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STARSHAPEDNESS IN CONVEXITY SPACES

Krzysztof Kołodziejczyk

Abstract

In terms of the *kernel* of starshaped sets characterizations are given of the property that a set is convex if and only if the hull of each pair in the set is in the set too, and of the so-called join-hull commutativity property. The setting is that of a convexity space.

1. Introduction

A *convexity space* is a pair (X, \mathcal{C}) where X is a nonempty set and \mathcal{C} is a family of subsets of X closed under arbitrary intersections and containing X and the empty set \emptyset . Members of \mathcal{C} are called \mathcal{C} -convex sets. For any $S \subset X$ the *convex hull* of S is defined as $\mathcal{C}(S) = \bigcap \{A \in \mathcal{C} : S \subset A\}$. The concept of convexity spaces was introduced by Levi [10] and has been extensively studied by many authors (see, among others, [3,5,6,12]).

The following two classes of convexity spaces are well-known.

A convexity space (X, \mathcal{C}) is said to be *join-hull commutative* (JHC) iff for each $p \in X$ and $S \subset X$ we have $\mathcal{C}(p \cup S) = \bigcup \{\mathcal{C}(p, x) : x \in \mathcal{C}(S)\}$.

A convexity space (X, \mathcal{C}) is said to be *domain finite* (DF) iff for each $S \subset X$, $\mathcal{C}(S) = \bigcup \{\mathcal{C}(T) : T \subset S \text{ and } \text{card } T < \infty\}$.

In this note some new classes of convexity spaces are defined and their relationships are studied. It is shown that the known characterization of convex sets for the class of JD (= JHC and DF) convexity spaces is in fact true in a greater class. Using this we add a remark to [11] where the solution of the *linearization problem* of a convexity space has been given.

2. \mathcal{C} -starshaped sets

First we mention that \mathcal{C} -starshaped sets are already studied by Soltan [13] but our approach is somewhat different.

A set $S \subset X$ is called *\mathcal{C} -starshaped* relative to a point $p \in S$ iff $\mathcal{C}(p, x) \subset S$ for each $x \in S$. The set of points with respect to which S is a \mathcal{C} -starshaped set is called the *\mathcal{C} -kernel* of S and is denoted by $\mathcal{C}\text{-ker}(S)$.

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Obviously each \mathcal{C} -convex set S is a \mathcal{C} -starshaped set relative to any point $p \in S$. A \mathcal{C} -star emanating from a point $q \in S$ is defined as the set of all points $y \in S$ such that $\mathcal{C}(q, y) \subset S$ and is denoted by $\mathcal{C}\text{-st}(q, S)$.

PROPERTY 2.1: $p \in \mathcal{C}\text{-ker}(S)$ iff $\cup \{ \mathcal{C}(p, x) : x \in S \} \subset S$.

PROPERTY 2.2: For each set $S \subset X$

$$\mathcal{C}\text{-ker}(S) = \cap \{ \mathcal{C}\text{-st}(q, S) : q \in S \}.$$

In general, in a convexity space singletons need not be convex. The next property shows that those points which are the \mathcal{C} -kernels of some \mathcal{C} -starshaped sets are \mathcal{C} -convex.

PROPERTY 2.3: If $\mathcal{C}\text{-ker}(S) = \{ p \}$ for some $S \subset X$, then $\{ p \} \in \mathcal{C}$.

PROOF: Suppose $\{ p \} \notin \mathcal{C}$. Then there is an element $y \in \mathcal{C}(p)$ such that $p \neq y$. However we have

$$\begin{aligned} \cup \{ \mathcal{C}(y, x) : x \in S \} &\subset \cup \{ \mathcal{C}(\mathcal{C}(p) \cup x) : x \in S \} \\ &= \cup \{ \mathcal{C}(p, x) : x \in S \} \subset S, \end{aligned}$$

and this implies that $y \in \mathcal{C}\text{-ker}(S)$, which is a contradiction.

3. B-convexity spaces

In 1913 Brunn [1] showed that in the ordinary convexity space $(\mathbb{R}^n, \text{conv})$ the kernel of every subset of \mathbb{R}^n is a convex set. This is also true for any linear space with the ordinary convex sets (cf. [15]), but is false for general convexity spaces, as the following example shows.

EXAMPLE 3.1: Let $X = \mathbb{R}^2$ and suppose \mathcal{C} consists of all ordinary convex closed subsets of X . (X, \mathcal{C}) is a convexity space. Now take the set $S = \{(x, y) : x^2 + y^2 < 1\}$. It is clear that S is a \mathcal{C} -starshaped set and $\mathcal{C}\text{-ker}(S) = S$, but S not belongs to \mathcal{C} .

A convexity space (X, \mathcal{C}) is said to be a **B-convexity space** iff for each $S \subset X$, $\mathcal{C}\text{-ker}(S)$ is a \mathcal{C} -convex set.

As already mentioned, $(\mathbb{R}^n, \text{conv})$ is a **B-convexity space**. A Bryant-Webster convexity space is also a **B-space**, this follows from [2] Theorem 13.

We are interested in the following question: What is the class of convexity spaces for which Brunn's theorem holds?

THEOREM 3.2: *Each \mathbf{B} -convexity space is DF.*

PROOF: Let us recall that (X, \mathcal{C}) is a DF-convexity space if and only if $\cup \mathcal{F} \in \mathcal{C}$ for each chain $\mathcal{F} \subset \mathcal{C}$ [4] (a chain being a nonempty family of sets totally ordered by inclusion). Let (X, \mathcal{C}) be a \mathbf{B} -convexity space and suppose that (X, \mathcal{C}) is not DF. Then there is a chain $\mathcal{F} \subset \mathcal{C}$ such that $\cup \mathcal{F} \notin \mathcal{C}$. Clearly, $\cup \mathcal{F}$ is a \mathcal{C} -starshaped set and $\mathcal{C}\text{-ker}(\cup \mathcal{F}) = \cup \mathcal{F}$. Hence, $\mathcal{C}\text{-ker}(\cup \mathcal{F}) \notin \mathcal{C}$, which is a contradiction.

THEOREM 3.3: *Any JD-convexity space is a \mathbf{B} -convexity space.*

PROOF: Let (X, \mathcal{C}) be a JD-convexity space and take any $S \subset X$. If S is not \mathcal{C} -starshaped, then $\mathcal{C}\text{-ker}(S) = \emptyset$ and we are done. So we may assume that S is \mathcal{C} -starshaped. Take any $x_1, x_2 \in \mathcal{C}\text{-ker}(S)$, $p \in \mathcal{C}(x_1, x_2)$, and $z \in S$. Always we have $\mathcal{C}(z, p) \subset \mathcal{C}(z \cup \mathcal{C}(x_1, x_2)) = \mathcal{C}(z, x_1, x_2) = \mathcal{C}(x_1, z, x_2) = \mathcal{C}(x_1 \cup \mathcal{C}(z, x_2))$. The starshapedness of S implies that $\mathcal{C}(x_1, q) \subset S$ for each $q \in \mathcal{C}(x_2, z)$ (since $\mathcal{C}(x_2, z) \subset S$). As (X, \mathcal{C}) is JHC it follows that

$$\mathcal{C}(x_1 \cup \mathcal{C}(x_2, z)) = \cup \{ \mathcal{C}(x_1, q) : q \in \mathcal{C}(x_2, z) \} \subset S.$$

Hence, $\mathcal{C}(z, p) \subset S$. Therefore, $\mathcal{C}(x_1, x_2) \subset \mathcal{C}\text{-ker}(S)$. Applying [6] Theorem 2 gives $\mathcal{C}\text{-ker}(S) \in \mathcal{C}$.

The example below shows that the converse of Theorem 3.3 is not true in general.

EXAMPLE 3.4: Let $X = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$ and we define \mathcal{C} as the family of the sets of the form $A \cap X$ where A is an ordinary convex set in \mathbb{R}^2 . We show that (X, \mathcal{C}) is a \mathbf{B} -convexity space. Take any $S \subset X$. We may of course suppose that $\mathcal{C}\text{-ker}(S) \neq \emptyset$. In \mathbb{R}^2 we define the set $S^* = \cup \{\text{conv}(x \cup \mathcal{C}\text{-ker}(S)) : x \in S\}$. It is easy to see that $S = S^* \cap X$ and $\mathcal{C}\text{-ker}(S) = \text{ker}(S^*) \cap X$. Now Brunn's theorem (in \mathbb{R}^2) and the definition of \mathcal{C} imply that $\mathcal{C}\text{-ker}(S) \in \mathcal{C}$ and, consequently, that (X, \mathcal{C}) is a \mathbf{B} -convexity space. However, it is easy to verify that (X, \mathcal{C}) does not satisfy JD because it does not satisfy JHC.

THEOREM 3.5: *Let (X, \mathcal{C}) be a \mathbf{B} -convexity space. The set $S \subset X$ is \mathcal{C} -convex if and only if $\mathcal{C}(x, y) \subset S$ for every $x, y \in S$.*

PROOF: The necessity is obvious. To prove sufficiency take any $x \in S$. Then by our assumption we have $\cup \{ \mathcal{C}(x, y) : y \in S \} \subset S$. This means that $x \in \mathcal{C}\text{-ker}(S)$ and consequently that $S \subset \mathcal{C}\text{-ker}(S)$. The reverse inclusion always holds, hence $S = \mathcal{C}\text{-ker}(S)$. Now the definition of \mathbf{B} -convexity space implies that $S \in \mathcal{C}$. This completes the proof.

COROLLARY 3.6: *Let (X, \mathcal{C}) be a \mathbf{B} -convexity space. Then for each $S \subset X$ we have $\mathcal{C}\text{-ker}(\mathcal{C}\text{-ker}(S)) = \mathcal{C}\text{-ker}(S)$.*

Our Theorem 3.5 and Corollary 3.6 are extensions (from \mathbf{JD} to \mathbf{B} -convexity space) of Theorem 2 in [6] and Corollary 1 in [13], respectively.

4. \mathbf{T} -convexity spaces

Toranzos [14] has characterized $\ker(S)$ for ordinary convexity space as the intersection of all maximal (in the sense of inclusion) convex subsets of S . Such a characterization of the \mathcal{C} -kernel is true for a wider class of convexity spaces.

EXAMPLE 4.1: Let $X = \{1, \dots, 7\}$ and let \mathcal{C} consists of all singletons and the system of Steiner triples on X (i.e. a family \mathcal{F} of 3-element subsets of X such that each pair in X belongs to exactly one triple; for instance $\mathcal{F} = \{\{1, 2, 3\}, \{1, 5, 7\}, \{1, 4, 6\}, \{2, 4, 5\}, \{2, 6, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$). It is easy to verify that for any set S , $\mathcal{C}\text{-ker}(S)$ is equal to the intersection of all maximal \mathcal{C} -convex subsets of S .

A convexity space (X, \mathcal{C}) is said to be a \mathbf{T} -convexity space iff for each $S \subset X$, $\mathcal{C}\text{-ker}(S)$ is the intersection of all maximal \mathcal{C} -convex subsets of S .

Obviously any \mathbf{T} -convexity space is a \mathbf{B} -space. The Example 3.4 shows that the converse is not true.

The answer to the question: Characterize the class of \mathbf{T} -convexity spaces, is the following theorem. A similar result can be found in Soltan [13]. A \mathbb{T}_1 convexity space has all singletons convex.

THEOREM 4.2: *Let (X, \mathcal{C}) be a \mathbb{T}_1 convexity space. (X, \mathcal{C}) is a \mathbf{T} -convexity space if and only if it is a \mathbf{JD} -convexity space.*

PROOF: Suppose first that (X, \mathcal{C}) is a \mathbf{JD} -space. Take any $S \subset X$. Our assumptions (with \mathbb{T}_1) imply that S can be described as the union of the maximal \mathcal{C} -convex subsets of S :

$$S = \cup \{M_a : a \in I_S\}.$$

It suffices to show that

$$\mathcal{C}\text{-ker}(S) = \cap \{M_a : a \in I_S\}. \quad (4.3)$$

The inclusion $\cap \{M_a : a \in I_S\} \subset \mathcal{C}\text{-ker}(S)$ can be shown as follows. Take any $p \in \cap \{M_a : a \in I_S\}$ and any $x \in S$. Then there is an $a' \in I_S$ such that $x \in M_{a'}$. Hence, $\{p, x\} \subset M_{a'}$, and so $\mathcal{C}(p, x) \subset M_{a'} \subset S$, which

implies that $p \in \mathcal{C}\text{-ker}(S)$. To prove the reverse inclusion take any $p \in \mathcal{C}\text{-ker}(S)$. Then $\mathcal{C}(p, x) \subset S$ for each $x \in S$. As (X, \mathcal{C}) is JHC it follows for each $a \in I_S$ that $\mathcal{C}(p \cup M_a) = \cup \{ \mathcal{C}(p, x) : x \in M_a \} \subset S$. The maximality of the M_a 's implies that $\mathcal{C}(p \cup M_a) = M_a$. Hence $p \in M_a$ for each $a \in I_S$, and we are done.

Now assume that (X, \mathcal{C}) is a T-convexity space. Then (X, \mathcal{C}) is a B-convexity space and by Theorem 3.2 it is a DF-space. We must show that (X, \mathcal{C}) satisfies JHC. Let $p \in X$ and $A \subset X$. Consider the set $S = \cup \{ \mathcal{C}(p, x) : x \in \mathcal{C}(A) \}$. It is clear that S is a \mathcal{C} -starshaped set with $p \in \mathcal{C}\text{-ker}(S)$. As $\mathcal{C}(A) \subset S$, there is a maximal convex subset $M_{a'}$ of S containing $\mathcal{C}(A)$ and p (since (4.3) holds). So we get

$$\begin{aligned} \mathcal{C}(p \cup A) &= \mathcal{C}(p \cup \mathcal{C}(A)) \\ &\subset M_{a'} \\ &\subset \cup \{ M_a : a \in I_S \} = S \\ &= \cup \{ \mathcal{C}(p, x) : x \in \mathcal{C}(A) \}. \end{aligned}$$

So (X, \mathcal{C}) is in fact JHC and the proof is complete.

5. Remarks

R-1. A well-known theorem of Krasnosel'skii [9,15] gives necessary and sufficient conditions for a compact set S in \mathbb{R}^n to be starshaped. In the proof of that theorem the following additional property of the kernel of a compact set S is established.

$$\mathcal{C}\text{-ker}(S) = \cap \{ \mathcal{C}(\mathcal{C}\text{-st}(x, S)) : x \in S \}. \tag{5.1}$$

By means of Krasnosel'skii's lemma (see [15]) it can be shown that any closed set also satisfies (5.1). Using (5.1) we define the following class of sets in an arbitrary convexity space.

A set S of X is called a K-set if and only if it satisfies the equality (5.1).

Obviously and \mathcal{C} -convex set is a K-set.

Note that K-sets play an important part in [8] where, using the ideas of [7], the "starshapedness number" of a convexity space is introduced.

R-2. A convexity space (X, \mathcal{C}) is said to be a K-convexity space iff any subset of X is a K-set.

EXAMPLE 5.2: $(X, 2^X)$ is a K-convexity space, but the space considered in Example 4.1 does not satisfy K.

EXAMPLE 5.3: Let D be a nonempty subset of a set X . Following [12] consider the family of sets $\mathcal{C} = \{X\} \cup \{A : D \not\subset A\}$. Clearly (X, \mathcal{C}) is a convexity space. It is easy to verify that (X, \mathcal{C}) is a K-space if and only if $\text{card } D \leq 2$.

Obviously any K-convexity space is also a B-space but not conversely as the example of the space $(\mathbb{R}^n, \text{conv})$ shows. Moreover, we remark that conditions K and T are independent.

R-3. In [11] necessary and sufficient conditions for the existence of a real linear structure for X such that the set of all convex sets of the resulting linear space is precisely \mathcal{C} , are given. This is a solution of the linearization problem of a convexity space.

Our approach enables us to give the solution for the larger class of convexity spaces. Namely, the main theorem in [11] can be formulated as follows (for definitions see [11]).

THEOREM 5.4: Let (X, \mathcal{C}) be a B-convexity space with the property that for all $x, y, z \in X$, $\mathcal{C}(x, y) = \mathcal{C}(z, y)$ implies $x = z$. A necessary and sufficient condition that \mathcal{C} is the family of all convex sets generated by a real linear structure for X is that X has a linearization family X^* .

PROOF: The proof goes just as in [11] but we apply Theorem 3.5 instead of Theorem 2 in [6].

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