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ON A DIOPHANTINE EQUATION INVOLVING QUADRATIC CHARACTERS

Karl Dilcher

1. Introduction

The equation

$$1^k + 2^k + \cdots + x^k = y^m \tag{1}$$

was first studied by E. Lucas (in a paper of 1877) for the case $k = m = 2$. J. Schäffer [15] proved that for fixed positive k and m, except for a few exceptional pairs (k, m), the equation (1) has a finite number of solutions in integers x and y.

K. Győry, R. Tijdeman, and M. Voorhoeve [9] extended this result by proving that for fixed integers r, $b \neq 0$, and $k \geqslant 2$, $k \notin \{3, 5\}$ the equation

$$1^k + 2^k + \cdots + x^k + r = by^z \tag{2}$$

has only finitely many solutions in integers x, $y \geqslant 1$ and $z \geqslant 2$, and upper bounds for x, y and z can be effectively determined. The result in [9] is actually somewhat stronger.

In [16], Győry, Tijdeman and Voorhoeve give a similar, though ineffective result, in which r is replaced by a polynomial $R(x)$ with integer coefficients. A recent theorem of B. Brindza [3] makes this result effective.

In this paper we are going to prove character analogues of the results concerning equation (2). Let χ be a primitive quadratic residue class character with conductor $f = f_\chi$. For fixed integers n and b we regard the equation

$$\chi(1)1^n + \chi(2)2^n + \cdots + \chi(xf)(xf)^n = by^z. \tag{3}$$

We are going to prove the following results.

THEOREM 1: *Let χ be a primitive quadratic character, and b and n fixed integers. If $b \neq 0$ and n is sufficiently large then equation* (3) *has only*

finitely many solutions in integers x, $y \geqslant 1$ *and* $z \geqslant 2$, *with effective upper bounds for* x, y, z.

THEOREM 2: *Let* χ *be a primitive odd quadratic character with conductor* f. *Then equation* (3) *has finitely many solutions in integers* x, $y \geqslant 1$ *and* $z \geqslant 2$ (*with effective upper bounds for* x, y, z) *in both the following cases.*

(a) $f = p \equiv 3 \pmod 8$ *is prime*, $n \geqslant 3$, *and* $n \not\equiv 0 \pmod 4$.

(b) $n = q$ *or* $n = q + 1$ $(n \neq 4)$, *where* q *is an odd prime such that* $\chi(q) = -1$, $q \neq f$, *and* $q \geqslant \frac{2}{5} f^{1/2} \log f$.

To prove the theorems, we follow the ideas of Györy, Tijdeman, and Voorhoeve in [9] and [16]. In analogy to their application of properties of Bernoulli polynomials, we are going to use those of generalized Bernoulli polynomials; Section 2 and 4 deal with these. In Section 3, the main lemmas are stated and the theorems are proved, and in Sections 5–7 we derive the necessary auxiliary results on the zeros of generalized Bernoulli polynomials. In Sections 8 and 9 we study the special case $f = 4$, in which case (3) has a particularly simple form. Finally, Section 10 contains a generalization of Theorem 1.

2. Generalized Bernoulli polynomials

Let χ be a primitive residue class character with conductor f. The complex numbers B_χ^m, defined by

$$\sum_{a=1}^{f} \frac{\chi(a) t\, \mathrm{e}^{at}}{\mathrm{e}^{ft} - 1} = \sum_{m=0}^{\infty} \frac{B_\chi^m}{m!} t^m,$$

are called generalized Bernoulli numbers belonging to χ. The generalized Bernoulli polynomials are then defined by

$$B_\chi^m(x) = \sum_{s=0}^{m} \binom{m}{s} B_\chi^s x^{m-s}.$$

We assume that χ is not the principal character, and denote $\delta = \delta_\chi = 0$ if χ is even (i.e. $\chi(-1) = 1$), and $\delta = 1$ if χ is odd (i.e. $\chi(-1) = -1$). The generalized Bernoulli numbers and polynomials have the following important properties (see e.g. [10, Sec. 2]).

$$B_\chi^m = 0 \quad \text{if } m \not\equiv \delta \pmod 2, \tag{4}$$

$$B_\chi^m \neq 0 \quad \text{if } m \equiv \delta \pmod 2 \text{ and } m \geqslant 1, \tag{4'}$$

$$B_\chi^m(0) = B_\chi^m, \tag{5}$$

$$B_\chi^m(x) = (-1)^{m-\delta} B_\chi^m(-x), \tag{6}$$

$$\left(B_\chi^m\right)'(x) = mB_\chi^{m-1}(x), \tag{7}$$

$$B_\chi^m(x+kf) - B_\chi^m(x) = m \sum_{a=1}^{kf} \chi(a)(a+x)^{m-1}, \tag{8}$$

where k is any positive integer.

3. Proof of the theorems

With (8) and (5) we get

$$\sum_{a=1}^{xf} \chi(a)a^n = \frac{1}{n+1}\left(B_\chi^{n+1}(xf) - B_\chi^{n+1}\right).$$

We denote the right hand side of this equation by $Q_\chi^n(s)$; hence (3) reduces to

$$Q_\chi^n(x) = by^z. \tag{9}$$

If χ is a quadratic character then the B_χ^n are rational numbers, and the Q_χ^n are polynomials with rational coefficients. Hence we can apply the following auxiliary results; see [9] and [16], and references therein.

LEMMA 1 (*Schinzel, Tijdeman*): *Let b be a nonzero integer, and let $P(x)$ be a polynomial with rational coefficients with at least two distinct zeros. Then the equation*

$$P(x) = by^z$$

in integers implies that $z < C$, where C is an effectively computable constant depending only on P and b.

The following lemma is a special case of a recent result of B. Brindza [3], which is an effective version of the theorem of Le Veque that was used in [16]. It is originally stated only for $b = 1$, $P(x) \in \mathbb{Z}[x]$, but we easily arrive at the more general form by using an argument given in [16, p. 2].

LEMMA 2: *Let $P(x) = a_0x^N + a_1x^{N-1} + \cdots + a_n = a_0\prod_{i=1}^k (x-\alpha_i)^{r_i}$ be a polynomial with rational coefficients, and $a_0 \neq 0$, $\alpha_i \neq \alpha_j$ for $i \neq j$. Let $b \neq 0$ and $m \geqslant 2$ be integers, and denote $s_i := m/gcd(m, r_i)$. Then the equation*

$$P(x) = by^m$$

has at most finitely many integer solutions (x, y) *(with effectively computable upper bounds for* x, y*) unless* $\{s_1, \ldots, s_k\}$ *is a permutation of one of the* k*-tuples* $\{s, 1, \ldots, 1\}$ $(s \geqslant 1)$ *or* $\{2, 2, 1, \ldots, 1\}$.

In the following sections we are going to prove

LEMMA 3: *Let* χ *be a primitive quadratic character with conductor* f*. Then* $Q_\chi^n(x)$ *has at least three simple zeros if*

(a) n *is sufficiently large, or*

(b) χ, f, *and* n *are as in* (*a*) *or* (*b*) *of Theorem* 2.

Now we are in a position to prove Theorems 1 and 2. We assume that hypotheses (a) or (b) of Lemma 3 hold. Then Q_χ^n has two distinct zeros, and we can aply Lemma 1 to (9). We find that z is bounded; we may therefore assume it fixed, say $z = m$. We apply now Lemma 2. Since Q_χ^n has at least three simple zeros, three of the numbers $s_1, \ldots, s_k$ are at least 2 for $m \geqslant 2$; so the two exceptional cases cannot occur. Hence equation (9) with $z = m$ has only a finite number of integer solutions in x and y, with effectively computable upper bounds for x and y. This completes the proof.

4. Analytic properties of $B_\chi^m(x)$

Denote $k := [m/2]$, the integral part of $m/2$. For $m \geqslant 1$ we define the periodic functions $\mathscr{B}_\chi^m(x)$ by means of the following Fourier expansions. For $m \equiv \delta \pmod 2$, set

$$\mathscr{B}_\chi^m(x) = (-1)^{k+1-\delta} \frac{2m! f^{m-1} i^\delta G(\chi)}{(2\pi)^m} \sum_{\nu=1}^{\infty} \frac{\bar{\chi}(\nu)}{\nu^m} \cos\left(\frac{2\pi\nu x}{f}\right), \quad (10)$$

and for $m \not\equiv \delta \pmod 2$,

$$\mathscr{B}_\chi^m(x) = (-1)^{k+1} \frac{2m! f^{m-1} i^\delta G(\chi)}{(2\pi)^m} \sum_{\nu=1}^{\infty} \frac{\bar{\chi}(\nu)}{\nu^m} \sin\left(\frac{2\pi\nu x}{f}\right), \quad (11)$$

where

$$G(\chi) = \sum_{a=1}^{f} \chi(a) \exp\left(\frac{2\pi i}{f} a\right)$$

is the Gaussian sum belonging to χ.

B.C. Berndt [1] defined the generalized Bernoulli polynomials $B_\chi^m(x)$ by putting

$$B_\chi^m(x) = \mathscr{B}_\chi^m(x) \quad \text{for } 0 < x < 1,$$

and by analytic continuation; he showed that this definition is equivalent to the usual one.

The following property exhibits the relationship between the generalized Bernoulli polynomials and their classical counterparts $B_n(x)$.

$$B_\chi^m(x) = f^{m-1} \sum_{a=1}^{f} \chi(a) B_m\left(\frac{a+x}{f}\right). \tag{12}$$

Furthermore, Berndt [1] proved in analogy to (12): for real x

$$\mathscr{B}_\chi^m(x) = f^{m-1} \sum_{a=1}^{f} \chi(a) \mathscr{B}_m\left(\frac{x+a}{f}\right), \tag{13}$$

where $\mathscr{B}_m(x) := B_m(x - [x])$.

The following Lemma extends the Fourier expansions (10) and (11).

LEMMA 4: *For all $x \geqslant 0$ we have*

$$B_\chi^m(x) = \mathscr{B}_\chi^m(x) + m \sum_{a=-[x]}^{0} \chi(a)(a+x)^{m-1}.$$

PROOF: First, assume $0 \leqslant x \leqslant f$. Let a be an integer, $1 \leqslant a \leqslant f$. If $a < f - x$ then $(a+x)/f < 1$, and

$$\mathscr{B}_m\left(\frac{x+a}{f}\right) = B_m\left(\frac{x+a}{f}\right).$$

If $a \geqslant f - x$ then $1 \leqslant (a+x)/f < 2$, so

$$\mathscr{B}_m\left(\frac{x+a}{f}\right) = \mathscr{B}_m\left(\frac{x+a}{f} - 1\right) = B_m\left(\frac{x+a}{f} - 1\right). \tag{14}$$

By a well-known property of the Bernoulli polynomials, we get

$$B_m\left(\frac{x+a}{f}\right) - B_m\left(\frac{x+a}{f} - 1\right) = m\left(\frac{x+a}{f} - 1\right)^{m-1}$$

for $f - x \leqslant a < f$. Now we combine this with (13) and (14):

$$\mathscr{B}_\chi^m(x) = f^{m-1} \sum_{a=1}^{f} \chi(a) B_m\left(\frac{x+a}{f}\right)$$

$$- f^{m-1} m \sum_{a=[f-x]}^{f} \chi(a)\left(\frac{x+a}{f} - 1\right)^{m-1},$$

and with (12) we get

$$\mathscr{B}_\chi^m(x) = B_\chi^m(x) - m \sum_{a=-[x]}^{0} \chi(a)(x+a)^{m-1} \tag{15}$$

Let $x \geqslant 0$ be an arbitrary real number, and write $x = kf + y$, $0 \leqslant y < f$, k an integer. Then by (8) we have

$$B_\chi^m(x) = B_\chi^m(kf+y) = B_\chi^m(y) + m \sum_{a=1}^{kf} \chi(a)(a+y)^{m-1},$$

and with (15),

$$B_\chi^m(x) = \mathscr{B}_\chi^m(x) + m \sum_{a=-[y]}^{kf} \chi(a)(y+a)^{m-1}$$

$$= \mathscr{B}_\chi^m(x) + m \sum_{a=-[x]}^{0} \chi(a)(x+a)^{m-1}$$

because $\mathscr{B}_\chi^m(x)$ is periodic with period f. This completes the proof.

Now denote

$$A_\chi^m(x) := \sum_{a=-[x]}^{0} \chi(a)(a+x)^{m-1} \quad \text{for } x > 1,$$

$$A_\chi^m(x) := 0 \quad \text{for } 0 \leqslant x \leqslant 1.$$

Then we have

LEMMA 5: *Let χ be a primitive quadratic character. Then for all $m > x - 1 > 2$,*

$$(-1)^\delta A_\chi^m(x) > 0.$$

PROOF: We have

$$A_\chi^m(x) = \sum_{a=0}^{[x]} \chi(-a)(x-a)^{m-1} = (-1)^\delta \sum_{a=0}^{[x]} \chi(a)(x-a)^{m-1}.$$

Hence

$$
\begin{aligned}
(-1)^{\delta} A_{\chi}^{m}(x) &\geqslant (x-1)^{m-1} - \sum_{a=2}^{[x]} (x-a)^{m-1} \\
&= (x-1)^{m-1} - \sum_{a=0}^{[x]-2} (x-[x]+a)^{m-1} \\
&\geqslant (x-1)^{m-1} - \int_{0}^{[x]-1} (x-[x]+t)^{m-1} \mathrm{d}t \\
&= (x-1)^{m-1} - \frac{1}{m}\left((x-1)^{m} - (x-[x])^{m}\right) \\
&\geqslant (x-1)^{m-1}\left(1 - \frac{x-1}{m}\right),
\end{aligned}
$$

and this completes the proof.

6. Proof of Lemma 3(a) for $n \equiv \delta \pmod 2$

In this case we have, by (4),

$$Q_{\chi}^{n}(x) = B_{\chi}^{n+1}(xf)/(n+1).$$

It suffices to regard $B_{\chi}^{m}(x)$, where we put $m := n+1$. First we note that because of (4) and (5), $B_{\chi}^{m}(x)$ has a zero at $x = 0$; it is simple because of (7) and (4′). On account of (6) it suffices to find one other simple zero. This is achieved through the following three lemmas. We always assume that χ is a primitive quadratic character with conductor f.

LEMMA 6: *Let m be such that $m \not\equiv \delta \pmod 2$, and denote $k := [m/2]$. Suppose that x belongs to*

(a) $(0, f/2]$ *if k is odd, and to*

(b) $[f/2, f]$ *or* $[3f/2, 2f]$ *if k is even.*

If m is sufficiently large (i.e. $m > 2f-1$) then $(-1)^{\delta} B_{\chi}^{m}(x) > 0$.

PROOF: We use a well-known result about Gaussian sums: $G(\chi) = \mathrm{i}^{\delta} f^{1/2}$ if χ is a quadratic character. We also have $\bar{\chi} = \chi$. Hence we get with Lemma 4 and with (11), for all $x \geqslant 0$,

$$B_{\chi}^{m}(x) = (-1)^{k-1+\delta} \frac{2m! f^{m-1/2}}{(2\pi)^{m}} \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu^{m}} \sin\frac{2\pi\nu x}{f} + m A_{\chi}^{m}(x). \tag{16}$$

Denote the infinite sum in (16) by $S_m(x)$, and write $y = x/f$. Then

$$S_m(fy) \geqslant \sin 2\pi y - \sum_{\nu=2}^{\infty} \nu^{-m} |\sin(2\pi\nu y)|.$$

For $0 \leqslant y \leqslant 1/4$ we have

$$|\sin(2\pi\nu y)| \leqslant 2\pi\nu y;\ \sin(2\pi y) \geqslant 4y.$$

Hence

$$\begin{aligned} S_m(x) &\geqslant 4y - 2\pi y \sum_{\nu=2}^{\infty} \nu^{-m+1} \\ &\geqslant 2y\left(2 - \pi \sum_{\nu=2}^{\infty} \nu^{-4}\right) = 2y\left(2 - \pi\left(\pi^4/90 - 1\right)\right) \\ &> 0 \end{aligned}$$

for $m \geqslant 5$ and $0 < y \leqslant 1/4$. In the same way we can see that $S_m(x) > 0$ for $f/4 \leqslant x \leqslant f/2$. Now (16) together with Lemma 5 gives assertion (a). To verify (b), we observe that $S_m(x) < 0$ if x is in the given intervals, and then we compare (16) with Lemma 5 again.

LEMMA 7: *Let $m \equiv \delta \pmod 2$ be sufficiently large. Then $B_\chi^m(x) \neq 0$ for $f/3 \leqslant x \leqslant 2f/3$.*

PROOF: With $\bar{\chi} = \chi$ and $G(\chi) = i^\delta \sqrt{f}$, (10) and Lemma 4 give

$$B_\chi^m(x) = (-1)^{k+1} \frac{2m! f^{m-1/2}}{(2\pi)^m} \sum_{\nu=1}^{\infty} \frac{\chi(\nu)}{\nu^m} \cos\frac{2\pi\nu x}{f} + mA_\chi^m(x). \quad (17)$$

Denote the infinite sum in (17) by $T_m(x)$. Using the fact that $\cos\alpha \leqslant -1/2$ if $2\pi/3 \leqslant \alpha \leqslant 4\pi/3$, we get with $y := x/f$,

$$\begin{aligned} T_m(fy) &\leqslant \cos(2\pi y) + \sum_{\nu=2}^{\infty} \nu^{-m} \\ &\leqslant -\tfrac{1}{2} + \sum_{\nu=2}^{\infty} \nu^{-4} = -\tfrac{1}{2} + \frac{\pi^4}{90} - 1 \\ &< -\tfrac{2}{5} \end{aligned}$$

for $m \geqslant 4$. On the other hand,

$$|A_\chi^m(x)| \leqslant \tfrac{2}{3} f \left(\tfrac{2}{3} f\right)^{m-1},$$

hence

$$|B_\chi^m(x)| \geqslant \frac{2m!f^{m-1/2}}{(2\pi)^m} \cdot \frac{2}{5} - m\left(\tfrac{2}{3}f\right)^m$$

$$= mf^m\left(\frac{4(m-1)!}{5(2\pi)^m f^{1/2}} - \left(\tfrac{2}{3}\right)^m\right) > 0$$

if m is sufficiently large.

LEMMA 8: *If $m \not\equiv \delta \pmod 2$ is sufficiently large then $B_\chi^m(x)$ has exactly one simple zero*

(i) *between $f/3$ and $f/2$ if k is even,*

(ii) *between $f/2$ and $2f/3$ if k is odd.*

PROOF: We use (16) to find estimates for $B_\chi^m(f/3)$ and $B_\chi^m(2f/3)$. We have

$$S_m(f/3) \geqslant \sin(2\pi/3) - \sum_{\nu=2}^{\infty} \nu^{-m}$$

$$\geqslant \tfrac{1}{2}\sqrt{3} - \sum_{\nu=2}^{\infty} \nu^{-4} > \tfrac{3}{4}$$

for $m \geqslant 4$, and similarly $S_m(2f/3) < -3/4$, while on the other hand, for $x \leqslant 2f/3$

$$|A_\chi^m(x)| \leqslant \left(\tfrac{2}{3}f\right)^m.$$

As in the proof of Lemma 7 we see that for $x = f/3$ and $x = 2f/3$

$$\left|\frac{2m!f^{m-1/2}}{(2\pi)^m} S_m(x)\right| > |mA_\chi^m(x)|$$

if m is sufficiently large. Hence, with (16),

$$(-1)^{k-1+\delta} B_\chi^m(f/3) > 0, \quad (-1)^{k-1+\delta} B_\chi^m(2f/3) < 0$$

if m is sufficiently large. If we compare this with Lemma 6, we get the existence of (an odd number of) zeros in the given intervals. Now assume that $B_\chi^m(x)$ has more than one zero or a multiple zero in the interval $[f/3, f/2]$ (resp. $[f/2, 2f/3]$). Then by (7), $B_\chi^{m-1}(x)$ would have a zero in the interval $[f/3, 2f/3]$, which however contradicts Lemma 7.

6. Proof of Lemma 3(a) for $n \not\equiv \delta \pmod 2$

In this case $B_\chi^{n+1} \neq 0$, by (4'). It sufficies to study

$$P_\chi^n(x) := B_\chi^{n+1}(x) - B_\chi^{n+1}.$$

It is now clear that $P_\chi^n(x)$ has a double zero at $x = 0$. We shall see that there is also a sufficient number of simple zeros.

LEMMA 9: *Let* $n \not\equiv \delta \pmod 2$ *and* $k := [(n+1)/2] \not\equiv \delta \pmod 2$. *If* n *is sufficiently large then* $P_\chi^n(x)$ *has exactly one simple zero in each of the intervals* $[f/2, f]$ *and* $[3f/2, 2f]$.

PROOF: Set $m = n + 1$ again. With (17) we get

$$P_\chi^n(x) = (-1)^{k+1} \frac{2m! f^{m-1/2}}{(2\pi)^m} \sum_{\nu=1}^{\infty} \left(\cos \frac{2\pi \nu x}{f} - 1 \right) + m A_\chi^m(x). \quad (18)$$

For $x = jf$, $j = 0, 1, \ldots$, the first term of the right-hand side of (18) vanishes, and with Lemma 5 we have therefore, if $m > 2f - 1$,

$$(-1)^\delta P_\chi^n(jf) > 0 \quad (j = 1, 2). \quad (19)$$

Denote the infinite sum in (18) by $U_m(x)$. Then for $j = 1, 3, \ldots$ and $m \geqslant 4$,

$$\begin{aligned} U_m\left(\frac{jf}{2}\right) &\leqslant \cos(\pi j) - 1 + \sum_{\nu=2}^{\infty} 2\nu^{-m} \\ &\leqslant -2 + 2 \sum_{\nu=2}^{\infty} \nu^{-4} = -2 + 2(\pi^4/90 - 1) \\ &< -9/5. \end{aligned}$$

Hence

$$(-1)^k P_\chi^n\left(\frac{jf}{2}\right) > \frac{18}{5} \frac{m! f^{m-1/2}}{(2\pi)^m} - m \left| A_\chi^m\left(\frac{jf}{2}\right) \right| > 0$$

if m is sufficiently large, just as in the proof of Lemma 7. If we compare this with (19), we find that for $k \not\equiv \delta \pmod 2$ there is at least one zero in each of the intervals $[f/2, f]$ and $[3f/2, 2f]$. By definition and by (7), the derivative of $P_\chi^n(x)$ is $(n+1)B_\chi^n(x)$. We note that $[n/2]$ is always even under the restrictions of the hypothesis of Lemma 9. Now we apply

Lemma 6(b), and we find that $P_\chi^n(x)$ has exactly one simple zero in each of the two given intervals.

LEMMA 10: *Let* $n \not\equiv \delta \pmod 2$ *and* $k := [(n+1)/2] \equiv \delta \pmod 2$. *If* n *is sufficiently large then* $P_\chi^n(z)$ *has exactly two simple complex conjugate zeros in the disk* $\{z: |z-f| < 1\}$.

The last two lemmas, together with (6), complete the proof of Lemma 3(a). To prove Lemma 10, we use the following criterion (see [7]).

LEMMA 11: *Let* $g(z) = c_0 + c_1 z + \cdots + c_n z^n$, $c_n \neq 0$. *If*

$$2|c_i| > \sum_{j=0}^{n} |c_j|$$

then $g(z)$ *has exactly* i *zeros in the interior of the unit circle, and none on the unit circle.*

We apply this lemma with $g(z) = P_\chi^n(z+f)$ and $i = 2$. With (8) and (5) we get

$$B_\chi^J(f) = B_\chi^J + j \sum_{a=1}^{f} \chi(a) a^{J-1},$$

and therefore, using a Taylor expansion and (7),

$$P_\chi^n(x+f) = \sum_{j=0}^{n+1} \binom{n+1}{j} \left(B_\chi^{n+1-J} + (n+1-j) \sum_{a=1}^{f} \chi(a) a^{n-J} \right) x^J - B_\chi^{n+1}.$$

It suffices to show

$$|c_2| > |c_0| + |c_1| + \sum_{j=3}^{n+1} |c_j|,$$

or

$$\binom{n+1}{2} |B_\chi^{n-1}| > \sum_{j=0}^{n} \binom{n+1}{j} (n+1-j) \left| \sum_{a=1}^{f} \chi(a) a^{n-J} \right| + \sum_{j=1}^{[n/2]} \binom{n+1}{2j} |B_\chi^{n+1-2J}|, \tag{20}$$

where we have taken (4) into account. Now the first term of the right hand side of (20) is less than

$$(n+1)\sum_{j=0}^{n}\binom{n}{j}\sum_{a=1}^{f-1}a^{n-j}=(n+1)\sum_{a=1}^{f-1}(a+1)^n<(n+1)f^{n+1}.$$

With (5) and (10) we get, taking into account $|G(\chi)|=\sqrt{f}$,

$$|B_\chi^{n+1-2j}|=\frac{2(n+1-2j)!}{(2\pi/f)^{n+1-2j}\sqrt{f}}\,|\sum_{\nu=1}^{\infty}\chi(\nu)\nu^{-n-1+2j}|.$$

Now for $a\geqslant 2$,

$$|\sum_{\nu=1}^{\infty}\chi(\nu)\nu^{-a}|\leqslant\sum_{\nu=1}^{\infty}\nu^{-2}=\pi^2/6<5/3;$$

hence

$$|B_\chi^{n+1-2j}|<\frac{10}{3\sqrt{f}}\,\frac{(n+1-2j)!}{(2\pi/f)^{n+1-2j}}\quad\text{for } 2j<n,$$

while from (12), (5), and $B_1(x)=x-\frac{1}{2}$ it follows that

$$|B_\chi^1|=|f^{-1}\sum_{a=1}^{f}\chi(a)a|<f.$$

Hence

$$\begin{aligned}
&\sum_{j=2}^{[n/2]}\binom{n+1}{2j}|B_\chi^{n+1-2j}|\\
&<\frac{10}{3\sqrt{f}}\sum_{j=2}^{[(n-1)/2]}\binom{n+1}{2j}\frac{(n+1-2j)!}{(2\pi/f)^{n+1-2j}}+(n+1)f\\
&<(n+1)!\,\frac{10}{3\sqrt{f}}\left(\frac{f}{2\pi}\right)^{n-3}\\
&\quad\times\tfrac{1}{24}\left(1+\tfrac{1}{30}\left(\frac{2\pi}{f}\right)^2+\tfrac{1}{1680}\left(\frac{2\pi}{f}\right)^4\exp\left(\frac{2\pi}{f}\right)\right)+(n+1)f\\
&<(n+1)!\,\frac{5}{27\sqrt{f}}\left(\frac{f}{2\pi}\right)^{n-3}+(n+1)f\quad\text{for } f\geqslant 3.
\end{aligned}$$

On the other hand,

$$\binom{n+1}{2}|B_\chi^{n-1}| > \frac{(n+1)n}{2}\frac{2(n-1)!}{(2\pi/f)^{n-1}\sqrt{f}}\left(1-\sum_{\nu=2}^{\infty}\nu^{1-n}\right)$$

$$> (n+1)!\frac{9}{10\sqrt{f}}\left(\frac{f}{2\pi}\right)^{n-1} \quad \text{for } n \geqslant 5.$$

So finally, (20) holds if

$$\frac{n!}{\sqrt{f}}\left(\frac{f}{2\pi}\right)^{n-3}\left(\tfrac{9}{10}\left(\frac{f}{2\pi}\right)^2-\frac{5}{27}\right) > f^{n+1}+f$$

which holds for all $f \geqslant 3$ and if n is sufficiently large. Hence by Lemma 11 there are two simple zeros or a double zero of $P_\chi^n(z)$ inside the circle about f with radius 1.

We note now that the infinite sum $U_m(x)$ in (18) vanishes at $x=0$, and that its derivative is $-2\pi S_n(x)/f$, with $S_n(x)$ as in the proof of Lemma 6. There we have shown that $S_n(x)>0$ if $n \not\equiv \delta \pmod 2$ and $0 < x \leqslant f/2$; hence $U_m(x)$ is decreasing for $0 < x \leqslant f/2$. The fact that it is an even and periodic function now implies $U_m(x) \leqslant 0$ for all x, with equality only at $x = jf$ for all integers j. With Lemma 5 we finally get $(-1)^k P_\chi^n(x) > 0$ for $0 < x < n+2$ because $k \equiv \delta \pmod 2$. This means that for $n \geqslant f$ the two zeros from above cannot be real; they are a pair of distinct complex conjugates because $P_\chi^n(x)$ has real coefficients.

7. Proof of Lemma 3(b)

In [5], the author has shown that the polynomials $Q_\chi^n(x)$, divided by the obvious factors x, respectively x^2, are irreducible if χ is odd and f and n satisfy the conditions given in (a) or (b) of Theorem 2. Hence if n is even then $Q_\chi^n(x)$ has $n-2$ simple zeros, and if n is odd then $Q_\chi^n(x)$ has n simple zeros. This proves Lemma 3(b).

8. A special case

The quadratic character with conductor $f=4$ is given by $\chi(1)=1$, $\chi(3)=-1$, $\chi(0)=\chi(2)=0$. Hence (3) takes the form

$$1-3^n+5^n-\cdots+(4x-3)^n-(4x-1)^n = by^z. \tag{21}$$

THEOREM 3: *Let $b \neq 0$ and $n \geqslant 3$, $n \notin \{4, 5\}$ be integers. Then* (21) *has at most finitely many solutions in integers x, $y \geqslant 1$ and $z \geqslant 2$, with effectively computable upper bounds for x, y and z.*

To prove this, we first need the following lemmas.

LEMMA 12: *If χ is the quadratic character with conductor $f=4$, and if $n \equiv 2 \pmod 4$ then $(B_\chi^{n+1}(x) - B_\chi^{n+1})x^{-2}$ has no real and no purely imaginary zeros.*

PROOF: It follows from the last part of the proof of Lemma 10 that $P_\chi^n(x) < 0$ for $0 < x < n+2$. Now we claim that $P_\chi^n(x)$ has no zeros $x \geqslant n+2$; this would prove the first assertion of Lemma 12. We are going to show a bit more. If we use (17) to get

$$B_\chi^{2j+1} = B_\chi^{2j+1}(0) = (-1)^{j+1}(2j+1)!\left(\frac{2}{\pi}\right)^{2j+1} S(j), \tag{22}$$

where

$$S(j) := \sum_{\nu=0}^{\infty} (-1)^\nu (2\nu+1)^{-2j-1},$$

then we have

$$P_\chi^n(x) = \sum_{j=0}^{n/2-1} \binom{n+1}{2j+1} B_\chi^{2j+1} x^{n-2j}$$

$$= (-1)^{k+1} \sum_{j=0}^{n/2-1} \frac{(n+1)!}{(n-2j)!}\left(\frac{2}{\pi}\right)^{2j+1} S(j)(-x^2)^{n/2-j}.$$

If we now put $x = i\alpha$, with α a nonzero real number then each summand in the last expression is positive, and $P_\chi^n(i\alpha)$ cannot vanish; this proves the second assertion of the lemma. We introduce now the new variable $z = -x^2$; the polynomial $(-1)^{k+1}x^{-2}P_\chi^n(x)$ can therefore be regarded as a polynomial in z with positive coefficients, and we can apply the theorem of Kakeya and Eneström (see [14, III. 23]). The quotients of consecutive coefficients are

$$q_j := \left(\frac{2}{\pi}\right)^2 \frac{(n-2j)!}{(n-2j-2)!} \frac{S(j+1)}{S(j)}$$

$$= \left(\frac{2}{\pi}\right)^2 (n-2j-1)(n-2j)\frac{S(j+1)}{S(j)}.$$

It is easy to verify that $\{S(j+1)/S(j)\}$ is a decreasing sequence; hence

$\{q_j\}$ is decreasing and therefore for all $j = 0, 1, \ldots, n/2 - 2$ we have

$$q_j \leqslant q_0 = \left(\frac{2}{\pi}\right)^2 (n-1)n\frac{S(3)}{S(1)}.$$

It is known that $S(3) = \pi^3/32$, $S(1) = \pi/4$ (one can obtain this also with (22) and (27) below), so $q_0 = (n-1)n/2$. With the theorem of Kakeya and Eneström and after returning to the variable x we find that all the zeros of $P_\chi^n(x)$ lie inside a circle with centre at the origin and radius $(n(n-1)/2)^{1/2}$. This proves the claim which concludes the proof of Lemma 12 (this proof also works for $f = 3$ and $f = 7$, but fails for higher conductors since the maximum modulus of the zeros depends on f).

The following lemma is based on a conjecture of L. Carlitz [4], saying that $|E_{2n}|$ is never a square for $n > 1$, where E_{2n} are the Euler numbers. Carlitz proved his conjecture for all $2n \not\equiv 2 \pmod 8$, and for a few more classes. We are going to show slightly more. For basic properties of E_{2n}, see [13, Ch. 2]; a table up to $2n = 120$ is contained in [11].

LEMMA 13: *$|E_{2n}|$ is not a square for $2n > 2$, $2n \not\equiv 2 \pmod{144}$.*

PROOF: Following Carlitz [4], we use the well-known Kummer congruence

$$E_m \equiv E_{m+p-1} \pmod p, \tag{23}$$

where p is a prime and $m \geqslant 2$ (see e.g. [8, p. 36]). We recall that $(-1)^n E_{2n} > 0$. First we have

$$|E_{16k+10}| = -E_{16k+10} \equiv -E_{10} = 50\,521 \equiv -3 \pmod{17};$$

but -3 is not a quadratic residue modulo 17, so $|E_{16k+10}|$ cannot be a square. Now only the case $2n \equiv 2 \pmod{16}$ remains open. We observe that

$$-E_{18k+4} \equiv -E_4 = -5 \pmod{19},$$

$$-E_{18k+16} \equiv -E_{16} = -19\,391\,512\,145 \equiv -7 \pmod{19},$$

$$-E_{36k+14} \equiv -E_{14} = 199\,360\,981 \equiv -14 \pmod{37},$$

$$-E_{72k+10} \equiv -E_{10} = 50\,521 \equiv 5 \pmod{73};$$

these, too, are quadratic non-residues, and they show that $|E_{2n}|$ cannot be a square for $2n \equiv 34, 50, 82, 98, 130 \pmod{144}$. Carlitz showed that

$|E_{24k+18}|$ is not a square, which takes care of the remaining cases $2n \equiv 18, 66, 114 \pmod{144}$. (Remark: it is easy to show with the same method that Lemma 13 holds for all $2n \not\equiv 2 \pmod{5040}$, or even more; however, as Carlitz remarked in [4], it seems doubtful that the general conjecture can be proved this way).

LEMMA 14: *Let χ be the quadratic character with conductor 4, $n \geqslant 3$.*
(a) *If n is odd, $n \neq 5$, then $Q_\chi^n(x)$ has only simple zeros.*
(b) *If $n = 16$ or $n \equiv 2 \pmod 4$ then $Q_\chi^n(x)$ has at least four simple zeros.*

PROOF: If n is odd then by (4) we have $B_\chi^{n+1} = 0$, since $\delta = 1$. It is easy to establish the following connection to the Euler polynomials (see [1, p. 425]).

$$B_\chi^{n+1}(2x-1) = -2^{n-1}(n+1)E_n(x), \ (n \geqslant 0) \tag{24}$$

and therefore for odd n

$$Q_\chi^n(x) = -2^{n-1}E_n\left(2x + \tfrac{1}{2}\right). \tag{25}$$

Brillhart [2] showed that the only Euler polynomial with a multiple zero is $E_5(x)$; this together with (25), proves (a). Thus for even n, $P_\chi^n(x)$ has only zeros of multiplicity at most 2.

If $n \equiv 2 \pmod 4$, it suffices to regard the polynomial $P_\chi^n(x)$, as defined in Section 6. It has a double zero at $x = 0$, and the remaining zeros occur in quadruplets, by (6) and Lemma 12. If $n \equiv 6 \pmod 8$ then there is an odd number of these quadruplets, so at least one of them has to consist of simple zeros only, for $n \neq 6$. But also, it is easy to compute (e.g. via (24)) that $2P_\chi^6(x) = -7x^2(x^4 - 15x^2 + 75)$, which clearly has four simple zeros. This proves (b) for $n \equiv 6 \pmod 8$.

If $n \equiv 2 \pmod 8$ then there is an even number of quadruplets of zeros. Since multiplicity can be at most two, then $P_\chi^n(x)$ has to be a square (up to a constant factor) if it has no simple zeros. By definition of $P_\chi^n(x)$ and $B_\chi^{n+1}(x)$ we have

$$\begin{aligned} -\frac{n+1}{2}P_\chi^n(x)x^{-2} &= x^{n-2} - \sum_{j=1}^{n/2-1} \frac{2}{n+1}\binom{n+1}{2j+1} B_\chi^{2j+1} x^{n-2j-2} \\ &= x^{n-2} + \sum_{j=1}^{n/2-1} \binom{n}{2j} E_{2j} x^{n-2j-2}, \end{aligned} \tag{26}$$

where we have used the well-known relation (see e.g. [8])

$$B_\chi^{2j+1} = -\tfrac{1}{2}(2j+1)E_{2j}. \tag{27}$$

If the polynomial on the right-hand side of (26) were a square then in particular its value for $x = 1$ had to be a square. But by the recursion formula for Euler numbers (see e.g. [13, p. 25]) we have

$$1 + \sum_{j=1}^{n/2-1} \binom{n}{2j} E_{2j} + E_n = 0.$$

However, by Lemma 13 we know that $-E_n$ is never a square if $n \equiv 0$ or 4 (mod 6), which leads to a contradiction in these cases. Another necessary condition for the right-hand side of (26) to be a square, is that its constant coefficient is a square. However, from (23) it follows that $E_{n-2} \equiv 2 \pmod 3$ for all even n; for $n \equiv 2 \pmod 6$ we have therefore

$$\tfrac{1}{2}n(n-1)E_{n-2} \equiv 2 \pmod 3,$$

which is not a quadratic residue modulo 3. Hence $P_\chi^n(x)$ can never be a constant multiple of a square; this proves (b) for $n \equiv 2 \pmod 8$.

Finally, to deal with $n = 16$, we adapt the proof of Lemma 9; (19) still holds for $n = 16$, i.e. we have $P_\chi^{16}(4) < 0$ and $P_\chi^{16}(8) < 0$. If we can show that there is some y, $4 < y < 8$, with $P_\chi^{16}(y) > 0$ then we are done because there have to be at least four zeros (taking (6) into account) of odd multiplicities; but they are necessarily simple zeros because the derivative of $P_\chi^{16}(x)$ has only simple zeros. We regard

$$U_{17}(5) = U_{17}\left(\tfrac{5}{4}f\right) \leqslant \cos\left(\tfrac{5}{2}\pi\right) - 1 + \sum_{\nu=2}^{\infty} 2\nu^{-17}$$

$$< -1 + 2\sum_{\nu=2}^{\infty} \nu^{-4} < -\tfrac{4}{5},$$

and

$$|A_\chi^{17}(5)| = \sum_{a=0}^{5} \chi(a)(5-a)^{16} = 4^{16} - 2^{16} < 4^{16}.$$

As in the proof of Lemma 9, and using Stirling's formula, we get therefore

$$P_\chi^n(5) > \frac{8}{5}\frac{17!4^{17-1/2}}{(2\pi)^{17}} - 17 \cdot 4^{16}$$

$$> 17 \cdot 4^{16}\left(\tfrac{32}{5}\left(\frac{2}{\pi}\right)^{1/2}\left(\frac{8}{\pi e}\right)^{16} - 1\right) > 0,$$

which completes the proof for $n = 16$, and therefore the proof of Lemma 14.

PROOF OF THEOREM 3. First, let $n \equiv 0 \pmod 4$; then Theorem 1 applies for all sufficiently large n, a lower bound of which can be found as follows. In the proof of Lemma 9 we see that $n = m - 1$ has to be such that $m > 7$ and

$$\frac{18}{5}\frac{m!4^{m-1/2}}{(2\pi)^m} - m\left(\tfrac{3}{2}\cdot 4\right)^m > 0.$$

By using Stirling's formula for an estimate of $n!$, we find that the above inequality holds if

$$\tfrac{3}{5}\left(\frac{2n}{\pi}\right)^{1/2}\left(\frac{n}{3\pi\,\mathrm{e}}\right)^n > 1,$$

which is true for all $n \geqslant 25$. For $n = 8$, 12, 20 and 24, Theorem 3 is a consequence of Theorem 2(b).

For the remaining cases $n = 16$ and $n \geqslant 3$ with $n \neq 5$, $n \neq 0 \pmod 4$, we proceed as in Section 3. By Lemma 14, $Q_\chi^n(x)$ has at least four simple zeros (three for $n = 3$); hence we apply first Lemma 1 to get an upper bound for x, and then Lemma 2. This completes the proof of Theorem 3.

9. Exceptional values for n

We study the cases not covered by Theorem 3, and show that they are indeed exceptions.

A: $n = 1$ is trivial. Let $n = 2$; we have $E_2(x) = x(x-1)$, and with (24) and the definition of $Q_\chi^n(x)$ we see that (21) becomes

$$-8x^2 = by^z.$$

For each z and for infinitely many choices of b this equation has infinitely many solutions in x and y.

B: Let $n = 4$, $z = 2$. We have $E_4(x) = x(x-1)(x^2 - x - 1)$, and (21) therefore becomes

$$-16x^2(8x^2 - 3) = by^2.$$

This equation has solutions if

$$8x^2 - dz^2 = 3 \tag{28}$$

is solvable, where $d := -b$. Let d be odd and not a perfect square. By the theory of Pell's equation (see e.g. [12, Ch. 8]), $u^2 - 2dv^2 = 1$ has infinitely many solutions, say (u_j, v_j), $j = 1, 2, \ldots$ Now it is easy to verify that if (x_0, z_0) is a solution of (28) then there are infinitely many more

solutions (x_j, z_j), $j = 1, 2, \ldots$, given by $x_j = x_0 u_j + \frac{1}{2} d z_0 v_j$, $z_j = z_0 u_j + 4 x_0 v_j$. x_j is indeed an integer since v_j has to be even if it is to be a solution of $u^2 - 2dv^2 = 1$. Once we have found a solution (x_j, z_j) of (28) with $z_j > 1$, we can regard the new equation $8x^2 - (dz_j^2)z^2 = 3$ which has $(x_j, 1)$ as a solution. As an example, we take $d = 5$; $(1, 1)$ is then a solution of (28), and (19, 6) is a solution of $u^2 - 2dv^2 = 1$. Then (28) has also the solution $(x_1, z_1) = (34, 43)$, and therefore it has an infinite class of solutions also for $d = 5 \cdot 43^2$, and so on. This shows that there is an infinite number of choices of b for which (21) (with $z = 2$) has infinitely many solutions in x and y.

C: Let $n = 5$, $z = 2$. Then $E_5(x) = (x - \frac{1}{2})(x^2 - x - 1)^2$, and (25) shows that (21) becomes

$$-2x(16x^2 - 5)^2 = by^2.$$

Now it is obvious that for any $b < 0$ this equation has infinitely many solutions: take $x = -2bt^2$, $t = 1, 2, \ldots$

D: Finally, let $n = 4$ or 5 and $z \geqslant 3$. Then the $Q_\chi^n(x)$ have three distinct zeros of multiplicities at most two. As in Section 3, we first apply Lemma 1. To apply Lemma 2, we observe that for $z = m \geqslant 3$ we have $s_i \geqslant 2$ for $i = 1, 2, 3$; therefore the two exceptional cases cannot occur. Hence (21) has at most finitely many solutions in x, $y \geqslant 1$, $z \geqslant 3$ for $n = 4$ or 5, with effective upper bounds for x, y, z.

10. A generalization

In analogy to the result in [16] which was mentioned in the introduction, we can show the following.

THEOREM 4: *Let χ and b be as in Theorem 1, and let $\{R_n(x)\}$ be a sequence of polynomials with integer coefficients, satisfying for any x*

$$\lim_{n \to \infty} |R_n(x)| (2\pi/f)^n/(n-1)! = 0. \tag{28}$$

If n is sufficiently large then

$$\chi(1)1^n + \chi(2)2^n + \cdots + \chi(xf)(xf)^n + R_n(x) = by^z \tag{29}$$

has only finitely many solutions in integers x, $y \geqslant 1$ and $z \geqslant 2$ if $n \equiv \delta$ (mod 2), resp. $z \geqslant 3$ if $n \not\equiv \delta$ (mod 2), with effective upper bounds for x, y, z.

PROOF: An adaptation of the lemmas in Sections 5 and 6 would give us explicit lower bounds for n, depending on f and $\{R_n\}$. Here we use a

different (but conceptionally similar) method which, however, fails to provide bounds for n.

We can rewrite (28) as

$$Q_\chi^n(x) = by^z,$$

where

$$Q_\chi^n(x) := \frac{1}{n+1}\left(B_\chi^{n+1}(xf) - B_\chi^{n+1}\right) + R_n(x)$$

We regard $P_\chi^n(x) := B_\chi^{n+1}(x) - B_\chi^{n+1} + (n+1)R_n(x/f)$. In [6] it was shown that for entire functions $S(x)$ the sequences of functions

$$B_\chi^{n+1}(x) - B_\chi^{n+1} + S(x)$$

after normalization converge uniformly on compact subsets of $\mathbb{C}$ to $\sin(2\pi x/f)$ if $B_\chi^{n+1} = 0$, and to $\cos(\pi x/f) - 1$ if $B_\chi^{n+1} \neq 0$. It is clear from the proofs in [6] that $S(x)$ can be replaced by a sequence of functions $S_n(x)$, with $|S_n(x)|(2\pi/f)^n/n! \to 0$ as $n \to \infty$, for all x. Because of (28), $(n+1)R_n(x/f)$ satisfies this condition. Now a well-known theorem of Hurwitz implies that the zeros of $\sin(2\pi x/f)$, resp. $\cos(2\pi x/f) - 1$ are limit points of the zeros of $P_\chi^n(x)$. Therefore, if n is sufficiently large, $P_\chi^n(x)$ and $Q_\chi^n(x)$ will have at least three simple zeros in the case $B_\chi^{n+1} = 0$ (i.e. $n \equiv \delta \pmod 2$ by (4)), and at least three zeros of multiplicity at most 2 in the case where $B_\chi^{n+1} \neq 0$ (i.e. $n \not\equiv \delta \pmod 2$). We now arrive at Theorem 4 using Lemmas 1 and 2 as before in Sections 3 and 9.D.

COROLLARY: *Theorem* 4 *holds for the equation*

$$\chi(1)1^n + \chi(2)2^n + \cdots + \chi(x)x^n + R_n(x) = by^z, \tag{30}$$

where x *is not necessarily a multiple of* f.

PROOF: For $j = 0, 1, \ldots, f-1$, we define polynomials

$$r_j^n(x) := \sum_{a=0}^{j} \chi(a)(xf + a)^n.$$

Now (30) can be rewritten as f equations

$$\chi(1)1^n + \chi(2)2^n + \cdots + \chi(\tilde{x}f)(\tilde{x}f)^n + r_j^n(\tilde{x}) + R_n(\tilde{x}f + j) = by^z \tag{31}$$

$(j = 0, 1, \ldots, f-1)$. Because of

$$|r_j^n(\tilde{x})| \leqslant j(\tilde{x}f + j)^n < f^{n+1}(\tilde{x} + 1)^n,$$

the polynomials $r_j^n(\tilde{x}) + R_n(\tilde{x}f + j)$ satisfy (28), and therefore by Theorem 4 each equation (31) has only finitely many solutions in $\tilde{x}$, y, z. But this means that (30) has only finitely many solutions in x, y, z.

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