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RAPIDLY DECREASING FUNCTIONS ON GENERAL SEMISIMPLE GROUPS

Rebecca A. Herb * and Joseph A. Wolf **

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§0. Introduction

In [4] we extended Harish-Chandra's Plancherel formula for semisimple Lie groups with finite center (see [2d,e,f]) to a class of reductive Lie groups which contains all connected semisimple real Lie groups, such as the simply connected semisimple groups associated to bounded symmetric domains, which are not in Harish-Chandra's class. See §1 for the precise definition of our class of groups.

Our proof of the Plancherel formula relied on computing explicit Fourier transforms of the orbital integrals of C^∞ compactly supported functions using character formulas for the representations of the group G and harmonic analysis on its Cartan subgroups. This is in contrast to Harish-Chandra's proof which relies on decomposing K-finite functions in the Schwartz space as sums of cusp forms and wave packets corresponding to the various series of induced representations and then exploiting the connection between the Plancherel densities and the c-functions giving the asymptotics of the Eisenstein integrals.

In this paper we extend our Plancherel formulas from C^∞ compactly supported functions to Schwartz class (rapidly decreasing) functions.

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There are actually two versions of the Plancherel theorem in [4]. One is a relative Plancherel formula for functions which transform by a character along the center and are compactly supported mod center. This is extended to a relative Schwartz space of functions with growth along the center controlled by the central character which are rapidly decreasing in the sense of Harish-Chandra mod center. The second (global) version of the Plancherel formula is for $C_c^\infty(G)$. This is extended to a global Schwartz space of functions which are rapidly decreasing in all directions, including that of the possibly infinite center.

In the case that G is of Harish-Chandra class, our global Schwartz space is the same as that defined by Harish-Chandra and our method gives a direct route to Harish-Chandra's Plancherel formula that does not rely on the machinery of Eisenstein integrals and wave packets needed for the decomposition of the Schwartz space.

Of course the decomposition of the Schwartz space is important despite the fact that it is no longer needed for the proof of the Plancherel formula. We do the first step in this decomposition for our general class of groups by showing how to break up the K-finite relative Schwartz space into the direct sum of the space of relative cusp forms and its orthogonal complement. Here the arguments rely heavily on Harish-Chandra's ideas although some of the analytic arguments are simplified by using the theory of leading characters of Casselman and Miličić [1]. These results cannot be extended in a simple way to the global Schwartz space since they depend on the use of K-finite and $\mathscr{Z}(\mathfrak{g})$-finite functions which do not exist in the global Schwartz space if the center of the derived group of G is infinite.

The organization of the paper is as follows:

In §1 we define our general class of groups, set up notation, and state the two versions (relative and global) of the Plancherel formula proved in [4].

In §2 we define the relative Schwartz space and establish its basic properties. In particular, we show that C^∞, compactly supported mod center, functions are dense in the relative Schwartz space and give a necessary and sufficient condition for a central, $\mathscr{Z}(\mathfrak{g})$-finite distribution to be tempered, i.e., extend continuously to the relative Schwartz space.

In §3 we show that the characters of all the representations of G appearing in the relative Plancherel formula are tempered and extend the relative Plancherel formula to the relative Schwartz space by continuity.

In §4 we do the first step in the decomposition of the K-finite relative Schwartz space by showing how it splits as the direct sum of the subspace corresponding to the relative discrete series representations of G and its orthogonal complement, the subspace corresponding to the various series of induced representations.

In §5 we relate the decomposition obtained in §4 to Harish-Chandra's theory of cusp forms.

In §6 we define the global Schwartz space for G and show it has the same basic properties as Harish-Chandra's Schwartz space.

In §7 we extend the global Plancherel theorem to the global Schwartz space. This is done via the results of §3 by showing that each Fourier coefficient of a global Schwartz class function with respect to a character of the center is in the corresponding relative Schwartz space.

The reader can proceed directly from the relative Plancherel theorem in §3 to the global Plancherel theorem in §7 by skipping over §§4 and 5.

§1. Preliminaries

As in [4] we work with reductive Lie groups G such that

$$\text{If } x \in G \text{ then } \mathrm{Ad}(x) \text{ is an inner automorphism of } \mathfrak{g}_{\mathbf{C}} \tag{1.1a}$$

and G has a closed normal abelian subgroup Z such that

$$\begin{cases} Z \text{ centralizes the identity component } G^0 \text{ of } G, \\ ZG^0 \text{ has finite index in } G, \text{ and} \\ Z \cap G^0 \text{ is co-compact in the center } Z_{G^0} \text{ of } G^0. \end{cases} \tag{1.1b}$$

This class of reductive groups contains every connected semisimple Lie group and is stable under passage to Levi components of cuspidal parabolic subgroups. The "Harish-Chandra class" of groups is the subclass for which G/G^0 and the center of $[G^0, G^0]$ are finite.

Recall [6] that a *Cartan involution* of G means an involutive automorphism θ such that the fixed point set $K = G^\theta$ is the full inverse image of a maximal compact subgroup of Ad(G). As in the usual case, every maximal compact subalgebra of $[\mathfrak{g}, \mathfrak{g}]$ corresponds to a unique Cartan involution of G, any two Cartan involutions are Ad(G^0)-conjugate, and the corresponding maximal compactly embedded subgroups meet every component of G.

Fix a Cartan involution θ of G and let K be the corresponding maximal compactly embedded subgroup. For any θ-stable Cartan subgroup J of G we will write $J = J_K J_p$ where $J_K = J \cap K$ and the Lie algebra of J_p is in the (-1)-eigenspace for θ. The set of roots of $\mathfrak{g}_{\mathbf{C}}$ with respect to $\mathfrak{j}_{\mathbf{C}}$ will be denoted by $\Phi = \Phi(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})$. The subset of Φ taking real values on $\mathfrak{j}$ will be denoted by $\Phi_R(\mathfrak{g}, \mathfrak{j})$. The corresponding Weyl groups will be denoted by $W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})$ and $W_R(\mathfrak{g}, \mathfrak{j})$. Φ^+ denotes a choice of positive roots in Φ, and $\rho(\Phi^+) = \frac{1}{2}\Sigma\alpha$, $\alpha \in \Phi^+$.

Let $W(G, J) = N_G(J)/J$ where $N_G(J)$ is the normalizer in G of J. Then $W(G, J)$ acts on $\mathfrak{j}$, but not necessarily on J since J need not be abelian. Write J_0 for the center of J and define $W(G, J_0) = N_G(J)/J_0$. In contrast, we write superscript ^{0} to denote the identity component. Let $L_J = Z_G(J_p)$, the centralizer in G of J_p, and write $L_J = M_J J_p$ in its Langlands decomposition.

Fix a θ-stable Cartan subgroup H of G and write $T = H_K$, $A = H_p$, $L = L_H$, $M = M_H$. There is a series of unitary representations of G associated to H as follows (for details see [6]). T^0 is a compact Cartan subgroup of the reductive group M^0. Let Φ^+ denote a choice of positive roots for $\Phi = \Phi(\mathfrak{m}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$, $\rho = \rho(\Phi^+)$. Let $L = \{\tau \in i\mathfrak{t}^*: \xi_{\tau-\rho}(\exp H) = \exp(\tau - \rho)(H)$ gives a well-defined character of $T^0\}$, $L' = \{\tau \in L: \langle \tau, \alpha \rangle \neq 0$ for all $\alpha \in \Phi\}$. For $\tau \in L'$, set $\epsilon(\tau) = \operatorname{sign} \prod_{\alpha \in \Phi^+} \langle \alpha, \tau \rangle$. Let $q_H = \frac{1}{2} \dim(M/M \cap K)$. Corresponding to each $\tau \in L'$ there is a discrete series representation π_τ of M^0 with character Θ_τ which is given on $(T^0)'$ by

$$\Theta_\tau(t) = \epsilon(\tau)(-1)^{q_H} \Delta^M(t)^{-1} \sum_{w \in W(M^0, T^0)} \det w\, \xi_{w\tau}(t) \tag{1.2}$$

where $\Delta^M(t) = \xi_\rho(t) \prod_{\alpha \in \Phi^+}(1 - \xi_{-\alpha}(t))$, $\rho = \rho(\Phi^+)$.

Note that if M^0 is not acceptable, $\Delta^M(t)$ and $\xi_{w\tau}(t)$, $w \in W(M^0, T^0)$, $\tau \in L'$, are not separately well-defined on T^0. However $\Delta^M(t)^{-1}\xi_{wt}(t)$ is well-defined for $t \in (T^0)'$ by the definition of L.

We will need formulas for Θ_τ on noncompact Cartan subgroups of M^0. As in [6] the formula of Harish-Chandra for this situation, which we state as (1.3), can be extended to the compact center case without difficulties.

Let J be a θ-stable Cartan subgroup of M^0, J_K^1 a connected component of J_K. We can assume that $J_K^1 \subseteq T^0$ and let y denote an element of $\operatorname{Int}(\mathfrak{m}_{\mathbf{C}})$ which gives the Cayley transform Ad y: $\mathfrak{t}_{\mathbf{C}} \to \mathfrak{j}_{\mathbf{C}}$. For $j \in J$, let $\Delta^M(j) = \xi_\rho(j) \prod_{\alpha \in {}^y\Phi^+}(1 - \xi_{-\alpha}(j))$, $\rho = \rho({}^y\Phi^+)$. Write $W_R = W_R(\mathfrak{m}, \mathfrak{j})$, $W_K = W_R \cap {}^yW(M^0, T^0)$. Then for $j_k \in J_K^1$ and $a \in J_p$ such that $j_k a \in J'$ we have

$$\begin{aligned} \Theta_\tau(j_k a) = \epsilon(\tau)(-1)^{q_H} \Delta^M(j_k a)^{-1} & \sum_{w \in W(M^0, T^0)} \det w\, \xi_{w\tau}(j_k) \\ & \times \sum_{s \in W_R/W_K} \det s\, c(s: w\tau: \Phi_R^+(j_k a)) \\ & \times \exp(s^y(w\tau)(\log a)), \end{aligned} \tag{1.3}$$

where $\Phi_R^+(j_k a) = \{\alpha \in \Phi_R(\mathfrak{m}, \mathfrak{j}): \xi_\alpha(j_k a) > 1\}$ and

$$c(s: w\tau: \Phi_R^+(j_k a)) = 0 \text{ unless } s^y(w\tau)(\log a) < 0. \tag{1.4}$$

Formulas for the constants $c(s: \tau: \Phi_R^+)$ are given in [3]; or see [4].

Let $M^\dagger = Z_M(M^0)M^0$. Note that $Z_{M^0} = Z_M(M^0) \cap M^0 = Z_M(M^0) \cap T^0$. For $\chi \in Z_M(M^0)\hat{}$, the set of irreducible unitary representations of $Z_M(M^0)$, let

$$L'_\chi = \left\{ \tau \in L' \colon \operatorname{tr} \chi|_{Z_{M^0}} = \deg \chi \xi_{\tau-\rho}|_{Z_{M^0}} \right\}, \ \rho = \rho(\Phi^+(\mathfrak{m}_\mathbf{C}, \mathfrak{t}_\mathbf{C})).$$

Then for $\chi \in Z_M(M^0)\hat{}$, $\tau \in L'_\chi$, there is a character of M, supported on $M^\dagger$, given by

$$\begin{aligned} \Theta_{\chi,\tau}(zm) &= \sum_{x \in M/M^\dagger} \operatorname{tr} \chi(xzx^{-1}) \Theta_\tau(xmx^{-1}), \\ z &\in Z_M(M^0),\ m \in M^0. \end{aligned} \tag{1.5}$$

It is the character of the discrete series representation $\pi_{\chi,\tau} = \operatorname{Ind}_{M^\dagger}^M(\chi \otimes \pi_\tau)$ of M.

Now for any $\nu \in \mathfrak{a}^*$, $\Theta_{\chi,\tau} \otimes e^{i\nu}$ is a character of $L = MA$. If $P = MAN$ is a parabolic subgroup with Levi factor L, we denote by $\Theta(H\colon \chi\colon \tau\colon \nu)$ the character of G induced from $\Theta_{\chi,\tau} \otimes e^{i\nu} \otimes 1$ on P, i.e., the character of $\pi(H\colon \chi\colon \tau\colon \nu) = \operatorname{Ind}_P^G(\pi_{\chi,\tau} \otimes e^{i\nu} \otimes 1)$. It is supported on the Cartan subgroups of G conjugate to those in L.

For $J \subseteq L$ a Cartan subgroup of G, let $J_1, \ldots, J_k$ denote a complete set of representatives for the L-conjugacy classes of Cartan subgroups of L which are conjugate to J in G. For $j \in J$, write $j_i = x_i j x_i^{-1}$ where $x_i \in G$ satisfies $x_i J x_i^{-1} = J_i$. Then for $j \in J'$,

$$\begin{aligned} \Theta(H\colon \chi\colon \tau\colon \nu)(j) = \sum_{i=1}^{k} [W(L, J_{i,0})]^{-1} |\Delta^G(j_i)|^{-1} \\ \times \sum_{w \in W(G, J_{i,0})} |\Delta^L(wj_i)| \big(\Theta_{\chi,\tau} \otimes e^{i\nu}\big)(wj_i). \end{aligned} \tag{1.6}$$

Here for any Cartan subgroup $J \subseteq L$, if $j \in J$ then

$$\Delta^G(j) = \xi_{\rho_G}(j) \prod_{\alpha \in \Phi^+(\mathfrak{g}_\mathbf{C}, \mathfrak{j}_\mathbf{C})} (1 - \xi_{-\alpha}(j))$$

and

$$\Delta^L(j) = \xi_{\rho_L}(j) \prod_{\alpha \in \Phi^+(\mathfrak{l}_\mathbf{C}, \mathfrak{j}_\mathbf{C})} (1 - \xi_{-\alpha}(j)),$$

$$\rho_G = \rho(\Phi^+(\mathfrak{g}_\mathbf{C}, \mathfrak{j}_\mathbf{C})),\ \rho_L = \rho(\Phi^+(\mathfrak{l}_\mathbf{C}, \mathfrak{j}_\mathbf{C})).$$

Δ^G and Δ^L may not be well defined on J, but the absolute values are well defined on $J \cap M^{\dagger}A$. Note that $|\Delta^G(j)|^2 = |D_l(j)|$ where, for any $x \in G$, $D_l(x)$ is the coefficient of t^l, $l = \text{rank } G$, in $\det(t + 1 - \text{Ad } x)$.

Write $\mathscr{Z} = \mathscr{Z}(\mathfrak{g})$ for the center of $\mathscr{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Let γ: $\mathscr{Z} \to I(\mathfrak{h}_{\mathbb{C}})$ be the canonical isomorphism of $\mathscr{Z}$ onto the Weyl group invariants in the symmetric algebra of $\mathfrak{h}_{\mathbb{C}}$. Then $\pi(H: \chi: \tau: \nu)$ has infinitesimal character $\chi_{\tau + i\nu}$ given by $\chi_{\tau+i\nu}(z) = \gamma(z)(\tau + i\nu)$, $z \in \mathscr{Z}$. In particular, if ω is the Casimir element of $\mathscr{Z}$, then $\chi_{\tau+i\nu}(\omega) = \|\tau\|^2 + \|\nu\|^2 - \|\rho\|^2$, $\rho = \rho(\Phi^+(\mathfrak{g}_{\mathbb{C}}\mathfrak{h}_{\mathbb{C}}))$.

Let Z be the abelian normal subgroup of G given in (1.1b). Then $Z \subseteq Z_M(M^0)$. For $\zeta \in \hat{Z}$ write $Z_M(M^0)^{\wedge}_{\zeta} = \{\chi \in Z_M(M^0)^{\wedge} : \chi|_Z$ contains a multiple of $\zeta\}$. Then for $\chi \in Z_M(M^0)^{\wedge}_{\zeta}$, $\tau = L'_{\chi}$, and $\nu \in \mathfrak{a}^*$, each irreducible component of $\pi(H: \chi: \tau: \nu)$ is in $\hat{G}_{\zeta} = \{\pi \in \hat{G} : \pi|_Z$ contains a multiple of $\zeta\}$. Let $C_c^{\infty}(G/Z, \zeta) = \{f \in C^{\infty}(G) : f(xz) = \zeta(z)^{-1}f(x)$ for all $x \in G$, $z \in Z$, and $|f|$ is compactly supported mod $Z\}$. Suppose now that Z is central in G. Then the character Θ_π of $\pi \in \hat{G}_{\zeta}$ satisfies $\Theta_\pi(xz) = \Theta_\pi(x)\zeta(z)$, $x \in G$, $z \in Z$, so we can define

$$\Theta_\pi(f) = \int_{G/Z} \Theta_\pi(x) f(x)\, dx,\; f \in C^{\infty}(G/Z, \zeta). \tag{1.7}$$

For arbitrary G of type (1.1) and $\pi \in \hat{G}$ with character Θ_π we define as usual

$$\Theta_\pi(f) = \int_G \Theta_\pi(x) f(x)\, dx,\; f \in C_c^{\infty}(G). \tag{1.8}$$

If Z is compact and central, the two definitions coincide for appropriate normalizations of the Haar measures.

For $\alpha \in \Phi_R(\mathfrak{g}, \mathfrak{h})$ let H_α^* be the element of $\mathfrak{a}$ dual to $\alpha^{\vee} = 2\alpha/\langle \alpha, \alpha \rangle$ under the Killing form. Let X_α, Y_α be elements of the root spaces $\mathfrak{g}_\alpha$, $\mathfrak{g}_{-\alpha}$ respectively so that $\theta(X_\alpha) = Y_\alpha$ and $[X_\alpha, Y_\alpha] = H_\alpha^*$. Write $Z_\alpha = X_\alpha - Y_\alpha$ and set $\gamma_\alpha = \exp(\pi Z_\alpha)$. Then $\gamma_\alpha \in Z_M(M^0)$.

For $\chi \in Z_M(M^0)^{\wedge}$ and $\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{h})$, it is proved in [4, Lemma 4.16] that $\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})$ is a scalar matrix of the form $(\eta_\alpha + \eta_\alpha^{-1})I_k$ where $k = \deg \chi$ and $|\eta_\alpha| = 1$. For $\chi \in Z_M(M^0)^{\wedge}$, $\nu \in \mathfrak{a}^*$, and $\alpha \in \Phi_R(\mathfrak{g}, \mathfrak{h})$, define

$$\bar{p}_\alpha(\chi: \nu) = \frac{\sinh \pi\nu_\alpha}{\cosh \pi\nu_\alpha - c_\alpha(\chi)} \tag{1.9}$$

where $c_\alpha(\chi) = \frac{1}{2}(\deg \chi)^{-1}\xi_{\rho_\alpha}(\gamma_\alpha)\, \text{tr}(\chi(\gamma_\alpha) + \chi(\gamma_\alpha^{-1})) = \frac{1}{2}\xi_{\rho_\alpha}(\gamma_\alpha)(\eta_\alpha + \eta_\alpha^{-1})$. Here $\nu_\alpha = 2\langle \nu, \alpha \rangle / \langle \alpha, \alpha \rangle$ and $\rho_\alpha = \rho(\Phi_\alpha^+)$ where

$$\Phi_\alpha^+ = \{\beta \in \Phi^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) : \beta|_{\mathfrak{a}} \text{ is a multiple of } \alpha\}.$$

The following theorems are proved in [4]. Here and throughout this paper all Haar measures are normalized as in [4].

THEOREM 1.10 (*relative Plancherel formula*): *Let G be a reductive group of type* (1.1) *with* $G = ZG^0$. *Suppose that* $Z \cap G^0 = Z_{G^0}$, *i.e. replace Z by* ZZ_{G^0} *if necessary. Let* $\zeta \in \hat{Z}$ *and* $f \in C_c^\infty(G/Z, \zeta)$. *Then*

$$f(1) = \sum_{J \in \mathrm{Car}(G)} \sum_{\chi \in Z_{M_J}(M_J^0)^\wedge_\zeta} \deg \chi \sum_{\tau \in L'_\chi} \times \int_{\mathfrak{i}_p^*} \Theta(J: \chi: \tau: \nu)(f) m(J: \chi: \tau: \nu)\, d\nu.$$

Here $\mathrm{Car}(G)$ denotes a complete set of representatives for conjugacy classes of Cartan subgroups of G, chosen to be θ-stable, and

$$m(J: \chi: \tau: \nu) = c(G, J)\left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{j}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle \times \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{j})} \bar{p}_\alpha(\chi: \nu) \right|.$$

The constant $c(G, J)$ is given explicitly in [4, Theorem 6.17]

THEOREM 1.11 (*global Plancherel formula*): *Let G be a reductive group in the class* (1.1). *For* $f \in C_c^\infty(G)$,

$$f(1) = [G/Z_G(G^0)G^0]^{-1} \sum_{J \in \mathrm{Car}(G)} \int_{\chi \in Z_{M_J}(M_J^0)^\wedge} \deg \chi \times \sum_{\tau \in L'_\chi} \int_{\mathfrak{i}_p^*} \Theta(J: \chi: \tau: \nu)(f) m(J: \chi: \tau: \nu)\, d\nu\, d\chi.$$

§2. The relative Schwartz space

Let G be a group in our general class (1.1). In this section we define and discuss the "Schwartz space" $\mathscr{C}(G/Z, \zeta)$ of C^∞ functions on G that transform on the right by a unitary character $\zeta \in \hat{Z}$ and, as sections of the associated line bundle over G/Z, are rapidly decreasing in the sense of Harish-Chandra.

Fix a Cartan involution θ of G and let K denote the corresponding maximal compactly embedded subgroup G^θ. The Killing form of G

defines the structure of riemannian symmetric space of noncompact type on $X = G/K$. Let $o = 1 \cdot K \in G/K$. Define $\sigma\colon G \to \mathbb{R}^+$ by

$$\sigma(x) = \text{distance}(o, x(o)). \tag{2.1}$$

In the Cartan decomposition $G = K \cdot \exp(\mathfrak{p})$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under θ, the $\{\exp_G(t\xi) \cdot o\}_{t \in \mathbb{R}}$, $\xi \in \mathfrak{p}$, are the geodesics through o and are globally minimizing. That gives us the classical properties of σ:

$$\sigma(k \cdot \exp \xi) = \|\xi\| \text{ where } k \in K,\ \xi \in \mathfrak{p}; \tag{2.2a}$$

$$\sigma(k_1 x k_2) = \sigma(x) \text{ where } k_i \in K \text{ and } x \in G; \tag{2.2b}$$

$$\sigma(xy) \leqslant \sigma(x) + \sigma(y); \text{ and} \tag{2.2c}$$

$$\begin{gathered}\text{if } \omega \subset G \text{ is compact there exist } d_1, d_2 > 0 \text{ such that} \\ d_1(1 + \sigma(xy)) \leqslant 1 + \sigma(x) \leqslant d_2(1 + \sigma(xy)) \\ \text{for } x \in G,\ y \in \omega K.\end{gathered} \tag{2.2d}$$

Note that (2.2a) is the classical definition of σ in the case where Z is compact, but differs from that when Z has vector space factors. Subadditivity (2.2c) is the triangle inequality on X. It implies (2.2d).

Fix an Iwasawa decomposition $G = NAK$ where $\mathfrak{n}$ is the sum of the positive $\mathfrak{a}$-root spaces. As usual, ρ is half the sum of the positive $\mathfrak{a}$-roots (with multiplicity) and we factor

$$\begin{gathered} x = n(x) \cdot \exp H(x) \cdot \kappa(x) \\ \text{where } n(x) \in N,\ H(x) \in \mathfrak{a},\ \kappa(x) \in K. \end{gathered} \tag{2.3}$$

Then the zonal spherical function on G for $0 \in \mathfrak{a}^*$ is

$$\Xi(x) = \int_{K/Z} e^{-\rho(H(kx))} \, d(kZ). \tag{2.4}$$

It is the lift of the corresponding function on the group $G/Z_G(G^0)$ of Harish-Chandra class. Thus [2d, Lemma 10.1] it satisfies

$$\Xi(k_1 x k_2) = \Xi(x) \text{ where } x \in G \text{ and } k_i \in K; \tag{2.5a}$$

$$\Xi(x) = \Xi(x^{-1}) \text{ where } x \in G; \tag{2.5b}$$

$$\begin{gathered}\text{if } \omega \subset G \text{ is compact and } x \in G \text{ there exists } c > 0 \\ \text{such that } \Xi(y_1 x y_2) \leqslant c\,\Xi(x) \text{ for } y_1^{-1}, y_2 \in \omega Z;\end{gathered} \tag{2.5c}$$

$$1 \leqslant h^{\rho} \Xi(h) \text{ where } h \in \exp(cl(\mathfrak{a}^+)); \tag{2.5d}$$

there is an integer $d \geqslant 0$ such that
$$\Xi(h) \preccurlyeq h^{-\rho}(1+\sigma(h))^{d} \text{ for h} \in \exp(\mathrm{cl}(\mathfrak{a}^{+}));\tag{2.5e}$$

there is a number $r \geqslant 0$ such that
$$\int_{G/Z} \Xi(x)^{2}(1+\sigma(x))^{-r}\, \mathrm{d}(xZ) < \infty.\tag{2.5f}$$

Fix a unitary character $\zeta \in \hat{Z}$, and let

$$C^{\infty}(G/Z, \zeta) = \left\{ f \in C^{\infty}(G) \colon f(xz) = \zeta(z)^{-1} f(x) \right\}.$$

Given $f \in C^{\infty}(G/Z, \zeta)$, D_1, $D_2 \in \mathscr{U}(\mathfrak{g})$ and $r \in \mathbb{R}$, we define

$$_{D_1}| f |_{r,D_2} = \sup_{x \in G} (1+\sigma(x))^{r} \Xi(x)^{-1} | f(D_1; x; D_2) |.\tag{2.6a}$$

The *relative Schwartz spaces* on G are the

$$\mathscr{C}(G/Z, \zeta) = \{ f \in C^{\infty}(G/Z, \zeta) \colon \text{if } r \in \mathbb{R} \text{ and } D_i \in \mathscr{U}(\mathfrak{g}) \text{ then } {}_{D_1}| f |_{r,D_2} < \infty \}.\tag{2.6b}$$

$\mathscr{C}(G/Z, \zeta)$ is a complete locally convex TVS with topology defined by the seminorms (2.6a). Its structure is preserved by the left translations $L(x)$, $x \in G$, and by the right translations $R(x)$, $x \in ZG^0$.

If the Lie algebra $\mathfrak{z}$ of Z is the whole center of $\mathfrak{g}$ then an easy argument shows that the seminorms $_{D_1}| f |_{r,D_2}$, $D_i \in \mathscr{U}([\mathfrak{g}, \mathfrak{g}])$, define the same topology on the relative Schwartz spaces.

In some circumstances it will be convenient to view

$$\mathscr{C}(G/Z, \zeta) = \{ f \in C^{\infty}(G/Z, \zeta) \colon (L(x_i) f)|_{ZG^0} \in \mathscr{C}(ZG^0/Z, \zeta) \text{ for } i = 1, \ldots, r \}\tag{2.6c}$$

where x_i runs over any set of coset representatives of G modulo ZG^0, in particular where $K = x_1(K \cap ZG^0) \cup \ldots \cup x_r(K \cap ZG^0)$. Then we will use the seminorms $_{D_1,i}| f |_{r,D_2} = {}_{D_1}|(L(x_i) f)|_{ZG^0})|_{r,D_2}$. Compare §6 below.

THEOREM 2.7: *If $\zeta \in \hat{Z}$, then $\mathscr{C}(G/Z, \zeta)$ is a dense subspace of $L_2(G/Z, \zeta)$, and the inclusion $\mathscr{C}(G/Z, \zeta) \to L_2(G/Z, \zeta)$ is continuous.*

PROOF: Let $r \geqslant 0$ be given by (2.5f). From (2.6),

$$| f(x) |^{2} \leqslant | f |_{r/2}^{2} (1+\sigma(x))^{-r} \Xi(x)^{2}$$

and the latter is integrable on G/Z by choice of r. The inclusion is continuous because

$$\| f_1 - f_2 \|_2^2 \leqslant | f_1 - f_2 |_{r/2}^2 \int_{G/Z} \Xi(x)^2 (1+\sigma(x))^{-r} \, d(xZ).$$

The image is dense because the subspace $C_c^\infty(G/Z, \zeta)$ already is L_2-dense in $L_2(G/Z, \zeta)$. QED

THEOREM 2.8: *$C_c^\infty(G/Z, \zeta)$ is dense in $\mathscr{C}(G/Z, \zeta)$.*

PROOF: Using (2.6c), it suffices to consider the case $G = ZG^0$. For $t > 0$ let $G_t = \{x \in G\colon \sigma(x) < t\}$ and $\bar{G}_t = G_t/Z$. Fix $s > 0$ and choose $\phi \in C_c^\infty(\bar{G}_s)$ such that $\int_{G/Z} \phi(x) \, d(xZ) = 1$. Let ψ_t denote the indicator (characteristic) function of $\bar{G}_t$ in G/Z and set

$$g_t = 1 - \phi * \psi_t * \phi \text{ (convolution on } G/Z). \tag{2.9a}$$

We compute, on G/Z,

$$\begin{aligned}(1-g_t)(D_1; \bar{u}; D_2) &= (\phi * \psi_t * \phi)(D_1; \bar{u}; D_2) \\ &= \int_{G/Z}\int_{G/Z} \phi(D_1; \bar{u}\bar{v}) \psi_t(\bar{w}) \phi(\bar{w}^{-1}\bar{v}^{-1}; D_2) \\ &\quad \times d\bar{v} \, d\bar{w}.\end{aligned}$$

In particular, $g_t \in C^\infty(G/Z)$ and

$$| g_t(D_1; \bar{u}; D_2) | \leqslant 1 + \int_{G/Z} | \phi(D_1; \bar{v}) | \, d\bar{v} \int_{G/Z} | \phi(\bar{w}; D_2) | \, d\bar{w}. \tag{2.9c}$$

Note also, in view of (2.2c) that

$$g_t(x) = 0 \text{ if } \sigma(x) \leqslant t - 2s, \ = 1 \textit{ if } \sigma(x) \geqslant t + 2s, \tag{2.9d}$$

so in particular $1 - g_t \in C_c^\infty(G/Z)$.

Fix $r \geqslant 0$ and $f \in \mathscr{C}(G/Z, \zeta)$. Set $f_t = (1 - g_t) f$. Then $f_t \in C_c^\infty(G/Z, \zeta)$, and $f - f_t = g_t f$. Now

$$\begin{aligned}&(1+\sigma(x))^r \Xi(x)^{-1} |(f - f_t)(D; x; \tilde{D})| \\ &\quad = (1+\sigma(x))^r \Xi(x)^{-1} \left| \sum g_t(D_i'; x; \tilde{D}_j') f(D_i; x; \tilde{D}_j) \right|\end{aligned}$$

$\left(\text{where } D_i', D_i \text{ and } \tilde{D}_j', \tilde{D}_j \text{ express } D \text{ and } \tilde{D} \text{ on } g_t f\right)$

$$\leqslant (1+\sigma(x))^r \Xi(x)^{-1} \sum a_{ij} \left| f\left(D_i ; x ; \tilde{D}_j\right) \right|$$

$$\left(\text{using (2.9c), where } a_{ij} \text{ is the bound for} \left| g_t\left(D_i' ; x ; \tilde{D}_j'\right) \right| \right).$$

Pulling out a factor $(1+\sigma(x))^{-1}$, this gives us

$$(1+\sigma(x))^r \Xi(x)^{-1} |(f-f_t)(D ; x ; \tilde{D})|$$

$$\leqslant (1+\sigma(x))^{-1} \sum a_{ij\,D_i}|f|_{r+1, \tilde{D}_j}. \tag{2.10}$$

Now consider the following three cases.

(i) $\sigma(x) > t+2s$, so $f(x)-f_t(x)=f(x)$.

We ignore g_t in the derivation of (2.10) and conclude

$$(1+\sigma(x))^r \Xi(x)^{-1} |(f-f_t)(D ; x ; \tilde{D})| \leqslant \frac{1}{1+t} \cdot {}_D|f|_{r+1, \tilde{D}}. \tag{2.11a}$$

(ii) $t+2s \geqslant \sigma(x) \geqslant t-2s$, say with $t>2s$. Then from (2.10),

$$(1+\sigma(x))^r \Xi(x)^{-1} |(f-f_t)(D ; x ; \tilde{D})|$$

$$\leqslant \frac{1}{1+t-2s} \sum a_{ij} \cdot {}_{D_i}|f|_{r+1, \tilde{D}_j}. \tag{2.11b}$$

(iii) $t-2s > \sigma(x)$, so $f(x)-f_t(x)=0$. Then

$$(1+\sigma(x))^r \Xi(x)^{-1} |(f-f_t)(D ; x ; \tilde{D})| = 0. \tag{2.11c}$$

From these three cases we conclude that ${}_D|f-f_t|_{r,\tilde{D}} \to 0$ as $t \to \infty$. Thus $\{f_t\} \to f$ in $\mathscr{C}(G/Z, \zeta)$. As $f_t \in C_c^\infty(G/Z, \zeta)$, since $1-g_t \in C_c^\infty(G/Z)$, this completes the proof of Theorem 2.8. QED

We will refer to continuous linear functionals on $C_c^\infty(G/Z, \zeta)$ as *ζ-distributions* on G. A ζ-distribution T will be called *tempered* if it has a continuous extension to $\mathscr{C}(G/Z, \zeta)$. Our next task is to give estimates that characterize the tempered ζ-distributions.

Let T be a central $\mathscr{Z}(\mathfrak{g})$-finite distribution on G. Then [2a] T is represented by a locally $L_1(G)$ function analytic on the regular set G', which we also denote by T. Note that T is a ζ-distribution if and only if $T(xz) = \zeta(z)T(x)$ for all $x \in G'$, $z \in Z$, and in that case $|T|$ is locally $L_1(G/Z)$. (Central ζ-distributions may not exist if Z is not central in G.)

Here is our variation on Harish-Chandra's condition that a distribution be tempered.

THEOREM 2.12: *Let T be a locally L_1 function on G such that $T(xz) = \zeta(z)T(x)$ a.e. Suppose that there is an integer $m \geqslant 0$ such that*

$$\operatorname*{ess\,sup}_{x \in G} (1+\sigma(x))^{-m} |D_l(x)|^{1/2} |T(x)| < \infty. \tag{2.13}$$

Then integration against T is a tempered ζ-distribution on G. In fact, if $f \in \mathscr{C}(G/Z, \zeta)$ then $|T(f)| \leqslant c|T|_m |f|_{r+m}$ where c, $r > 0$ depend only on G and where $|T|_m$ is given by (2.13).

In particular, if T is a central $\mathscr{Z}(\mathfrak{g})$-finite ζ-distribution on G, and if there is an integer $m \geqslant 0$ such that

$$\sup_{x \in G'} (1+\sigma(x))^{-m} |D_l(x)|^{1/2} |T(x)| < \infty, \tag{2.14}$$

then T is tempered with $|T(f)| \leqslant c|T|_m |f|_{r+m}$ as above.

PROOF: Let $f \in \mathscr{C}(G/Z, \zeta)$ and compute the integral of f against the function T:

$$\begin{aligned}\left|\int_{G/Z} f(x)T(x)\,\mathrm{d}(xZ)\right| &\leqslant \int_{G/Z} |f(x)||T(x)|\,\mathrm{d}(xZ) \\ &\leqslant |T|_m |f|_{r+m} \int_{G/Z} (1+\sigma(x))^{-r-m} \Xi(x) \\ &\qquad \times (1+\sigma(x))^m |D_l(x)|^{-1/2}\,\mathrm{d}(xZ) \\ &= c|T|_m |f|_{r+m}\end{aligned}$$

where $|T|_m$ is given by (2.13) — or by (2.14) when applicable — and where r and c are given by

$$c = \int_{G/Z} (1+\sigma(x))^{-r} \Xi(x) |D_l(x)|^{-1/2}\,\mathrm{d}(xZ).$$

Since D_l, Ξ and σ are the G-lifts of the corresponding functions on $G/Z_G(G^0)$, Harish-Chandra's result [2d, Lemma 13.1] holds for us in the form

there exists a number $r \geqslant 0$ such that

$$\int_{G/Z} (1+\sigma(x))^{-r} \Xi(x) |D_l(x)|^{-1/2}\,\mathrm{d}(xZ) < \infty. \tag{2.15}$$

Now with r as in (2.15) we have $c < \infty$ and $|T(f)| \leqslant c|T|_m|f|_{r+m}$ as asserted. QED

We remark without proof that (2.14) is necessary, as well as sufficient, for a central $\mathscr{Z}(\mathfrak{g})$-finite ζ-distribution to be tempered. In effect, the argument in [5, §8.3.8] goes through in our situation, as it is used in the proof of necessity.

Similarly, the argument in [5, §8.3.7], or more simply that of [2d, Lemma 14.2], goes through in our case and gives us Rader's result

$$\mathscr{C}(ZG^0/Z, \zeta) \text{ is a topological algebra under convolution.} \quad (2.16)$$

Here note that convolution in $\mathscr{C}(G/Z, \zeta)$ only makes sense when Z is central in G. We can, however, enlarge ZG^0 in (2.16) to the centralizer of Z in G.

Finally, one checks that the argument of [5, 8.3.7.8] goes through without change to show that

$$\begin{array}{l}\text{The left and right regular representations of}\\ ZG^0 \text{ on } \mathscr{C}(ZG^0/Z, \zeta) \text{ are differentiable.}\end{array} \quad (2.17a)$$

It follows as in [5, p. 161] that

$$\begin{array}{l}\text{If } f \in \mathscr{C}(ZG^0/Z, \zeta) \text{ then } \sum_{\delta_1} \alpha_{\delta_1} * f \to f,\ \sum_{\delta_2} f * \alpha_{\delta_2} \to f\\ \text{and } \sum_{\delta_1, \delta_2} \alpha_{\delta_1} * f * \alpha_{\delta_2} \to f\end{array} \quad (2.17b)$$

where $\delta_i \in (K \cap ZG^0)^{\wedge}_{\zeta}$, α_{δ_i} denotes normalized character, and the convergence is absolute convergence in $\mathscr{C}(ZG^0/Z, \zeta)$.

§3. The relative Plancherel formula

In this section we will extend the relative Plancherel Theorem proved in [4] for functions in $C_c^\infty(G/Z, \zeta)$ to the Schwartz space $\mathscr{C}(G/Z, \zeta)$, $G = ZG^0$. We will first show that all the characters which occur in the Plancherel formula are tempered.

LEMMA 3.1: *Let $H = TA$ be a θ-stable Cartan subgroup of G. Write $M = M_H$. Then there is a constant $c \geqslant 0$ so that for all $\chi \in Z_M(M^0)^{\wedge}$, $\tau \in L'_\chi$ and $\nu \in \mathfrak{a}^*$, $\sup_{x \in G'} |D_l(x)|^{1/2} |\Theta(H: \chi: \tau: \nu)(x)| \leqslant c$.*

PROOF: Because $|D_l(x)|^{1/2}|\Theta(H: \chi: \tau: \nu)(x)|$ is a class function on G, we have

$$\sup_{x\in G'} |D_l(x)|^{1/2}|\Theta(H: \chi: \tau: \nu)(x)|$$

$$= \max_{J\in \mathrm{Car}(G)} \sup_{j\in J'} |\Delta^G(j)||\Theta(H: \chi: \tau: \nu)(j)|.$$

Thus it is enough to show that for each $J\in \mathrm{Car}(G)$, there is a constant c_J so that $\sup_{j\in J'}|\Delta^G(j)|\,|\Theta(H: \chi: \tau: \nu)(j)| \leqslant c_J$. Recall that $\Theta(H: \chi: \tau: \nu)$ is supported on Cartan subgroups of G conjugate to those in $L=MA$. Thus we may as well assume that $J\subseteq L$ as $c_J=0$ otherwise. Using the notation of (1.6) we have for $j\in J'$,

$$|\Delta^G(j)||\Theta(H: \chi: \tau: \nu)(j)|$$

$$\leqslant \sum_{i=1}^{k} \sum_{w\in W(G,J_{i,0})} |\Delta^L(wj_i)||\Theta_{\chi,\tau}\otimes e^{i\nu}(wj_i)|.$$

For $1\leqslant i\leqslant k$ and $w\in W(G, J_{i,0})$, $(\Theta_{\chi,\tau}\otimes e^{i\nu})(wj_i)=0$ unless $wj_i\in J_i^\dagger = J_i\cap M^\dagger A$. For $wj_i\in J_i^\dagger$ we have, using (1.5),

$$|\Delta^L(wj_i)||\Theta_{\chi,\tau}\otimes e^{i\nu}(wj_i)|$$

$$\leqslant \sum_{y\in M/M^\dagger} |\Delta^L(ywj_iy^{-1})||\mathrm{tr}\ \chi\otimes\Theta_\tau\otimes e^{i\nu}(ywj_iy^{-1})|.$$

For $y\in M/M^\dagger$ (chosen to normalize J_i), write $ywj_iy^{-1}=zj_Ma$ where $z\in Z_M(M^0)$, $j_M\in J_i\cap M^0$ and $a\in A$. Then

$$|\Delta^L(ywj_iy^{-1})||\mathrm{tr}\ \chi\otimes\Theta_\tau\otimes e^{i\nu}(ywj_iy^{-1})|$$

$$=|\mathrm{tr}\ \chi(z)\Delta^M(j_M)\Theta_\tau(j_M)a^{i\nu}| \leqslant \deg\chi|\Delta^M(j_M)\Theta_\tau(j_M)|.$$

Here ZZ_{M^0} is a normal abelian subgroup of finite index in $Z_M(M^0)$, so $\deg\chi\leqslant[Z_M(M^0)/ZZ_{M^0}]<\infty$.

Finally, if $j_M=j_Ka_1$ where $j_K\in J_{i,K}$ and $a_1\in J_{i,p}\cap M$, we have using (1.3),

$$|\Delta^M(j_M)\Theta_\tau(j_M)|$$

$$\leqslant \sum_{w\in W(M^0,T^0)} \sum_{s\in W_R/W_K} |c(s: w\tau: \Phi_R^+(j_Ka_1))$$

$$\times\exp|(s^y(w\tau)(\log a_1))|.$$

But $c(s\colon w\tau\colon \Phi_R^+(j_K a_1)) = 0$ unless $s^y(w\tau)(\log a_1) < 0$, and $c(s\colon w\tau\colon \Phi_R^+(j_K a_1))$ assumes only finitely many possible values. This is because only finitely many Weyl group elements s and root systems Φ^+ occur, and for fixed s and Φ^+, $c(s\colon \tau\colon \Phi^+)$ depends only on the chamber of τ with respect to Φ^+. In fact it is easy to see using the explicit formulas given in [3] that $|c(s\colon \tau\colon \Phi^+)| \leqslant [W(\Phi)]2^n$ if $n = \text{rank } \Phi$.

Combining the above estimates we find that

$$\sup_{j \in J'} |\Delta^G(j)\Theta(H\colon \chi\colon \tau\colon \nu)(j)| \leqslant c_J \text{ where if } n = \text{rank } \Phi_R(\mathfrak{m}, \mathfrak{j}),$$

$$c_J = k[W(G, J)][M/ZM^0][W(M^0, T^0)][W_R(\mathfrak{m}, \mathfrak{j})]^2 2^n.$$

QED

Corollary 3.2: *Assume $G = ZG^0$. Let $\zeta \in \hat{Z}$. Then for all $\chi \in Z_M(M^0)^\wedge_\zeta$, $\tau \in L'_\chi$, and $\nu \in \mathfrak{a}^*$, $\Theta(H\colon \chi\colon \tau\colon \nu)$ is a tempered ζ-distribution. In fact, if $f \in \mathscr{C}(G/Z, \zeta)$, $|\Theta(H\colon \chi\colon \tau\colon \nu)(f)| \leqslant c|f|_r$ where c is independent of χ, τ, and ν, and r is as in* (2.14).

Proof: Lemma 3.1 shows that $\Theta(H\colon \chi\colon \tau\colon \nu)$ satisfies the estimate required in (2.14) with $m = 0$. QED

Lemma 3.3: *There is an integer $m \geqslant 0$ and a constant C so that for all $\chi \in Z_M(M^0)^\wedge$, $\tau \in L'_\chi$, and $\nu \in \mathfrak{a}^*$,*

$$m(H\colon \chi\colon \tau\colon \nu) = c(G, H)\left| \prod_{\alpha \in \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})} \langle \alpha, \tau + i\nu \rangle \times \prod_{\alpha \in \Phi_R^+(\mathfrak{g}, \mathfrak{h})} \bar{p}_\alpha(\chi\colon \nu) \right| \leqslant C\left(1 + \|\tau\|^2 + \|\nu\|^2\right)^m.$$

Here $\bar{p}_\alpha(\chi\colon \nu)$ is defined as in (1.9).

Proof: Write $\Phi^+ = \Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}})$ and $\Phi_R^+ = \Phi_R^+(\mathfrak{g}, \mathfrak{h})$. Then

$$\prod_{\alpha \in \Phi^+} \langle \alpha, \tau + i\nu \rangle \prod_{\alpha \in \Phi_R^+} \bar{p}_\alpha(\chi\colon \nu) = \prod_{\alpha \in \Phi^+ \setminus \Phi_R^+} \langle \alpha, \tau + i\nu \rangle \times \prod_{\alpha \in \Phi_R^+} \frac{i\langle \alpha, \alpha \rangle}{2} \nu_\alpha \bar{p}_\alpha(\chi\colon \nu).$$

But, using (1.9), $\nu_\alpha \bar{p}_\alpha(\chi\colon \nu) = \nu_\alpha \sinh \pi\nu_\alpha/\cosh \pi\nu_\alpha - c_\alpha(\chi)$ where $c_\alpha(\chi) = \frac{1}{2}\xi_{\rho_\alpha}(\gamma_\alpha)(\eta_\alpha + \eta_\alpha^{-1})$ with $|\eta_\alpha| = 1$. Thus $|c_\alpha(\chi)| \leqslant |\eta_\alpha| = 1$. Now for any c with $|c| \leqslant 1$, $g(\nu, c) = \nu \sinh \pi\nu/\cosh \pi\nu + c$ satisfies $\sup_\nu(1 + |\nu|)^{-1}|g(\nu, c)| \leqslant \sup_\nu(1 + |\nu|)^{-1}|g(\nu, 1)| < \infty$. The result is now clear since $\prod_{\alpha \in \Phi^+ \setminus \Phi_R^+}\langle \alpha, \tau + i\nu\rangle$ is a polynomial in τ and ν. QED

For $\zeta \in \hat{Z}$, $f \in C_c^\infty(G/Z, \zeta)$, $G = ZG^0$, define

$$\Psi(H\colon \zeta)(f) = \sum_{\chi \in Z_M(M^0)\hat{}_\zeta} \deg \chi \sum_{\tau \in L'_\chi} \int_{\mathfrak{a}^*} \Theta(H\colon \chi\colon \tau\colon \nu)(f) \times m(H\colon \chi\colon \tau\colon \nu) d\nu. \tag{3.4}$$

LEMMA 3.5: *For all $H \in \mathrm{Car}(G)$, $\Psi(H\colon \zeta)$ is tempered ζ-distribution.*

PROOF: Let Ω be the element of $\mathscr{Z}(\mathfrak{g})$ given by $\Omega = 1 + \|\rho\|^2 + \omega$ where ω is the Casimir element and $\rho = \rho(\Phi^+(\mathfrak{g}_{\mathbf{C}}, \mathfrak{h}_{\mathbf{C}}))$. Then for any integer $n \geqslant 0$, $f \in \mathscr{C}(G/Z, \zeta)$, $\Theta(H\colon \chi\colon \tau\colon \nu)(\Omega^n f) = (1 + \|\tau\|^2 + \|\nu\|^2)^n \Theta(H\colon \chi\colon \tau\colon \nu)(f)$. Write $L'_\zeta = \{\tau \in L'\colon \zeta|_{Z \cap T^0} = \xi_{\tau - \rho}|_{Z \cap T^0}\}$, where $\rho = \rho(\Phi^+(\mathfrak{m}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}}))$, and for $\tau \in L'_\zeta$ let $Z_M(M^0)\hat{}_{\zeta,\tau} = \{\chi \in Z_M(M^0)\hat{}_\zeta\colon \tau \in L'_\chi\}$. Then we can rewrite

$$\Psi(H\colon \zeta)(f) = \sum_{\tau \in L'_\zeta} \sum_{\chi \in Z_M(M^0)\hat{}_{\zeta,\tau}} \deg \chi \times \int_{\mathfrak{a}^*} \Theta(H\colon \chi\ \tau\colon \nu)(f) m(H\colon \chi\colon \tau\colon \nu)\, d\nu$$

where

$$\left[Z_M(M^0)\hat{}_{\zeta,\tau}\right] \leqslant \left[Z_M(M^0)/ZZ_{M^0}\right] < \infty.$$

Pick n large enough so that

$$M = \sum_{\tau \in L'_\zeta} \int_{\mathfrak{a}^*} \left(1 + \|\tau\|^2 + \|\nu\|^2\right)^{-n+m} d\nu < \infty$$

where m is defined as in (3.3). Then using (3.2) and (3.3), for $f \in \mathscr{C}(G/Z, \zeta)$,

$$|\Psi(H\colon \zeta)(f)| \leqslant \sum_{\tau \in L'_\zeta} \sum_{\chi \in Z_M(M^0)\hat{}_{\zeta,\tau}} \deg \chi \times \int_{\mathfrak{a}^*} |\Theta(H\colon \chi\colon \tau\colon \nu)(\Omega^n f)| \left(1 + \|\tau\|^2 + \|\nu\|^2\right)^{-n} \times m(H\colon \chi\colon \tau\colon \nu)\, d\nu$$

$$\leqslant C_{\Omega^n}|f|_r \sum_{\tau \in L'_\zeta} \sum_{\chi \in Z_M(M^0)^\wedge_{\zeta,\tau}} \deg \chi$$

$$\times \int_{\mathfrak{a}^*} \left(1 + \|\tau\|^2 + \|\nu\|^2\right)^{-n+m} d\nu$$

$$\leqslant C_{\Omega^n}|f|_r M\left[Z_M(M^0)/ZZ_{M^0}\right]^2$$

where C is the product of the constants given by (3.2) and (3.3). Thus $\Psi(H\colon \chi)$ extends continuously to $f \in \mathscr{C}(G/Z, \zeta)$. QED

THEOREM 3.6 (*relative Plancherel Theorem*): *Let G be a reductive group of type* (1.1) *with $G = ZG^0$. Replace Z by ZZ_{G^0} if necessary, so that $Z \cap G^0 = Z_{G^0}$. Let $\zeta \in \hat{Z}$ and $f \in \mathscr{C}(G/Z, \zeta)$. Then*

$$f(1) = \sum_{J \in \mathrm{Car}(G)} \sum_{\chi \in Z_{M_J}(M_J^0)^\wedge_\zeta} \deg(\chi)$$

$$\times \sum_{\tau \in L'_\chi} \int_{\mathfrak{j}_p^*} \Theta(J\colon \chi\colon \tau\colon \nu)(f) m(J\colon \chi\colon \tau\colon \nu)\, d\nu$$

as in Theorem 1.10.

§4. Splitting off the relative discrete spectrum

The relative Plancherel formula says that any $f \in \mathscr{C}(G/Z, \zeta)$, $G = ZG^0$, can be decomposed as $f = \sum_{H \in \mathrm{Car}(G)} f_H$ where

$$f_H(x) = \Psi(H\colon \zeta)\big(L(x^{-1})f\big),$$

$$x \in G,\ H \in \mathrm{Car}(G)\ \big(\text{see } (3.4)\big). \tag{4.1}$$

We know from the direct integral decomposition of $L^2(G/Z, \zeta)$ that each $f_H \in L^2(G/Z, \zeta)$. In fact, using the argument in (3.5) for left translates of f we see that the sums over L'_χ, $\chi \in Z_M(M^0)^\wedge_\zeta$ and integral over $\mathfrak{a}^*$ in the definition of $\Psi(H\colon \zeta)(L(x^{-1})f)$ converge absolutely, uniformly for x in any compact subset of G, so that each f_H is C^∞ and $f_H(D_1; x; D_2) = (D_1 f D_2)_H(x)$ for $D_1, D_2 \in \mathscr{U}(\mathfrak{g})$, $x \in G$.

We would like to know that each $f_H \in \mathscr{C}(G/Z, \zeta)$. This is essentially the problem of showing that "wave packets" are Schwartz functions, which will be deferred to another paper. It would give an orthogonal decomposition $\mathscr{C}(G/Z, \zeta) = \sum_{H \in \mathrm{Car}(G)} \mathscr{C}_H(G/Z, \zeta)$ where $\mathscr{C}_H(G/Z, \zeta) = \{f \in \mathscr{C}(G/Z, \zeta)\colon f = f_H\}$. As a preliminary step in this direction we will prove that $f_T \in \mathscr{C}(G/Z, \zeta)$ if f is "K-finite" and T is a relatively compact Cartan subgroup of G.

For arbitrary G of class (1.1), $\mathscr{C}(G/Z, \zeta)$ is stable under the left regular action of G but is only preserved by the right regular action of ZG^0. Thus by a K-finite function in $L^2(G/Z, \zeta)$ we will mean a function f such that $\{L(k)f, R(k')f: k \in K, k' \in K \cap ZG^0\}$ span a finite-dimensional subspace of $L^2(G/Z, \zeta)$. Since $[K/K \cap ZG^0] < \infty$, the difference between K and $K \cap ZG^0$ is not significant. Write $\mathscr{C}(G/Z, \zeta)^K$ for the set of K-finite functions in $\mathscr{C}(G/Z, \zeta)$.

THEOREM 4.2: *Suppose $f \in L^2(G/Z, \zeta)$ is K-finite and $\mathscr{Z}$-finite. Then $f \in \mathscr{C}(G/Z, \zeta)^K$.*

Suppose Z is central in G. For $\delta \in \hat{K}$, let $\alpha_\delta = \deg \delta \operatorname{tr} \delta$ and for F any finite subset of $\hat{K}$, let $\alpha_F = \Sigma_{\delta \in F}\, \alpha_\delta$. Let $\check{F}$ denote the set of representations contragredient to those in F. Suppose F is a finite subset of $\hat{K}_\zeta = \{\delta \in \hat{K} \mid \delta(kz) = \zeta(z)\delta(k)$ for all $z \in Z,\ k \in K\}$. Then for $\pi \in \hat{G}_\zeta$ and $f \in \mathscr{C}(G/Z, \zeta)$, the convolutions $\Theta_\pi * \alpha_F$ and $f * \alpha_{\check{F}}$ are well-defined as integrals over K/Z.

THEOREM 4.3: *Assume that $G = ZG^0$. Fix a finite subset $F \subseteq \hat{K}_\zeta$. Then there are only finitely many $\pi \in \hat{G}_{\zeta\text{-disc}}$ so that $\Theta_\pi * \alpha_F \neq 0$.*

The proofs of Theorems 4.2 and 4.3 will be given later in this section. They have the following important corollaries.

COROLLARY 4.4. *Suppose $f \in \mathscr{C}(G/Z, \zeta)^K$ and T is a relatively compact Cartan subgroup of $G = ZG^0$. Then $f_T \in \mathscr{C}(G/Z, \zeta)^K$ where*

$$f_T(x) = \sum_{\pi \in \hat{G}_{\zeta\text{-disc}}} \mathrm{d}(\pi)\Theta_\pi(L(x^{-1})f).$$

PROOF: Since f is right K-finite there is a finite subset F of $\hat{K}_\zeta$ so that $f * \alpha_{\check{F}} = f$. But then for any $\pi \in \hat{G}_{\zeta\text{-disc}}$, $\Theta_\pi(L(x^{-1})f) = (\Theta_\pi * \alpha_F)(L(x^{-1})f)$. But by (4.3), $\Theta_\pi * \alpha_F = 0$ for all but finitely many $\pi \in \hat{G}_{\zeta\text{-disc}}$. Thus the sum defining f_T is finite so that f_T is $\mathscr{Z}$-finite. But $f_T \in L^2(G/Z, \zeta)$ and clearly inherits K-finiteness from f. Thus by (4.2), $f_T \in \mathscr{C}(G/Z, \zeta)^K$.

QED

Note that $f_T = f_{\text{disc}}$, the projection of $f \in L^2(G/Z, \zeta)$ into $L^2(G/Z, \zeta)_{\text{disc}}$. Thus (4.4) says that if $G = ZG^0$, $f \in \mathscr{C}(G/Z, \zeta)^K$ implies $f_{\text{disc}} \in \mathscr{C}(G/Z, \zeta)^K$. This result can be extended to arbitrary groups of class (1.1) as follows. Let $f \in \mathscr{C}(G/Z, \zeta)^K$. Then

$$L(x)f|_{ZG^0} \in \mathscr{C}(ZG^0/Z, \zeta)^{K \cap ZG^0} \text{ for all } x \in G,$$

so that

$$(L(x)f|_{ZG^0})_{\text{disc}} \in \mathscr{C}(ZG^0/Z, \zeta)^{K\cap ZG^0}.$$

But using (5.11),

$$(L(x)f|_{ZG^0})_{\text{disc}} = (L(x)f)_{\text{disc}}|_{ZG^0}$$

$$= L(x)(f_{\text{disc}})|_{ZG^0} \in \mathscr{C}(ZG^0/Z, \zeta)^{K\cap ZG^0}.$$

Thus $f_{\text{disc}} \in \mathscr{C}(G/Z, \zeta)^K$.

Let

$$\mathscr{C}_{\text{disc}}(G/Z, \zeta)^K = \mathscr{C}(G/Z, \zeta)^K \cap L^2(G/Z, \zeta)_{\text{disc}}$$

and

$$\mathscr{C}_{\text{cont}}(G/Z, \zeta)^K = \mathscr{C}(G/Z, \zeta)^K \cap L^2(G/Z, \zeta)_{\text{cont}}.$$

Corollary 4.5: $\mathscr{C}(G/Z, \zeta)^K = \mathscr{C}_{\text{disc}}(G/Z, \zeta)^K \oplus \mathscr{C}_{\text{cont}}(G/Z, \zeta)^K$.

Remark: For general G of class (1.1) and $H \in \text{Car}(G)$ we can define $\mathscr{C}_H(G/Z, \zeta)$ to be $\mathscr{C}(G/Z, \zeta) \cap L^2(G/Z, \zeta)_H$ where $L^2(G/Z, \zeta)_H$ denotes the subspace of $L^2(G/Z, \zeta)$ corresponding to the direct integral over $\{\pi(H\colon \chi\colon \tau\colon \nu)\colon \chi \in Z_M(M^0)^{\wedge}_{\zeta},\ \tau \in L'_{\chi},\ \nu \in \mathfrak{a}^*\}$. Then if G has real rank one (4.5) gives the complete decomposition of

$$\mathscr{C}(G/Z, \zeta)^K = \sum_{H \in \text{Car}(G)} \mathscr{C}_H(G/Z, \zeta)^K.$$

We now turn to the proof of (4.2). We will use the theory of leading characters of Casselman and Miličić [1]. Although their results are stated only for groups of Harish-Chandra class, it is easy to check that they remain valid for groups of our class (1.1) if Z and hence K are compact. But once we have Theorem 4.2 in that generality, it extends to all groups of class (1.1), in two steps, as follows.

(4.6) *Step 1*: Extension to class (1.1) groups of the form $G = ZG^0$. Given $\zeta \in \hat{Z}$ we set

$$G[\zeta] = \{S \times ZG^0\}/\left\{\left(\zeta(z)^{-1}, z\right)\colon z \in Z\right\}$$

where $S = \{e^{i\theta}\}$ is the circle group. The projection $S \times ZG^0 \to G[\zeta]$ restricts to a homomorphism.

$$q\colon ZG^0 \to G[\zeta].$$

This is the construction of [6, §3.3]. It replaces Z by the circle group S. In [6, §3.3] it is shown that $f \to f \circ q$ gives an equivariant isometry of $L_2(G[\zeta]/S, 1)$ onto $L_2(ZG^0/Z, \zeta)$, where $1 \in \hat{S}$ denotes the character $e^{i\theta} \to e^{i\theta}$.

Let $f \in L_2(ZG^0/Z, \zeta)$ be $(K \cap ZG^0)$-finite and $\mathscr{Z}(\mathfrak{g})$-finite. Express $f = f' \circ q$. Then $f' \in L_2(G[\zeta]/S, 1)$ is $K[\zeta]$-finite and $\mathscr{Z}(\mathfrak{g}[\zeta])$-finite. Since $G[\zeta]$ is a connected group of class (1.1) with compact center, now $f' \in \mathscr{C}(G[\zeta]/S, 1)$. But in the definition of the relative Schwartz spaces, (2.6), we note

$$\sigma_{ZG^0}(x) = \sigma_{G[\zeta]}(q(x)) \text{ and } \Xi_{ZG^0}(x) = \Xi_{G[\zeta]}(q(x)).$$

Thus, if $D_1, D_2 \in \mathscr{U}(\mathfrak{g})$ and $r \in \mathbb{R}$,

$$_{D_1}|f|_{r,D_2} = {}_{q_*D_1}|f'|_{r,q_*D_2} < \infty$$

so $f \in \mathscr{C}(ZG^0/Z, \zeta)$.

(4.7) *Step 2*: Extension to all groups of class (1.1). Let $f \in L_2(G/Z, \zeta)$ be K-finite and $\mathscr{Z}(\mathfrak{g})$-finite. Since K meets every component of G we have $G = \cup x_i ZG^0$ with $x_i \in K$. Each $(L(x_i)f)|_{ZG^0}$ is $(K \cap ZG^0)$-finite and $\mathscr{Z}(\mathfrak{g})$-finite, hence contained in $\mathscr{C}(ZG^0/Z, \zeta)$. That says exactly that $f \in \mathscr{C}(G/Z, \zeta)$.

Thus it suffices to prove

LEMMA 4.8: Suppose G is a group of class (1.1) with Z compact. Suppose $f \in L^2(G)$ is both K-finite and $\mathscr{Z}$-finite. Then $f \in \mathscr{C}(G)$.

PROOF: Let $\tau = (\tau_1, \tau_2)$ denote the representation of $K \times K$ on $L^2(K \times K)$ given by $[\tau_1(k_1)F\tau_2(k_2)](k_1', k_2') = F(k_1^{-1}k_1', k_2'k_2^{-1})$, $k_1, k_2, k_1', k_2' \in K$. Fix $f \in L^2(G)$ as above and define $F(x) \in L^2(K \times K)$ by $F(x)(k_1, k_2) = f(k_1^{-1}xk_2^{-1})$, $x \in G$, $k_1, k_2 \in K$. Then because f is K-finite, the subspace E of $L^2(K \times K)$ spanned by the $F(x)$, $x \in G$, is finite-dimensional. Further, since f is $\mathscr{Z}$-finite, so is F, and both are C^∞. Thus the Casselman-Miličić theory of leading characters can be applied to F. Let A and ρ be defined as in (2.3) and let $cl(A^-)$ denote the closed negative Weyl chamber. Since $\int \| F(x) \|_2^2 \, dx = \int |f(x)|^2 \, dx < \infty$, we have by Theorem 7.5 of [1] that every leading character ν of F satisfies $|\nu(a)| < e^{\rho(\log a)}$ for all $a \in cl(A^-)$, $a \neq 1$. But there are only finitely many leading characters so that in fact there is a constant $c > 1$ so that $|\nu(a)| \leqslant e^{c\rho(\log a)}$ for all $a \in cl(A^-)$. But now, by Theorem 7.1 of [1], there are $M \geqslant 0$ and $m \geqslant 0$ so that $\| F(a) \| \leqslant M \, e^{c\rho(\log a)}(1 + \sigma(a))^m$ for $a \in cl(A^-)$.

Thus for any $r \geqslant 0$, since $\| F(x) \|_2$, $\Xi(x)$ and $\sigma(x)$ are all K-biinvariant, and using (2.5d),

$$\begin{aligned}
&\sup_{x \in G} \| F(x) \|_2 \Xi^{-1}(x)(1+\sigma(x))^r \\
&\quad = \sup_{a \in cl(A^-)} \| F(a) \|_2 \Xi^{-1}(a)(1+\sigma(a))^r \\
&\quad \leqslant M \sup_{a \in cl(A^-)} e^{c\rho(\log A)} \Xi^{-1}(a)(1+\sigma(a))^{r+m} \\
&\quad \leqslant M \sup_{a \in cl(A^-)} e^{(c-1)\rho(\log a)}(1+\sigma(a))^{r+m} < \infty
\end{aligned}$$

But

$$\begin{aligned}
&\sup_{x \in G} | f(x) | \Xi^{-1}(x)(1+\sigma(x))^r \\
&\quad = \sup_{x \in G, k_1, k_2 \in K} \left| f\left(k_1^{-1} x k_2^{-1}\right) \right| \Xi^{-1}(x)(1+\sigma(x))^r \\
&\quad = \sup_{x \in G} \| F(x) \|_\infty \Xi^{-1}(x)(1+\sigma(x))^r < \infty
\end{aligned}$$

since $\| \cdot \|_2$ and $\| \cdot \|_\infty$ are equivalent norms on the finite-dimensional vector space $E \subseteq L^2(K \times K) \cap C^\infty(K \times K)$.

We still need a similar estimate for $D_1 f D_2$, D_1, $D_2 \in \mathscr{U}(\mathfrak{g})$. For this we need a result from [5, 8.3.9.2] which can be extended to the compact center groups and says there are ψ_1, $\psi_2 \in C_c^\infty(G)$ so that $f = \psi_1 * f * \psi_2$. Write $\omega_i = \operatorname{supp} \psi_i$, $i = 1, 2$. Then

$$\begin{aligned}
&\sup_{x \in G} | f(D_1 ; x ; D_2) | \Xi^{-1}(x)(1+\sigma(x))^r \\
&\quad = \sup_{x \in G} | D_1 \psi_1 * f * \psi_2 D_2(x) | \Xi^{-1}(x)(1+\sigma(x))^r \\
&\quad \leqslant \sup_{x \in G} \int_{G \times G} | \psi_1(D_1 ; v) f(v^{-1} x w^{-1}) \psi_2(w ; D_2) | \\
&\qquad \times dv\, dw \Xi^{-1}(x)(1+\sigma(x))^r \\
&\quad \leqslant \| D_1 \psi_1 \|_\infty \| \psi_2 D_2 \|_\infty \sup_{x \in G} \sup_{(v,w) \in \omega_1 \times \omega_2} \\
&\qquad \times | f(x) | \Xi^{-1}(vxw)(1+\sigma(vxw))^r \\
&\quad \leqslant C \| D_1 \psi_1 \|_\infty \| \psi_2 D_2 \|_\infty \sup_{x \in G} | f(x) | \Xi^{-1}(x)(1+\sigma(x))^r
\end{aligned}$$

by (2.2d) and (2.5c). QED

The proof of (4.3) requires the following proposition which again is proved using the theory of leading characters.

PROPOSITION 4.9: *Suppose Z is central in G and compact. Let $\pi \in \hat{G}_{\zeta\text{-disc}}$ and let F be a finite subset of $\hat{K}_{\zeta}$. Then*

$$\Theta_{\pi,F} = \Theta_\pi * \alpha_F \in \mathscr{C}(G/Z, \zeta).$$

PROOF: As in [5, 8.3.8.6], we see that $\Theta_{\pi,F}$ is C^∞ on G and satisfies the weak inequality $|\Theta_{\pi,F}(x)| \leqslant C\Xi(x)(1+\sigma(x))^m$ for some C, $m \geqslant 0$. Let $\tau = (\tau_1, \tau_2)$ be the unitary representation of $K \times K$ on $L^2(K \times K)$ defined as in (4.8). For all $x \in G$ define $f(x) \in L^2(K \times K)$ by $f(x)(k_1, k_2) = \Theta_{\pi,F}(k_1^{-1} x k_2^{-1})$, $k_1, k_2 \in K$. As in (4.8) the theory of leading characters applies to f.

Let T be a compact Cartan subgroup of G. Since Θ_π is the character of a discrete series representation of G we know by [6] that there are $\chi \in Z_G(G^0)\hat{}$, $\lambda \in L'_\chi$, so that $\Theta_\pi = \Theta(T: \chi: \lambda)$ (see §1). Then Θ_π has infinitesimal character χ_λ so that $zf = \chi_\lambda(z) f$ for all $z \in \mathscr{Z}(\mathfrak{g})$. Now by Proposition 5.4 of [1] we conclude that the possible leading characters of f are $e^{\rho + {}^y(s\lambda)}$ where $y \in \mathrm{Int}(\mathfrak{g}_{\mathbf{C}})$ gives the Cayley transform ${}^y(\mathfrak{t}_{\mathbf{C}}) = \mathfrak{h}_{\mathbf{C}}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$, and $s \in W(\mathfrak{g}_{\mathbf{C}}, \mathfrak{t}_{\mathbf{C}})$. But the weak inequality for $\Theta_{\pi,F}$ implies that $|f(a)| \leqslant M\, e^{\rho(\log a)}(1+\sigma(a))^m$ for all $a \in cl(A^-)$ so that by Theorem 7.1 of [1], the leading characters ν of f all satisfy $|\nu(a)| \leqslant e^{\rho(\log a)}$ for all $a \in A^-$. Thus $e^{\rho + {}^y(s\lambda)}$ is a leading character of f only if ${}^y(s\lambda)(\log a) \leqslant 0$ for all $a \in A^-$. But the regularity of λ now implies that ${}^y(s\lambda)(\log a) < 0$ for all $a \neq 1$ in $cl(A^-)$. Thus $e^{(\rho + {}^y(s\lambda))(\log a)} < e^{\rho(\log a)}$ for all $a \neq 1$ in $cl(A^-)$. Now by Theorem 7.5 of [1], f, and hence $\Theta_{\pi,F}$, are square-integrable on G. Hence by (4.2),

$$\Theta_{\pi,F} \in \mathscr{C}(G/Z, \zeta). \qquad \text{QED}$$

COROLLARY 4.10: *Suppose Z is central in G and compact. Fix a finite subset $F \subseteq \hat{K}_\zeta$. Then there are only finitely many $\pi \in \hat{G}_{\zeta\text{-disc}}$ so that $\Theta_\pi * \alpha_F \neq 0$.*

PROOF: We repeat the argument of Harish-Chandra [2c, Lemma 70]. For $\lambda \in L'$, let $\hat{G}(\lambda) = \{\pi \in \hat{G}_{\zeta\text{-disc}}: z\Theta_\pi = \chi_\lambda(z)\Theta_\pi \text{ for all } z \in \mathscr{Z}(\mathfrak{g})\}$. Let $Y_1, \ldots, Y_p$ and $Z_1, \ldots, Z_q$ be bases for $\mathfrak{k}$ and $\mathfrak{p}$ respectively which are "orthonormal" with respect to the canonical bilinear form on $\mathfrak{g}$. Put $\omega_1 = -Y_1^2 - \ldots - Y_p^2$ and $\omega_2 = Z_1^2 + \ldots + Z_q^2$, $\omega = \omega_1 + \omega_2$. Then $\omega \in \mathscr{Z}(\mathfrak{g})$ and $\omega_1 \in \mathscr{Z}(\mathfrak{k})$. Further, for $\lambda \in L'$, $\chi_\lambda(\omega) = |\lambda|^2 - |\rho|^2$ and for $\delta \in \hat{K}$, $\omega_1 \alpha_\delta = \chi_\delta(\omega_1)\alpha_\delta$ where $\chi_\delta(\omega_1) \geqslant 0.$.

Now for $\pi \in \hat{G}(\lambda)$, $\lambda \in L'$, and $\delta \in \hat{K}$, $f = \Theta_\pi * \alpha_\delta \in \mathscr{C}(G) \subseteq L^2(G)$

by (4.9), and $\omega f = (|\lambda|^2 - |\rho|^2)f$, $\omega_1 f = \chi_\delta(\omega_1)f$. Thus $(|\lambda|^2 - |\rho|^2)\|f\|_2^2 = \langle f, \omega f\rangle_2 = \chi_\delta(\omega_1)\|f\|_2^2 + \langle f, \omega_2 f\rangle_2$. But

$$\langle f, \omega_2 f\rangle_2 = \sum_{i=1}^{q} \langle f, Z_i^2 f\rangle = \sum_{i=1}^{q} \langle -Z_i f, Z_i f\rangle$$

$$= -\sum_{i=1}^{q} \|Z_i f\|^2 \leqslant 0.$$

Thus $|\lambda|^2 \|f\|^2 \leqslant (|\rho|^2 + \chi_\delta(\omega_1))\|f\|_2^2$, so that if $f \neq 0$, $|\lambda|^2 \leqslant |\rho|^2 + \chi_\delta(\omega_1)$. Thus for each $\delta \in \hat{K}$ there is a constant $c(\delta) > 0$ so that if $\pi \in \hat{G}(\lambda)$ and $\Theta_\pi * \alpha_\delta \neq 0$, $|\lambda| \leqslant C(\delta)$. But there are only finitely many $\lambda \in L'$ with $|\lambda| \leqslant C(\delta)$. Further $[\hat{G}(\lambda)] < \infty$ for all $\lambda \in L'$. Thus there are only finitely many $\pi \in \hat{G}_{\zeta\text{-disc}}$ with $\Theta_\pi * \alpha_\delta \neq 0$. The same is clearly true for F any finite subset of $\hat{K}_\zeta$. QED

In order to complete the proof of Theorem 4.3 we need to remove the restriction that Z be compact in (4.10). We use the notation of (4.6). Thus $\zeta \in \hat{Z}$ and q: $ZG^0 \to G[\zeta]$ where $G[\zeta]$ has compact center. Then it is shown in [6, §3.3] that $[\psi] \to [\psi \circ q]$ is a bijection of $G[\zeta]\hat{}_1$ onto $(ZG^0)\hat{}_\zeta$ which maps $G[\zeta]\hat{}_{1\text{-disc}}$ onto $(ZG^0)\hat{}_{\zeta\text{-disc}}$. In that bijection distribution characters are related by $\Theta_{\psi \circ q} = \Theta_\psi \circ q$. Clearly if $q_K = q|_{ZG^0 \cap K}$, we have a bijection $[\delta] \to [\delta \circ q_K]$ of $K[\zeta]\hat{}_1$ onto $(K \cap ZG^0)\hat{}_\zeta$ which satisfies $\alpha_{\delta \circ q_K} = \alpha_\delta \circ q_K$. Thus for any $[\psi] \in G[\zeta]\hat{}_{1\text{-disc}}$, $[\delta] \in K[\zeta]\hat{}_{1\text{-disc}}$, $\Theta_{\psi \circ q} * \alpha_{\delta \circ q_K} = (\Theta_\psi \circ q) * (\alpha_\delta \circ q_K) = (\Theta_\psi * \alpha_\delta) \circ q$. Thus (4.10) applied to $G[\zeta]$ implies Theorem 4.3.

§5. Relative cusp forms

In this section we will show how the decomposition of the relative Schwartz space started in §4 is related to Harish-Chandra's theory of cusp forms.

For $f \in \mathscr{C}(G/Z, \zeta)$ and $P = MAN$ a parabolic group of G, define

$$f^P(x) = \int_N f(xn)\mathrm{d}n, \qquad x \in G \tag{5.1a}$$

$$f^{(P)}(m: a) = \mathrm{e}^{\rho(\log a)} f^P(ma), \qquad m \in M,\ a \in A; \tag{5.1b}$$

$$f_\nu^{(P)}(m) = \int_A f^{(P)}(m: a) a^{\mathrm{i}\nu} \mathrm{d}a,\ \nu \in \mathfrak{a}^*,\ m \in M. \tag{5.1c}$$

LEMMA 5.2: *Fix $r > s \geqslant 0$. Then for all $m \in M$, $a \in A$,* $\mathrm{e}^{\rho(\log a)} \int_N \Xi(man) \times (1 + \sigma(man))^{-(r+2d)}\, \mathrm{d}n \leqslant C\Xi_M(m)(1 + \sigma(ma))^{-s}$.

PROOF: This is a result about G/Z and so follows immediately from the corresponding theorem of Harish-Chandra [2d, Lemma 10.2].

QED

Using the standard arguments (see eg. [5, §8.5.3]), the above estimate yields the following

COROLLARY 5.3: For $f \in \mathscr{C}(G/Z, \zeta)$, the integral defining $f^P(x)$ converges absolutely, uniformly for x in compact subsets of G. Further:

For all $a \in A$ and $\nu \in \mathfrak{a}^*$, $f \to f^{(P)}(\cdot\,; a)$ and $f \to f_\nu^{(P)}$ are continuous mappings of $\mathscr{C}(G/Z, \zeta)$ into $\mathscr{C}(M/Z, \zeta)$. (5.3a)

For all $m \in M$, $f \to f^{(P)}(m : \cdot)$ is a continuous mapping of $\mathscr{C}(G/Z, \zeta)$ into $\mathscr{C}(A)$, the Euclidean Schwartz space of A. (5.3b)

For all $z \in \mathscr{Z}(\mathfrak{g})$, $(zf)^{(P)} = \mu_P(z) f^{(P)}$ where μ_P is the canonical embedding of $\mathscr{Z}(\mathfrak{g})$ into $\mathscr{Z}(\mathfrak{m}) \otimes S(\mathfrak{a})$. (5.3c)

We now define the space of relative cusp forms corresponding to $\zeta \in \hat{Z}$ by

$$°\mathscr{C}(G/Z, \zeta) = \{f \in \mathscr{C}(G/Z, \zeta) : f^P(x) = 0 \text{ for all } x \in G \text{ and all proper parabolic subgroups } P \text{ of } G\}. \tag{5.4}$$

LEMMA 5.5: *$°\mathscr{C}(G/Z, \zeta)$ is a closed subspace of $\mathscr{C}(G/Z, \zeta)$, stable under left translations by G.*

PROOF: Clearly for $y \in G$, $[L(y)f]^P(x) = f^P(y^{-1}x)$, $x \in G$. The fact that $°\mathscr{C}(G/Z, \zeta)$ is closed in $\mathscr{C}(G/Z, \zeta)$ follows easily from the fact that $\Xi(n)(1 + \sigma(n))^{-r}$ is integrable on N for sufficiently large $r > 0$.

QED

LEMMA 5.6: *Suppose $f \in \mathscr{C}(G/Z, \zeta)$ is $\mathscr{Z}$-finite. Then $f \in °\mathscr{C}(G/Z, \zeta)$.*

PROOF: Fix a proper parabolic subgroup $P = MAN$ of G. Then $\mathscr{Z}(\mathfrak{m}) \otimes S(\mathfrak{a})$ is a finite module over $\mu_P(\mathscr{Z}(\mathfrak{g}))$ [2a, Lemma 21] so that (5.3c) implies that $f^{(P)}$ is $\mathscr{Z}(\mathfrak{m}) \otimes S(\mathfrak{a})$-finite. Fix $m \in M$. Then by (5.3b), $g(a) = f^{(P)}(m : a)$ is a Schwartz function on A. But the only $S(\mathfrak{a})$-finite element in $\mathscr{C}(A)$ is zero. Thus $f^P(ma) = 0$ for all $m \in M$, $a \in A$. But replacing f by left translates we see that in fact $f^P(x) = 0$ for all $x \in G$.

QED

COROLLARY 5.7: *Every K-finite matrix coefficient of a relative discrete series representation is a sum of relative cusp forms.*

PROOF: Write $G = \bigcup_{i=1}^{r} x_i Z G^0$ and for $1 \leqslant i \leqslant r$ define $\zeta_i(z) = \zeta(x_i^{-1} z x_i)$. Then any K-finite matrix coefficient of $\pi \in \hat{G}_{\zeta\text{-disc}}$ is a sum $f = \sum_{i=1}^{r} f_i$ where each f_i is a $\mathscr{Z}$-finite, K-finite element of $L^2(G/Z, \zeta_i)$, hence in $\mathscr{C}(G/Z, \zeta_i)$. Now by (5.6), $f_i \in {}^\circ\mathscr{C}(G/Z, \zeta_i)$ for $1 \leqslant i \leqslant r$. QED

THEOREM 5.8: ${}^\circ\mathscr{C}(G/Z, \zeta) = \mathscr{C}_{\text{disc}}(G/Z, \zeta)$.

COROLLARY 5.9: *The space of relative cusp forms is discretely decomposable under the left regular action of G.*

The proof of theorem 5.8 will be done in two stages. We will first prove (5.8) when $G = ZG^0$ and then show how to extend the result to arbitrary groups of class (1.1).

LEMMA 5.10: *Assume that Z is central in G. Let $H = TA$ be a θ-stable Cartan subgroup of G, $P = MAN$ a parabolic subgroup with split component A. Let $\chi \in Z_M(M^0)^\wedge_\zeta$, $\tau \in L'_\chi$, $\nu \in \mathfrak{a}^*$. Then for all $f \in \mathscr{C}(G/Z, \zeta)$, $\Theta(H{:}\ \chi{:}\ \tau{:}\ \nu)(f) = \Theta_{\chi,\tau}((f_K)_\nu^{(P)})$ where $\Theta_{\chi,\tau}$ is the character of the relative discrete series representation of M corresponding to χ and τ and*

$$f_K(x) = \int_{K/Z} f(kxk^{-1})\, \mathrm{d}(kZ), \quad x \in G.$$

PROOF: This is proved in [6, 4.3.11b] for $f \in C_c^\infty(G/Z, \zeta)$. But both sides of the equation give tempered ζ-distributions so that the equality holds for all $f \in \mathscr{C}(G/Z, \zeta)$. QED

PROOF *of* (5.8) *when* $G = ZG^0$: Let $f \in {}^\circ\mathscr{C}(G/Z, \zeta)$, $P = MAN \neq G$. Then for $x \in G$,

$$(f_K)^P(x) = \int_N \int_{K/Z} f(kxnk^{-1})\, \mathrm{d}(kZ)\, \mathrm{d}n.$$

Since the integral over N converges absolutely and K/Z is compact, we can switch the order of integration to obtain

$$(f_K)^P(x) = \int_{K/Z} \left\{ \int_{kNk^{-1}} f(kxk^{-1}n)\, \mathrm{d}n \right\} \mathrm{d}(kZ)$$

$$= \int_{K/Z} f^{kPk^{-1}}(kxk^{-1})\, \mathrm{d}(kZ) = 0$$

since for each $k \in K$, kPk^{-1} is a proper parabolic subgroup of G with unipotent radical kNk^{-1}.

Let $H = TA \in \text{Car}(G)$, $\chi \in Z_M(M^0)\hat{}_\zeta$, $\tau \in L'_\chi$, $\nu \in \mathfrak{a}^*$. Fix $f \in {}^\circ\mathscr{C}(G/Z, \zeta)$. Then by (5.10),

$$\Theta(H: \chi: \tau: \nu)\big(L(x^{-1})f\big) = \Theta_{\chi,\tau}\Big(\big(L(x^{-1})f\big)^{(P)}_{K,\nu}\Big).$$

But $L(x^{-1})f \in {}^\circ\mathscr{C}(G/Z, \zeta)$ for all $x \in G$ so that $(L(x^{-1})f)^{(P)}_{K,\nu} \equiv 0$ if P is a proper parabolic subgroup of G, i.e. if $A \neq \{1\}$. Thus if $H = TA \in \text{Car}(G)$ and $f \in {}^\circ\mathscr{C}(G/Z, \zeta)$, $f_H = 0$ unless H is relatively compact. Since $f = \Sigma_{H \in \text{Car}(G)} f_H$, this proves that $f = 0$ if rank $G >$ rank K. Also if rank $G =$ rank K and $H = T$ is relatively compact, then $f = f_T \in \mathscr{C}_T(G/Z, \zeta)$.

Conversely, suppose $f \in \mathscr{C}_T(G/Z, \zeta)$. If f is K-finite, then as in (4.4), f is $\mathscr{Z}$-finite. Thus by (5.6), $f \in {}^\circ\mathscr{C}(G/Z, \zeta)$. This shows that $\mathscr{C}_T(G/Z, \zeta)^K \subseteq {}^\circ\mathscr{C}(G/Z, \zeta) \subseteq \mathscr{C}_T(G/Z, \zeta)$. But as in (2.16), $\mathscr{C}_T(G/Z, \zeta)^K$ is dense in $\mathscr{C}_T(G/Z, \zeta)$ and ${}^\circ\mathscr{C}(G/Z, \zeta)$ is closed, so that $\mathscr{C}_T(G/Z, \zeta) = {}^\circ\mathscr{C}(G/Z, \zeta)$.

QED

In order to finish the proof of Theorem 5.8 we need the following result which was also used in extending (5.5) from ZG^0 to G.

THEOREM 5.11: *Let $\zeta \in \hat{Z}$ and $f \in L_2(G/Z, \zeta)$. Then $f \in L_2(G/Z, \zeta)_{\text{disc}}$ if, and only if, $(L(x)f)|_{ZG^0} \in L_2(ZG^0/Z, \zeta)_{\text{disc}}$ for all $x \in G$; and $f \in L_2(G/Z, \zeta)_{\text{cont}}$ if, and only if, $(L(x)f)|_{ZG^0} \in L_2(ZG^0/Z, \zeta)_{\text{cont}}$ for all $x \in G$.*

PROOF: Each of these conditions is equivalent to the next:

(i) $f \in L_2(G/Z, \zeta)_{\text{disc}}$;
(ii) f is an $L_2(G/Z, \zeta)$-limit of $\mathscr{Z}(\mathfrak{g})$-finite functions;
(iii) the restriction of f to any coset of ZG^0 is an L_2 mod Z limit of $\mathscr{Z}(\mathfrak{g})$-finite functions there;
(iv) every $(L(x)f)|_{ZG^0}$ is an $L_2(ZG^0/Z, \zeta)$ limit of $\mathscr{Z}(\mathfrak{g})$-finite functions, and
(v) every $L(x)f|_{ZG^0} \in L_2(ZG^0/Z, \zeta)_{\text{disc}}$.

If $h \in L_2(G/Z, \zeta)$ is zero except on one coset of ZG^0, and there is in the discrete spectrum, then $h \in L_2(G/Z, \zeta)_{\text{disc}}$. Combine this with the fact that, for both G and ZG^0, the continuous spectrum is the orthocomplement of the discrete spectrum. Then the second assertion follows from the first. QED

COROLLARY 5.13. *Let G be a reductive group in the class* (1.1). *Then $f \in \mathscr{C}_{\text{disc}}(G/Z, \zeta)$ if and only if for all $x \in G$,*

$$L(x)f|_{ZG^0} \in \mathscr{C}_{\text{disc}}(ZG^0/Z, \zeta).$$

PROOF *of* (5.8) *for general* G: Clearly $f \in {}^\circ\mathscr{C}(G/Z, \zeta)$ if and only if all $L(x)f|_{ZG^0} \in {}^\circ\mathscr{C}(ZG^0/Z, \zeta)$. Thus the theorem follows from (5.13) together with the fact that ${}^\circ\mathscr{C}(ZG^0/Z, \zeta) = \mathscr{C}_{\text{disc}}(ZG^0/Z, \zeta)$.

QED

§6. The global Schwartz space

In order to formulate and prove the Plancherel formula for rapidly decreasing functions, we must make a mild restriction on the class (1.1) of reductive Lie groups. This restriction allows the use of techniques of tempered analysis along Z; the point is that growth along Z will no longer be controlled by some unitary character $\zeta \in \hat{Z}$. Thus, from now on, G is a reductive Lie group such that

$$G \text{ satisfies the conditions } (1.1), \text{ and} \tag{6.1a}$$

$$\hat{Z} \text{ is a Lie group, i.e. } Z/Z^0 \text{ is finitely generated.} \tag{6.1b}$$

We note that (6.1) is a hereditary class in the sense of

LEMMA 6.2: *The Levi component of every cuspidal parabolic subgroup of* G *satisfies* (6.1).

Our first task in defining the Schwartz space of G is to replace the polynomial growth function σ of §§2–5 by a function $\tilde{\sigma}$ that detects distance along Z.

LEMMA 6.3: $K \cap ZG^0$ *has a unique maximal compact subgroup* $K^\vee$ *and has global structure*

$$K \cap ZG^0 = K^\vee \times V \times E, \tag{6.4a}$$

where V *is a vector group such that*

$$K \cap G^0 \text{ has finite index in } K^\vee \times V \text{ and } Z \cap V \text{ is cocompact in } V, \tag{6.4b}$$

and where E *is a finitely generated free abelian group such that*

$$Z = \{(Z \cap G^0) \cdot (\text{finite abelian})\} \times E. \tag{6.4c}$$

ZG^0/G^0 has a unique maximal compact subgroup. Let $G^\vee$ denote its inverse image under $ZG^0 \to ZG^0/G^0$. Then

$$G^\vee = G^0 K^\vee \text{ and } ZG^0 = G^\vee \times E. \tag{6.4d}$$

In particular, $(e, v, k^{\vee}, \xi) \mapsto evk^{\vee} \cdot \exp(\xi)$ is a diffeomorphism of $E \times V \times K^{\vee} \times \mathfrak{p}$ onto ZG^0.

PROOF: Let us generically use F's for finite abelian groups, E's for finitely generated free abelian groups, T's for tori, and V's for vector groups. Then $Z/Z^0 = F_1 \times E_1$ from (6.1b). As $Z^0 = T_2 \times V_2$ is divisible, and Z is abelian, now $Z = T_2 \times V_2 \times F_2 \times E_2$ where F_2, E_2 map isomorphically onto F_1, E_1. Now $T_2 \times F_2$ is the unique maximal compact subgroup of Z. Note $ZG^0/G^0 = Z/Z \cap G^0$, so the former has a unique maximal compact subgroup, and $G^{\vee}$ is well defined. E_2 splits as the product of $E_2 \cap G^0$ and E, and evidently $V_2 \subset G^0$, so $Z \cap G^{\wedge} = T_2 \times V_2 \times F_2 \times (E_2 \cap G^0)$ and $Z = (Z \cap G^{\vee}) \times E$. Thus $ZG^0 = G^{\vee} \times E$. In particular, $K \cap ZG^0 = (K \cap G^{\vee}) \times E$.

$K \cap ZG^0 = (K \cap G^0)Z$, and $K \cap G^0$ is connected. Since $(K \cap G^0)/(Z \cap G^0)$ is compact, now $K \cap G^0 = [K^0, K^0] \cdot Z_{K \cap G^0}$ and $Z_{K \cap G^0} = F_3 \times T_3 \times V$. Now $K \cap G^0$ has unique maximal compact subgroup $[K^0, K^0] \cdot (F_3 \times T_3)$. It follows that $K \cap ZG^0$ has a unique maximal compact subgroup $K^{\vee}$, generated by $[K^0, K^0]$, F_2, F_3, T_2 and T_3. We may assume $V_2 \subset V$. Now $K \cap ZG^0$ is generated by $K^{\vee}$, V and E, and that is a direct product.

We now have (6.4a, b, c) and the second part of (6.4d). Since $K^{\vee} \times V$ maps into the maximal compact subgroup of ZG^0/G^0, it is in $G^{\vee}$. Compare $ZG^0 = G^{\vee} \times E$ with (6.4a) to see $G^{\vee} = G^0 K^{\vee}$.

Finally, we of course have the diffeomorphism $(K \cap ZG^0) \times \mathfrak{p} \to ZG^0$ by $(k, \xi) \mapsto k \cdot \exp(\xi)$. The last assertion just combines this with (6.4a).

QED

View the group E of Lemma 6.3 as a lattice in a vector group U, and define a norm $\|ev\|$ on $E \times V$ as induced by a positive definite inner product on $U \oplus V$. Now, using the diffeomorphism of Lemma 6.3, we define

$$\tilde{\sigma}: ZG^0 \to \mathbb{R}^+ \text{ by } \tilde{\sigma}(evk^{\vee} \cdot \exp \xi) = \|ev\| + \|\xi\| \tag{6.5}$$

where $e \in E$, $v \in V$, $k^{\vee} \in K^{\vee}$ and $\xi \in \mathfrak{p}$. The following are immediate consequences of the definition:

$$\text{if } K \text{ is compact then } \tilde{\sigma} = \sigma|_{ZG^0}; \tag{6.6a}$$

$$\text{if } x \in ZG^0 \text{ then } \tilde{\sigma}(x) \geqslant \sigma(x); \tag{6.6b}$$

$$\tilde{\sigma}(k_1 x k_2) = \tilde{\sigma}(x) \text{ for } k_i \in K^{\vee} \text{ and } x \in ZG^0; \tag{6.6c}$$

$$\left\{\begin{array}{l}\text{if } G \text{ is of Harish-Chandra class then } \tilde{\sigma} \text{ is equivalent}\\ \text{to the restriction of Harish-Chandra's } \sigma \text{ to } ZG^0.\end{array}\right. \tag{6.6d}$$

The analog of (2.2c) is a little more subtle. Denote

$$X^{\vee} = (U \times G^{\vee})/(\{0\} \times K^{\vee}) = U \times (G^{\vee}/K^{\vee}) \tag{6.7a}$$

with invariant riemannian metric such that $\mathfrak{u} \perp \mathfrak{v}$, $(\mathfrak{u} + \mathfrak{v}) \perp \mathfrak{p}$, and the scalar product on $\mathfrak{p}$ comes from the Killing form. Then the projection

$$q\colon X^{\vee} \to X = ZG^0/(K \cap ZG^0) \text{ by } (u, gK^{\vee}) \to g(K \cap ZG^0) \tag{6.7b}$$

is a reimannian submersion with fibre $U \times \{\dot{V}/(V \cap Z)\}$. In particular $U \times V$ is a totally geodesic manifold of $X^{\vee}$. So, if o is the base point $\{0\} \times 1 \cdot K^{\vee}$ in $X^{\vee}$, then

$$\text{distance}(o, wo) = \|w\| \text{ for all } w \in U \times V. \tag{6.7c}$$

Now we are going to verify that

$$\text{if } \xi, \eta, \zeta \in \mathfrak{p} \text{ and } \exp(-\zeta)\exp(\xi)\exp(\eta) = \mathrm{wk} \text{ with } \mathrm{w} \in \mathrm{U} \times \mathrm{V} \text{ and } \mathrm{k} \in \mathrm{K}^{\vee} \text{ then } \|w\| \leqslant \|\xi\| + \|\eta\| + \|\zeta\|. \tag{6.8}$$

Write d for distance, e^{ξ} for $\exp(\xi)$, etc. First, from the triangle inequality on $X^{\vee}$,

$$\begin{aligned} d(o, e^{-\zeta} e^{\xi} o) &\leqslant d(o, e^{-\zeta} o) + d(e^{-\zeta} o, e^{-\zeta} e^{\xi} o) \\ &\leqslant \text{length}\{e^{-t\zeta} o\}_{0 \leqslant t \leqslant 1} + \text{length}\{e^{-\zeta} e^{t\xi} o\}_{0 \leqslant t \leqslant 1} \\ &= \text{length}\{e^{-t\zeta} o\}_{0 \leqslant t \leqslant 1} + \text{length}\{e^{t\xi} o\}_{0 \leqslant t \leqslant 1} \\ &= \text{length}\{q(e^{-t\zeta} o)\}_{0 \leqslant t \leqslant 1} + \text{length}\{q(e^{t\xi} o)\}_{0 \leqslant t \leqslant 1} \\ &= \|\zeta\| + \|\xi\|. \end{aligned}$$

Second, again from the triangle inequality,

$$\begin{aligned} d(o, e^{-\zeta} e^{\xi} e^{\eta} o) &\leqslant d(o, e^{-\zeta} e^{\xi} o) + d(e^{-\zeta} e^{\xi} o, e^{-\zeta} e^{\xi} e^{\eta} o) \\ &= d(o, e^{-\zeta} e^{\xi} o) + d(o, e^{\eta} o) \\ &\leqslant d(o, e^{-\zeta} e^{\xi} o) + \|\eta\|. \end{aligned}$$

Since $\|w\| = d(o, wo) = d(o, wko)$, these two inequalities combine to give (6.8). Now we will use (6.8) to prove that

$$\text{if } x, y \in ZG^0 \text{ then } \tilde{\sigma}(xy) \leqslant 3(\tilde{\sigma}(x) + \tilde{\sigma}(y)). \tag{6.9a}$$

In effect, let $x = w_1 k_1 \, e^{\xi}$ and $y = w_2 k_2 \, e^{\eta}$ with $w_i \in U \times V$, $k_i \in \check{K}$ and $\xi, \eta \in \mathfrak{p}$. Then $xy = (w_1 w_2)(k_1 k_2) \, e^{\xi'} e^{\eta}$ where $\xi' = \mathrm{Ad}(w_2 k_2)^{-1} \xi \in \mathfrak{p}$ has $\| \xi' \| = \| \xi \|$. Express $e^{\xi'} e^{\eta} = wk \, e^{\zeta}$ with $w \in U \times V$, $k \in \check{K}$ and $\zeta \in \mathfrak{p}$. Then $xy = (w_1 w_2 w)(k_1 k_2 k) \, e^{\zeta}$. Compute, using (6.8),

$$\begin{aligned}
\tilde{\sigma}(xy) &= \| w_1 w_2 w \| + \| \zeta \| \\
&\leqslant \| w_1 \| + \| w_2 \| + \| w \| + \| \zeta \| \\
&\leqslant \| w_1 \| + \| w_2 \| + \| \xi \| + \| \eta \| + 2 \| \zeta \| \\
&= \tilde{\sigma}(x) + \tilde{\sigma}(y) + 2 \| \zeta \| \\
&= \tilde{\sigma}(x) + \tilde{\sigma}(y) + 2 \sigma(xy) \\
&\leqslant \tilde{\sigma}(x) + \tilde{\sigma}(y) + 2(\sigma(x) + \sigma(y)) \\
&\leqslant 3(\tilde{\sigma}(x) + \tilde{\sigma}(y)),
\end{aligned}$$

which is the statement of (6.9a). Note the immediate consequence

$$\begin{cases} \text{if } \omega \subset G \text{ is compact there exist } \mathrm{d}_1, \mathrm{d}_2 > 0 \text{ such that} \\ \mathrm{d}_1(1 + \tilde{\sigma}(xy)) \leqslant 1 + \tilde{\sigma}(x) \leqslant \mathrm{d}_2(1 + \tilde{\sigma}(xy)) \text{ for } x \in \\ \mathrm{G},\ y \in \omega. \end{cases} \tag{6.9b}$$

As before, Ξ denotes the spherical function of (2.4). Since $\tilde{\sigma}(x) \geqslant \sigma(x)$ we still have (2.5e) with σ replaced by $\tilde{\sigma}$.

The group E of Lemma 6.3 need not be normal in K, so we don't have a decomposition of K similar to the splitting (6.4a) of $K \cap ZG^0$. As a result, we do not have a convenient extension of $\tilde{\sigma}$ from ZG^0 to G. Thus we define the Schwartz space $\mathscr{C}(G)$ by behavior of functions on cosets of ZG^0, as in (2.6c). Here are the details.

Given $f \in C^{\infty}(ZG^0)$, D_1, $D_2 \in \mathscr{U}(\mathfrak{g})$, and $r \in \mathbb{R}$, we define

$$_{D_1}\| f \|_{r, D_2} = \sup_{x \in ZG^0} \, (1 + \tilde{\sigma}(x))^r \, \Xi(x)^{-1} | f(D_1 ; x ; D_2) |. \tag{6.10a}$$

The *Schwartz space* on ZG^0 is

$$\mathscr{C}(ZG^0) = \{ f \in C^{\infty}(ZG^0) \colon \text{if } r \in \mathbb{R} \text{ and } D_i \in \mathscr{U}(\mathfrak{g}) \text{ then } {}_{D_1}\| f \|_{r, D_2} < \infty \}. \tag{6.10b}$$

It is a complete locally convex *TVS* with topology defined by the seminorms (6.10a). Left and right translations by elements of ZG^0 preserve its structure. Thus the *Schwartz space on G*,

$$\mathscr{C}(G) = \{ f \in C^\infty(G) \colon \text{if } x \in G \\ \text{then} \quad (L(x)f)|_{ZG^0} \in \mathscr{C}(ZG^0) \} \tag{6.10c}$$

is well defined, using any system of representatives $\{x\}$ of G/ZG^0. Again, $\mathscr{C}(G)$ is a complete locally convex *TVS*. Its topology is defined by the seminorms ${}_{D_1,i}\| f \|_{r,D_2} = {}_{D_1}\|((L(x_i)f)\mid_{ZG^0})\|_{r,D_2}$ as x_i runs over a set of representatives of G/ZG^0.

THEOREM 6.11: *$\mathscr{C}(G)$ is a dense subspace of $L_2(G)$, and the inclusion $\mathscr{C}(G) \to L_2(G)$ is continuous.*

PROOF: We need the analog of (2.5f), which is

there is a number $r \geqslant 0$ such that

$$\int_{ZG^0} \Xi(x)^2 (1 + \tilde{\sigma}(x))^{-r} \, dx < \infty. \tag{6.12}$$

Then Theorem 5.11 follows as in the relative case (2.7).

To prove (6.12), assume $G = ZG^0$. If $x \in G$ and $w \in E \times V$ then $\tilde{\sigma}(wx) \geqslant \sigma(wx) = \sigma(x)$, so if $r \geqslant 0$ then

$$(1 + \tilde{\sigma}(wx))^{-r} \leqslant (1 + \sigma(x))^{-r/2} (1 + \tilde{\sigma}(wx))^{-r/2}.$$

Now

$$\int_G \Xi(x)^2 (1 + \tilde{\sigma}(x))^{-r} \, dx = \int_{G^\vee/V} \Xi(x)^2 \int_{E\times V} (1 + \tilde{\sigma}(wx))^{-r} \, dw \, d\dot{x}$$

$$\leqslant \int_{G^\vee/V} \Xi(x)^2 (1 + \sigma(x))^{-r/2} \times \int_{E\times V} (1 + \tilde{\sigma}(wx))^{-r/2} \, dw \, d\dot{x}.$$

Given $x \in G$, say $x = w_1 k_1 \exp(\xi)$, $\tilde{\sigma}(wx) = \| ww_1 \| + \| \xi \| \geqslant \| ww_1 \|$, so

$$\int_{E\times V} (1 + \tilde{\sigma}(wx))^{-r/2} \, dw \leqslant \int_{E\times V} (1 + \| ww_1 \|)^{-r/2} \, dw$$

$$= \int_{E\times V} (1 + \| w \|)^{-r/2} \, dw,$$

which is finite for sufficiently large r. Thus

$$\int_G \Xi(x)^2(1+\tilde{\sigma}(x))^{-r}\,\mathrm{d}x \leqslant A(r)\int_{G^\vee/V} \Xi(x)^2(1+\sigma(x))^{-r/2}\,\mathrm{d}\dot{x}$$

where $A(r)<\infty$ for r sufficiently large, as just seen, and

$$\int_{G^\vee/V} \Xi(x)^2(1+\sigma(x))^{-r/2}\,\mathrm{d}\dot{x} = \int_{G/Z} \Xi(x)^2(1+\sigma(x))^{-r/2}\,\mathrm{d}(xZ)$$

is finite by (2.5f) for sufficiently large r. QED

THEOREM 6.13: *$C_c^\infty(G)$ is dense in $\mathscr{C}(G)$.*

PROOF: We may assume $G=ZG^0$ and follow the argument of Theorem 2.8. There we replace σ by $\tilde{\sigma}$ and integrate over G rather than G/Z. The only changes in the analogs of (2.9)–(2.11) come from use of (6.9a) in place of (2.3c). Thus the correct analog of (2.9d) is

$$g(x)=0 \text{ for } \tilde{\sigma}(x)\leqslant t_0 \text{ where } t_0=t/9-4s/3,$$

$$g(x)=1 \text{ for } \tilde{\sigma}(x)\geqslant t_1 \text{ where } t_1=9t+12s,$$

and the ranges for the three cases of (2.11) are $\tilde{\sigma}(x)>t_1$, $t_1\geqslant\tilde{\sigma}(x)\geqslant t_0$ with $t_0>0$, and $t_0>\tilde{\sigma}(x)$. QED

We will, of course, say that a distribution on G is *tempered* if it has a continuous extension from $C_c^\infty(G)$ to $\mathscr{C}(G)$.

THEOREM 6.14: *Let T be a locally L_1 function on G. Suppose that there is an integer $m\geqslant 0$ such that*

$$\sup_{1\leqslant i\leqslant u}\ \operatorname*{ess\,sup}_{x\in x_iZG^0} \left(1+\tilde{\sigma}(x_i^{-1}x)\right)^{-m}|D_l(x)|^{1/2}|T(x)|<\infty \tag{6.15}$$

where $G=x_1ZG^0\cup\ldots\cup x_uZG^0$ with $x_i\in K$. Then integration against T is a tempered distribution on G. In fact, if $f\in\mathscr{C}(G)$ then $|T(f)|\leqslant c\|T\|_m\Sigma_i\|f\|_{r+m}$ where c, $r>0$ depend only on G and where $\|T\|_m$ is given by (6.15).

In particular, if T is a central $\mathscr{Z}(\mathfrak{g})$-finite distribution on G, and if there is an integer $m\geqslant 0$ such that

$$\sup_{1\leqslant i\leqslant u}\ \sup_{x\in x_iZG^0\cap G'} \left(1+\tilde{\sigma}(x_i^{-1}x)\right)^{-m}|D_l(x)|^{1/2}|T(x)|<\infty \tag{6.16}$$

then T is tempered with $|T(f)|\leqslant c\|T\|_m\Sigma_i\|f\|_{r+m}$ as above.

PROOF: As in the relative case, let $f \in \mathscr{C}(G)$ and compute

$$\left|\int_G f(x)T(x)\,\mathrm{d}x\right|$$

$$\leqslant \sum_{i=1}^{u} \int_{ZG^0} |f(x_i x)||T(x_i x)|\,\mathrm{d}x$$

$$\leqslant \|T\|_m \sum_{i=1}^{u} \int_{ZG^0} |f(x_i x)|(1+\tilde{\sigma}(x))^m |D_l(x_i x)|^{-1/2}\,\mathrm{d}x$$

$$\leqslant \|T\|_m \sum_{i=1}^{u} \left\{ {}_i\|f\|_{r+m} \int_{ZG^0} (1+\tilde{\sigma}(x))^{-r-m} \Xi(x)(1+\tilde{\sigma}(x))^m \right.$$

$$\left. \times |D_l(x_i x)|^{-1/2}\,\mathrm{d}x \right\}$$

$$\leqslant c\|T_m\| \textstyle\sum {}_i\|f\|_{r+m}$$

where c is the maximum of the numbers

$$c_i = \int_{ZG^0} (1+\tilde{\sigma}(x))^{-r} \Xi(x)|D_l(x_i x)|^{-1/2}\,\mathrm{d}x.$$

Since each $x_i \in K$, the proof of Theorem 6.14 is now reduced to that of

if $k \in K$ then there is a number $r \geqslant 0$ such that

$$\int_{ZG^0} (1+\tilde{\sigma}(x))^{-r} \Xi(x)|D_l(kx)|^{-1/2}\mathrm{d}x < \infty. \tag{6.17}$$

The proof of (6.17) is a matter of using (2.15) with the argument for (6.12). In effect, since $\Xi(xz) = \Xi(x)$ and $D_l(kxz) = D_l(kx)$ for $x \in G$ and $z \in Z$, we have

$$\int_{ZG^0} (1+\tilde{\sigma}(x))^{-r} \Xi(x)|D_l(kx)|^{-1/2}\,\mathrm{d}x$$

$$= \int_{ZG^0/Z} \Xi(x)|D_l(kx)|^{-1/2} \int_Z (1+\tilde{\sigma}(xz))^{-r}\,\mathrm{d}z\,\mathrm{d}(xZ)$$

$$\leqslant \int_{ZG^0/Z} (1+\sigma(x))^{-r/2} \Xi(x)|D_l(kx)|^{-1/2}$$

$$\times \int_Z (1+\tilde{\sigma}(xz))^{-r/2}\,\mathrm{d}z\,\mathrm{d}(xZ)$$

$$= \int_{ZG^0/Z} (1+\sigma(kx))^{-r/2}\Xi(kx)|D_l(kx)|^{-1/2}$$

$$\times \int_Z (1+\tilde{\sigma}(xz))^{-r/2}\, \mathrm{d}z\, \mathrm{d}(xZ)$$

If r is sufficiently large then (2.15) holds for $r/2$ and we use it on the coset kZG^0/Z of ZG^0/Z in G/Z. If r is sufficiently large then $\int_{E\times V} (1+\|w\|)^{-r/2}\, \mathrm{d}w < \infty$, which implies

$$\int_Z (1+\tilde{\sigma}(xz))^{-r/2}\, \mathrm{d}z \leqslant \int_Z (1+\tilde{\sigma}(z))^{-r/2}\, \mathrm{d}z < \infty.$$

Thus (6.17) holds for r sufficiently large, and so Theorem 6.14 is proved. QED

The argument of [5, 8.3.7.8] goes through and gives us

the left and right regular representations of G on $\mathscr{C}(G)$ are differentiable. (6.18)

Similarly, one can go through the proof of [5, 8.3.7.14] and see that no essential change is needed here. Thus (6.18) extends to

$\mathscr{C}(G)$ is a topological algebra under convolution. (6.19)

Density of K-finite functions cannot be expected, unless of course K is compact. But we can at least conclude from (6.18) that

$$\text{if } f\in\mathscr{C}(G) \text{ then } \sum_{\delta_1}\alpha_{\delta_1} * f \to f,\ \sum_{\delta_2} f * \alpha_{\delta_2} \to f$$

$$\text{and } \sum_{\delta_1,\delta_2} \alpha_{\delta_1} * f * \alpha_{\delta_2} \to f \qquad (6.20)$$

where $\delta_i \in (K^{\vee})^{\wedge}$, α_{δ_i} is normalized character, and the convergence is absolute convergence in $\mathscr{C}(G)$. Thus the $K^{\vee}$-finite functions are dense in $\mathscr{C}(G)$.

§7. The global Plancherel formula

In this section we will extend the global Plancherel formula, proved in [4] for functions in $C_c^\infty(G)$, to the global Schwartz space $\mathscr{C}(G)$.

For $f\in\mathscr{C}(G)$ and $\zeta\in\hat{Z}$ we define

$$f_\zeta(x) = \int_Z f(xz)\zeta(z)\, \mathrm{d}z. \qquad (7.1)$$

THEOREM 7.2: *If $\zeta \in \hat{Z}$ then $f \mapsto f_\zeta$ is a continuous map $\mathscr{C}(G) \to \mathscr{C}(G/Z, \zeta)$. In fact, given a system $\{x_i\} \subset K$ of coset representatives of G modulo ZG^0, there are constants c, $d > 0$ independent of f, ζ and i such that, if D_1, $D_2 \in \mathscr{U}(\mathfrak{g})$ and $r \in \mathbb{R}$, then*

$$_{D_1,i}| f_\zeta |_{r,D_2} \leqslant c \cdot {}_{D_1,i}\| f \|_{r+d,D_2}. \tag{7.3}$$

PROOF: Let $d > 0$ such that $\int_Z (1 + \tilde{\sigma}(z))^{-d}\, \mathrm{d}z < \infty$. Since Z centralizes G^0 we have

$$f_\zeta(D_1 ; x ; D_2) = (D_1 f D_2)_\zeta(x).$$

Now compute

$$_{D_1,i}| f_\zeta |_{r,D_2} = \sup_{x \in ZG^0} (1 + \sigma(x))^r \Xi(x)^{-1} \left| f_\zeta\left(D_1 ; x_i^{-1}x ; D_2\right)\right|$$

$$= \sup_{x \in ZG^0} (1 + \sigma(x))^r \Xi(x)^{-1}$$

$$\times \left| \int_Z f\left(D_1 ; x_i^{-1}xz ; D_2\right)\zeta(z)\, \mathrm{d}z \right|$$

$$\leqslant \sup_{x \in ZG^0} (1 + \sigma(x))^r \Xi(x)^{-1} \int_Z \left| f\left(D_1 ; x_i^{-1}xz ; D_2\right)\right| \mathrm{d}z.$$

Since $f \in \mathscr{C}(G)$ we have

$$\left| f\left(D_1 ; x_i^{-1}xz ; D_2\right)\right| \leqslant c_i (1 + \tilde{\sigma}(xz))^{-r-d} \Xi(x)$$

where $c_i = {}_{D_1,i}\| f \|_{r+d,D_2}$. Now we have

$$_{D_1,i}| f_\zeta |_{r,D_2} \leqslant c_i \sup_{x \in ZG^0} (1 + \sigma(x))^r \int_Z (1 + \tilde{\sigma}(xz))^{-r-d}\, \mathrm{d}z.$$

Since $\tilde{\sigma}(xz) \geqslant \sigma(xz) = \sigma(x)$ this gives

$$_{D_1,i}| f_\zeta |_{r,D_2} \leqslant c_i \sup_{x \in ZG^0} \int_Z (1 + \tilde{\sigma}(xz))^{-d}\, \mathrm{d}z.$$

If $x = z_0 k_0 p_0$ with $z_0 \in E \times V$, $k_0 \in K^{\vee}$ and $p_0 \in \exp(\mathfrak{p})$, then $\tilde{\sigma}(xz) = \tilde{\sigma}(z_0 z) + \tilde{\sigma}(p_0)$ so

$$\int_Z (1 + \tilde{\sigma}(xz))^{-d}\,\mathrm{d}z = \int_Z (1 + \tilde{\sigma}(p_0) + \tilde{\sigma}(z_0 z))^{-d}\,\mathrm{d}z$$

$$\leqslant \int_Z (1 + \tilde{\sigma}(z))^{-d}\,\mathrm{d}z.$$

Now finally, ${}_{D_1,\iota}|f_\zeta|_{r,D_2} \leqslant c \cdot c_\iota$ where $c = \int_Z (1 + \tilde{\sigma}(z))^{-d}\,\mathrm{d}z$.

QED

LEMMA 7.4: *Let $H = T \times A$ be a θ-stable Cartan subgroup of G and $M = M_H$. If $\chi \in Z_M(M^0)^{\wedge}$, $\tau \in L'_\chi$ and $\nu \in \mathfrak{a}^*$, then $\Theta(H{:}\ \chi{:}\ \tau{:}\ \nu)$ is a tempered distribution on G. In fact, there are constants c, $s > 0$ depending only on G, such that if $f \in \mathscr{C}(G)$ then*

$$|\Theta(H{:}\ \chi{:}\ \tau{:}\ \nu)(f)| \leqslant c \cdot \sum_i {}_i\|f\|_s \tag{7.5}$$

where $\{x_\iota\} \subset K$ is a set of representatives of G modulo ZG^0.

PROOF: This follows from (6.14) using the estimate of Lemma 3.1.

QED

It would require additional machinery to prove the global analog of Lemma 3.5 directly. Instead, we proceed as in [4, §6], integrating the relative Plancherel formula to obtain a global Plancherel formula for ZG^0, and then extending the formula from ZG^0 to G.

Let $f \in \mathscr{C}(ZG^0)$. Then $f = \int_{\hat{Z}} f_\zeta\,\mathrm{d}\zeta$ in the sense that $f(x) = \int_{\hat{Z}} f_\zeta(x)\,\mathrm{d}\zeta$ for all $x \in ZG^0$. Theorem 7.2 and Theorem 3.6 give us

$$f_\zeta(1) = \sum_{J \in \mathrm{Car}(G)} \sum_{\chi^0 \in Z_{M_J \cap ZG^0}(M_J^0)^{\wedge}_\zeta} \deg(\chi^0)$$

$$\times \sum_{\tau \in L'_{\chi^0}} \int_{\mathfrak{j}_p^*} \Theta^0(J \cap ZG^0{:}\ \chi^0{:}\ \tau{:}\ \nu)(f_\zeta)$$

$$\times m^0(J \cap ZG^0{:}\ \chi^0{:}\ \tau{:}\ \nu)\,\mathrm{d}\nu$$

where Θ^0, m^0 refer to ZG^0. The integral over $\hat{Z}$ moves past $\sum_{J \in \mathrm{Car}(G)}$ and combines with the sum over $Z_{M_J \cap ZG^0}(M_J^0)^{\wedge}_\zeta$ to give a sum over $Z_{M_J \cap ZG^0}(M_J^0)^{\wedge}$. The result is

LEMMA 7.5: *Let G be a reductive group in the class* (6.1). *If* $f \in \mathscr{C}(ZG^0)$ *then*

$$f(1) = \sum_{J \in \mathrm{Car}(G)} \int_{\chi^0 \in Z_{M_J \cap ZG^0}(M_J^0)^\wedge} \deg(\chi^0) \sum_{\tau \in L'_{\chi^0}}$$

$$\times \int_{\mathfrak{i}_p^*} \Theta^0(J \cap ZG^0 : \chi^0 : \tau : \nu)(f) m^0(J \cap ZG^0 : \chi^0 : \tau : \nu)$$

$$\times \mathrm{d}\nu \, \mathrm{d}\chi^0$$

where, using Lemma 7.4, *distribution characters are evaluated by integration over* ZG^0.

Now mimic the calculation in [4, §6]. *The calculation is formal and algebraic, and it leads from Lemma* 7.5 *to*

THEOREM 7.6 (*global Plancherel Theorem*). *Let G be a reductive group in the class* (6.1). *If* $f \in \mathscr{C}(G)$ *then*

$$f(1) = \left[G/Z_G(G^0)G^0\right]^{-1} \sum_{J \in \mathrm{Car}(G)} \int_{\chi \in Z_{M_J}(M_J^0)^\wedge} \deg(\chi)$$

$$\times \sum_{\tau \in L'_\chi} \int_{\mathfrak{i}_p^*} \Theta(J : \chi : \tau : \nu)(f) m(J : \chi : \tau : \nu) \mathrm{d}\nu \mathrm{d}\chi$$

as in Theorem 1.11.

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