\nu)$ and proceed as before. This gives terms arising from $\mathscr{D}_\alpha^1$, the action of X_α effects the connecting homomorphism (see (4) of (3.2). Cases (ii) and (iii) are similar.

3.9. For the present the above algorithm seems too complicated for practical calculations. Ideally we should like to replace it by an algorithm which only involves the $\Omega_*(w)$. The trouble is then that ambiguities arise. What we show here is that these ambiguities can be eliminated if one can independently prove that $Y_i \cap Y_{i+1} = \emptyset$, $\forall i \in \mathbb{N}$ (notation 2.7). At least in low rank cases this condition can be verified by certain self-consisting requirements – thus the string module decomposition of $H_*(\mathfrak{m}, F(w\lambda))$

puts a useful constraint on the corresponding weights in $\Omega_*(w)$, the computed value of $\Omega_*(w)$ must be independent of the reduced decomposition of w and finally $\Omega_*(1)$ and $\Omega_*(w_0)$ are known by Bott–Kostant.

3.10. Fix $\alpha \in B$ and let $F(\mu, \nu)$ be a string module for $\mathfrak{c}$ (notation 3.1). Obviously $\mu = \nu + l\alpha$, for some $l \in \mathbb{N}$. If $\mu - s_\alpha \nu \in \mathbb{N}\alpha$, then $\mathscr{D}_\alpha^0 F(\mu, \nu) \neq 0$, $\mathscr{D}_\alpha^1 F(\mu, \nu) = 0$ and we shall say that $F(\mu, \nu)$ is of 0-type. If $\mu + \alpha - s_\alpha \nu \in -\mathbb{N}^+\alpha$; then $\mathscr{D}_\alpha^0 F(\mu, \nu) = 0$, $\mathscr{D}_\alpha^1 F(\mu, \nu) \neq 0$ and we shall say that $F(\mu, \nu)$ is of 1-type. Finally if $\mu + \alpha = s_\alpha \nu$, then $\mathscr{D}_\alpha^0 F(\mu, \nu) = \mathscr{D}_\alpha^1 F(\mu, \nu) = 0$ and we shall say that $F(\mu, \nu)$ is of vanishing type.

Consider the problem of determining $H_*(\mathfrak{n}, E)$ from $H_*(\mathfrak{n}, F)$ under the hypotheses of 3.2. As we noted above we can clearly assume without loss of generality that every weight of $H_*(\mathfrak{n}, F)$ takes the form $\mu + \mathbb{Z}\alpha$ for some fixed $\mu \in P(R)$. It is convenient to assume $(\mu, \alpha) = 0$ but then we have to allow for possible half-integer multiples of α. Then ignoring μ which will play no further role we can assume that every weight of $H_*(\mathfrak{n}, F)$ takes the form $l\alpha$ where either $l \in \mathbb{Z}$ or $l + \frac{1}{2} \in \mathbb{Z}$.

Set $p = e^\alpha$ and consider the Laurent polynomial $P_F(q, p)$ defined by

$$P_F(q, p) = \sum_{j=0}^{\infty} \sum_{l \in \frac{1}{2}\mathbb{Z}} (-q)^j \dim H_j(\mathfrak{n}, F)_{l\alpha} p^l.$$

Our aim is deduce from $P_F(q, p)$ the polynomial $P_E(q, p)$ defined by replacing F by E in the above. In this we consider $P_F(q, p)$ as a sum of terms of the form $(-q)^j p^l$. By 3.3 each such term represents relative to string module decomposition either a lowest weight vector in $H_j(\mathfrak{m}, F)$ of weight $l\alpha$ or a highest weight vector in $H_{j-1}(\mathfrak{m}, F)$ of weight $(l-1)\alpha$. We define the following three operations on each such term.

Γ_1). If $(-q)^j p^l$ represents a highest weight vector of weight $(l-1)\alpha$ in a string submodule of $H_{j-1}(\mathfrak{m}, F)$ of 0-type, replace $(-q)^j p^l$ by $-(-q)^{j-1} p^l$. If $(-q)^j p^l$ represents a lowest weight vector of weight $l\alpha$ in a string submodule of $H_j(\mathfrak{m}, F)$ of 1-type replace $(-q)^j p^l$ by $-(-q)^{j+1} p^l$. The terms corresponding to vanishing type may be included in either one of the above or simply dropped.

Γ_2). Replace p^l by $(p^l + p^{-l})(1 - p^{-1})^{-1}$.

Γ_3). Replace p^l: $l \in \mathbb{N}$ by qp^l.

PROPOSITION: *Under the hypothesis of* 3.2 *one has* $\Gamma_3\Gamma_2\Gamma_1 P_F(q, p) = P_E(q, p)$.

REMARKS: Only Γ_1 is not completely defined and its proper definition needs some knowledge (though not complete knowledge) of the $\mathfrak{c}$ module structure of $H_*(\mathfrak{m}, F)$. The operation Γ_2 gives a Laurent polynomial in y because it is always applied to differences $p^s - p^t$: $s - t \in \mathbb{N}^+$. For example

$$\Gamma_2(p^s - p^t) = \frac{p^s - p^t}{(1-p^{-1})} + \frac{p^{-s} - p^{-t}}{(1-p^{-1})},$$

$$= (p^s + p^{s-1} + \cdots + p^{t+1})$$

$$-(p^{-(s-1)} + p^{-s} + \cdots + p^{-t}). \qquad (*)$$

The case $s = -t$ corresponds to a string module of vanishing type. Finally Γ_3 recovers the $\mathfrak{h}$ module structure of $H_*(\mathfrak{n}, E)$ from the $\mathfrak{r}$ module structure of $H_*(\mathfrak{m}, E)$.

The proof is straightforward though somewhat dull. First assume that the pair $(-q)^{j_1}p^{l_1}$, $(-q)^{j_2}p^{l_2}$ represent respectively highest and lowest weight vectors of a string module $F(\mu, \nu)$ of 0-type in $H_j(\mathfrak{m}, F)$. We must have $j_1 = j+1$, $j_2 = j$, $l_1 - l_2 \in \mathbb{N}^+$ and we can write $\mu = (l_1 - 1)\alpha$, $\nu = l_2\alpha$. Set $s = l_1$, $t = l_2$. We must consider two cases:

a) $t \geqslant 0$.

In this case by (I, 2.4) the highest weight vectors of the $\mathfrak{r}$ module $\mathscr{D}^0_\alpha F(\mu, \nu)$ take the form $(s-1)\alpha, (s-2)\alpha, \ldots, t$. Recalling the shift in α it follows that the above pair give rise to the term

$$(-q)^{j+1}(p^s + p^{s-1} + \cdots + p^{t+1}) + (-q)^j(p^{-(s-1)} + \cdots + p^{-t})$$

in $P_E(q, p)$. Comparison with $(*)$ above gives the required assertion.

b) $t \leqslant 0$.

In this case by (I, 2.4) we must first replace t by $-t$ and then proceed as before. Since only Γ_2 involves p and $\Gamma(p^t) = \Gamma(p^{-t})$ the assertion also obtains in this case.

Now assume that the pair $q^{j_1}p^{l_1}$, $q^{j_2}p^{l_2}$ represent respectively highest and lowest weight vectors of a string submodule $F(\mu, \nu)$ of 1-type in $H_j(\mathfrak{m}, F)$. As before $j_1 = j+1$, $j_2 = j$; but now $l_1 - l_2 \in \mathbb{N}^-$ and we can write $\mu = (l_1 - 1)\alpha$, $\nu = l_2\alpha$. Set $s = l_1$, $t = l_2$. Let Q denote the string module which is the extension of $F(s_\alpha(\nu + \alpha), \mu + \alpha)$ by $F(\mu, \nu)$. One checks that $\mathscr{D}^i_\alpha Q = 0$, $\forall i$ and so we obtain that $\mathscr{D}^1_\alpha F(\mu, \nu) \xrightarrow{\sim} \mathscr{D}^0_\alpha F(s_\alpha(\nu + \alpha), \mu + \alpha)$. It follows that we obtain the same result as before if we take $\mu = -l_2\alpha - \alpha = -(t+1)\alpha$, $\nu = l_1\alpha = s\alpha$. This corresponds to interchanging s, t and replacing s by $-s$. Neither operation affects Γ_2. Finally we must recall that $\mathscr{D}^1_\alpha H_j(\mathfrak{m}, F)$ contributes to $H_{j+1}(\mathfrak{m}, E)$ and so j_1 must be increased by one. This proves the proposition.

3.11. One may note that 3.2 gives unambiguously the Euler characteristic associated to $H_*(\mathfrak{m}, E)$ from $H_*(\mathfrak{m}, F)$. Indeed set

$$\chi(\mathfrak{m}, E) = \sum_{k=0}^{\infty} (-1)^k \text{ ch } H_k(\mathfrak{m}, E)$$

with a similar definition for $\chi(\mathfrak{m}, F)$. Then from 3.2 we obtain

$$\begin{aligned}\chi(\mathfrak{m}, E) &= \sum_{k=0}^{\infty} (-1)^k \text{ ch } H_k(\mathfrak{m}, E)\\ &= \sum_{k=0}^{\infty} (-1)^k (\text{ch } \mathscr{D}_\alpha^0 H_k(\mathfrak{m}, F) - \text{ch } \mathscr{D}_\alpha^1 H_k(\mathfrak{m}, F))\\ &= \sum_{k=0}^{\infty} (-1)^k \Delta_\alpha \text{ ch } H_k(\mathfrak{m}, F), \quad \text{by (I, 5.4(iv))}\\ &= \Delta_\alpha \chi(\mathfrak{m}, F).\end{aligned}$$

In this computation any terms corresponding to weight spaces in adjacent homologies having the same weight space must cancel. One might anticipate that if there were no such occurrences, then $H_*(\mathfrak{m}, E)$ could be unambiguously deduced from $H_*(\mathfrak{m}, F)$. We say that $H_*(\mathfrak{m}, E)$ is *cancellation-free* if whenever $H_j(\mathfrak{n}, E)_\mu \neq 0$ and $H_{j+1}(\mathfrak{n}, E)_\nu \neq 0$ one has $\mu \neq \nu$. Now from 3.10 (because of the ambiguity in Γ_1) we obtain a number of possible solutions for $H_*(\mathfrak{n}, E)$ depending on the different possible choices for the $\mathfrak{c}$ module structure of $H_*(\mathfrak{m}, F)$ consistent with a knowledge of $H_*(\mathfrak{n}, F)$. We shall say that a solution is *cancellation-free* if the resulting $H_*(\mathfrak{n}, E)$ is cancellation-free. It can happen that there are no cancellation free solutions. For example suppose $P_F(q, p) = q^2p^3 - qp^2 - q + p$. Then q^2p^3 must be paired with $-q$ because $-q$ cannot be paired with p. Then $\Gamma_1 P_F(q, p) = qp^3 - q - p^2 + p$, so $\Gamma_2\Gamma_1 P_F(q, p) = q\{(p^3 + p^2 + p) - (p^{-2} + p^{-1} + 1)\} - (p^2 - p^{-1})$ and then $P_E(q, p) = q\{q(p^3 + p) - (p^{-2} + 1)\} + (q - 1)(qp^2 - p^{-1})$ which is not cancellation-free. Again $P_F(q, p) = q^2p^3 - qp^2 - qp + 1$ admits a cancellation-free solution (pair q^2p^3, $-qp^2$) and one which is not cancellation-free (pair q^2p^3, $-qp$). Nevertheless one has the

PROPOSITION: *If $P_F(q, p)$ admits a cancellation-free solution then it is unique.*

REMARK: In other words though there may be several possible different $\mathfrak{c}$ module structures on $H_*(\mathfrak{m}, F)$ consistent with a knowledge of $H_*(\mathfrak{n}, F)$, all those giving a cancellation-free solution give the same solution.

The proof is by induction on $\Sigma_j \dim H_j(\mathfrak{m}, F) = P_F(-1, 1)$ which is always even. Roughly speaking we pick a term with a known interpretation and pair it with as close as possible a neighbour and show that this gives the unique cancellation free solution.

If the term $(-q)^j p^s$ occurs in $P_F(q, p)$ with (of course) a positive integer coefficient we shall simply write $(-q)^j p^s \in P_F(q, p)$. Set

$$s_0 = \max\left\{ |s| \text{ such that } (-q)^j p^s \in P_F(q, p)\right\}.$$

We let $(-q)^j p^s$ denote a term with $|s| = s_0$. We must consider two cases:

(1) $s = s_0$.

In this case $(-q)^j p^s$ represents a highest weight (of weight $(s-1)\alpha$) of a string submodule $F(\mu, \nu)$ of $H_{j-1}(\mathfrak{m}, F)$. We can write $\nu = t\alpha$ where $s - |t| \in \mathbb{N}^+$ and then this term gives the contribution $(-q)^{j-1}p^t$ to $P_F(q, p)$. Now consider possible terms in $P_F(q, p)$ of the form $(-q)^{j-1}p^l$ with $s \geqslant |l| > |t|$. If there are no such terms we can identify the term $(-q)^{j-1}p^t$ as the uniquely determined monomial in $P_F(x, y)$ with exponent in q equal to $j-1$ and exponent in p having the maximum possible modulus. Let us show that we can always assume that there are no such terms without loss of generality. Indeed suppose we having a term in $P_F(q, p)$ of the form $(-q)^{j-1}p^l$ with $s \geqslant |l| > |t|$ and assume $|l|$ is as large as possible. This term can represent (up to the appropriate translation by α) four possible types of weight spaces.

(i) A highest weight of a string module of 0-type in $H_{j-2}(\mathfrak{m}, F)$.

(ii) A lowest weight of a string module of 0-type in $H_{j-1}(\mathfrak{m}, F)$.

(iii) A highest weight of a string module of 1-type in $H_{j-2}(\mathfrak{m}, F)$.

(iv) A lowest weight of a string module of 1-type in $H_{j-1}(\mathfrak{m}, F)$.

(The weights of string modulus of vanishing type can be included in any one of the above).

If (i) or (iv) occurs one checks that $P_E(q, p)$ is not cancellation free. The cancellation is essentially the same in both cases and similar to that given in the above example. Suppose (ii) holds. Then the term $(-q)^{j-1}p^l$ must be paired to a term of the form $(-q)^j p^{l'}$ with $|l'| > |l|$. Then $s \geqslant |l'| > |l|$ by definition of s and $|l'| > |l| > t$ by choice of l. Thus instead of the above pairing, we can consider that $(-q)^j p^s$, $(-q)^{j-1}p^l$ are paired and so are $(-q)^j p^{l'}$, $(-q)^{j-1}p^s$. From the form of Γ_1 it is obvious that both choices of pairings give the same solution. A similar conclusion holds in type (iii) and to all the remaining terms in $P_F(q, p)$ of the form $(-q)^{j-1}p^l$ with $s \geqslant |l| > |t|$. This proves our assertion.

(2) $s = -s_0$. A similar argument shows that if a cancellation free solution exists then we can assume that $(-q)^j p^s$ is paired to the term $(-q)^{j+1}p^l$ where $|l|$ is as large as possible amongst the exponents of y in the coefficient of $(-q)^{j+1}$.

This completes the proof of the proposition.

3.12. Remember that we ignored a factor of e^{μ} with $(\mu, \alpha) = 0$, the operation Γ_2: $p \mapsto (p^s + p^{-s})(1 - p^{-1})^{-1}$ can be interpreted as the operation $e^{\mu} \mapsto (e^{\mu} + e^{s_\alpha \mu})(1 - e^{-\alpha})^{-1}$. If we divide $P_F(q, p)$ and $P_E(q, p)$ by the factor

$$\prod_{\alpha \in R^+} (1 - e^{\alpha})$$

then this operation is just Demazure's Δ_α (notation I, 1.4).

4. On a theorem of Tolpygo

4.1. Let V be a non-zero finite dimensional $U(\mathfrak{b})$ module (with a semi-simple action of $\mathfrak{h}$) and let $\Omega(V)$ denote its set of weights. Let d denote the boundary operator in $\mathfrak{n}$ homology. Comparison of 2.9 with a recent result of Tolpygo [2] motivates the following

LEMMA: *Fix $z \in W$ and set*

$$a_z = \bigwedge_{\alpha \in S(z)} X_\alpha .$$

Choose $\delta \in \Omega(V)$ such that $\delta + \alpha \notin \Omega(V)$, $\forall \alpha \in S(z)$ and $\delta - \alpha \notin \Omega(V)$. $\forall \alpha \in R^+ \setminus S(z)$ (for example if $(\delta, z^{-1}\rho)$ takes a minimal value). Then for each $e \in V_\delta$ one has*
(i) $d(e \otimes a_z) = 0$.
(ii) $e \otimes a_z \in \operatorname{Im} d$ implies $e = 0$.

REMARK: Compare * with the construction of ([2], 2.1):
(i) Take $\alpha \in S(z)$. Since $\alpha + \delta \notin \Omega(V)$ we have $X_\alpha e = 0$. Yet β, $\gamma \in S(z)$, $\beta + \gamma \in R^+$ implies $\beta + \gamma \in S(z)$, so this gives (i).
(ii) Suppose $e \otimes a_z \in \operatorname{Im} d$. We can write $e \otimes a_z = db$ where

$$b = \sum f_i \otimes \bigwedge_{\alpha \in S_i} X_\alpha$$

with $f_i \in V$ and $S_i \subset R^+$ satisfying card $S_i = 1 + l(z)$. Since $\beta, \gamma \notin S(z)$, $\beta + \gamma \in R^+$ implies $\beta + \gamma \notin S(z)$ we cannot write $S(z)$ in the form $S_i \cap \{\beta + \gamma\} \setminus \{\beta, \gamma\}$. We conclude that there exist $\alpha_i \in R^+ \setminus S(z)$ such that $\sum X_{\alpha_i} f_i = e$. If $e \neq 0$ it follows that there exists $\alpha \in R^+ \setminus S(z)$ such that $\delta - \alpha \in \Omega(V)$. This contradicts the choice of δ.

4.2. Retain the above hypotheses. In [2], Tolpygo shows that $\dim H_*(\mathfrak{n}, V) \geqslant \operatorname{card} W$. This is obviously an immediate consequence of 4.1 which is a stronger statement incorporating part of 2.9. Moreover the apparently harder result below also follows easily from 4.1.

THEOREM: (Tolpygo [2]). *Suppose* dim $H_*(\mathfrak{n}, V) = \text{card } W$. *Then there exists* $\lambda \in \mathfrak{h}^*$ *such that* $V \otimes \mathbb{C}_\lambda$ *admits the structure of a simple finite dimensional* $U(\mathfrak{g})$ *module.*

The hypothesis and 4.1 implies that dim $H_0(\mathfrak{n}, V) = 1$. Hence V is a cyclic $U(\mathfrak{n})$ module generated by a weight vector e_{μ_0} of weight μ_0. For each $\alpha_i \in B$ let k_i be the largest integer $\geqslant 0$ such that $e_i := X_{\alpha_i}^{k_i} e_{\mu_0} \neq 0$. Then e_i has weight $\mu_0 + k_i\alpha_i$. Take $\beta \in R^+ \setminus \{\alpha_i\}$ and suppose that $\mu_0 + k_i\alpha_i - \beta \in \Omega(V)$. Then if f is a corresponding weight vector in V we can write $f = a_\nu e_{\mu_0}$ where $a_\nu \in U(\mathfrak{n})$ has weight $k_i\alpha_i - \beta$. This is of course impossible. Similarly $\mu_0 + (k_i + 1)\alpha_i \notin \Omega(V)$. From 4.1 we conclude that $e_i \otimes X_{\alpha_i} \in H_1(\mathfrak{n}, V)$. Now the hypothesis implies that dim $H_1(\mathfrak{n}, V) = \text{card } B$ and so we conclude that the above elements form a basis for $H_1(\mathfrak{n}, V)$. It follows that V is isomorphic to the quotient of $U(\mathfrak{n}) \otimes \mathbb{C}_{\mu_0}$ by its submodule

$$\sum_{\alpha_i \in B} \mathbb{C} X_{\alpha_i}^{k_i+1} \otimes \mathbb{C}_{\mu_0},$$

as required.

REMARK: The last step above is just the last step of [2]; but we have been able to avoid Tolpygo's complicated geometric arguments.

I should like to thank J. Dixmier and J. Humphreys for drawing my attention to [2].

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