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BEN LICHTIN
DIANE MEUSER

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POLES OF A LOCAL ZETA FUNCTION AND NEWTON POLYGONS

Ben Lichtin and Diane Meuser

This article continues the investigation begun in [7] of a local zeta function defined by Igusa [3] as follows. Let K be a nonarchimedean local field of characteristic zero, R its ring of integers, and $f \in R[x_1, \dots, x_n]$. Let Φ be a Schwartz-Bruhat function on K^n and s a complex number with $\mathrm{Re}(s) > 0$. Define

$$Z_f(s, \Phi) = \int_{K^n} |f|^2 \Phi |dx|$$

where $|dx|$ is the usual Haar measure on K^n, and $|\ |$ is the usual absolute value on K.

As in [7], the aim of this article is to detect the poles of the meromorphic continuation of $Z_f(s, \Phi)$ into $\mathrm{Re}(s) < 0$. The only strategy seemingly available to accomplish this task is to take a resolution $\pi: X \to K^n$ of f and study the resolution data $\{(N_\iota, n_\iota)\}$ in which N_ι = multiplicity of f along divisor D_ι, $n_\iota - 1$ = multiplicity of $\det(d\pi)$ along D_ι. The set of radios $\{-n_\iota/N_\iota\}_\iota \cup \{-1\}$ contains the poles of $Z_f(s, \Phi)$ as observed in [3], but for the known examples most of these ratios are not actually poles. The problem is to determine the actual poles.

Here this is accomplished for a certain class of reducible plane curves with exactly one singularity at the point $(0, 0) \in K \times K$ which is "toroidal". Toroidal singularities are a useful class because a resolution can be easily constructed via a monomial transformation π, obtained from a study of the Newton polygon associated to a defining function for the curve in a given coordinate system. The poles of $Z_f(s, \Phi)$ are determined by the singularity of f so we shall assume without loss of generality that Φ is the characteristic function of $R \times R$. We then fix f and denote $Z_f(s, \Phi)$ by $Z(s)$.

Section 1 recalls this "toroidal resolution" following [6]. In Section 2 arithmetical information is obtained on the numerical data. Section 3 shows why most of the ratios $-n_\iota/N_i$ cannot be poles of $Z(s)$. Moreover the small number of its genuinely possible poles is described in terms of the polygon for f. Section 4 contains the affirmative result describing conditions imposed both upon f and the polygon itself which imply the non-vanishing of the residue of $Z(s)$ at each of the ratios in this smaller set of good candidates for poles of the zeta function. A simple description of the largest pole is given in terms of the polygon which is identical to that given in [10] by Varcenko when the local field is $\mathbb{R}$ and Vasiliev [12] when the local field is $\mathbb{C}$. It is also interesting to note here that the negative of the value obtained by Varcenko and Vasiliev also has an interpretation. As shown by Ehlers and Lo in [2], it is the smallest of the exponents associated to the mixed Hodge structure on the vanishing cohomology of the Milnor fiber. This follows from Varcenko [9].

1. Toroidal resolutions

For more detailed descriptions of toroidal resolution [1], [6], [8] or [11] may be consulted. Here, a brief summary of techniques and results will be given for the case of two variables only.

For fixed coordinates (x_1, x_2) in R and $f \in R[x_1, x_2]$ define $\mathrm{Supp}(f) = \{I = (i_1, i_2) \in \mathbb{N}^2 \colon x_1^{i_1} x_2^{i_2} = x^I$ appears with a non-zero coefficient in the expression for $f\}$. Define $S = \bigcup_{I \in \mathrm{Supp}(f)} (I + \mathbb{R}_+^2)$ and $\Gamma_+(f) =$ the boundary of the convex hull of S. $\Gamma = \Gamma_+(f)$ is the Newton polygon for f with respect to the (x_1, x_2) coordinates. We may write $f = f_\Gamma +$ (higher order terms) where $f_\Gamma(x_1, x_2) = \sum_{I \in \Gamma} a_I x^I$ is the "principal part" of f.

One can dualize Γ as follows. In the dual space $(\mathbb{R}_+^*)^2$ of covectors in the first quadrant define the "first meet locus" of a covector a as $K_a = \{x \in \Gamma \colon a \cdot x = m(a)\}$ where $m(a) = \inf\{a \cdot y \colon y \in \Gamma\}$. K_a is either a closed face of Γ or a vertex. On the set of covectors define an equivalence relation by $a^1 \sim a^2$ iff $K_{a^1} = K_{a^2}$. The equivalence classes consist of a finite set of open cones whose boundaries are also equivalence classes consisting of covectors dual to a one dimensional face of Γ. There is an evidence refinement process allowing one to obtain a finite set of closed cones of the form $\langle a^1, a^2 \rangle = \{a \colon a = \alpha a^1 + \beta a^2;\ \alpha, \beta \geqslant 0\}$ such that

i) $\det \begin{vmatrix} a^1 \\ a^2 \end{vmatrix} = \pm 1$ and

ii) a^1, a^2 are "primitive", that is, if $a^i = (a_1^i, a_2^i)$, then g.c.d.$(a_1^i, a_2^i) = 1$ $i = 1, 2$.

To each "unit" cone $\sigma = \langle a^1, a^2 \rangle$ one associates an affine chart $K \times K = K^2(\sigma)$ and defines a map $\pi(\sigma)\colon K^2(\sigma) \to K^2$ by $x_i \circ \pi(\sigma)(y_1, y_2) = y_1^{a_i^1} y_2^{a_i^2}$, $i = 1, 2$; i.e. if $P = (p_1, p_2)$ denotes the point on Γ corresponding to the monomial $x_1^{p_1} x_2^{p_2}$ we have $x_1^{p_1} x_2^{p_2} \circ \pi(\sigma) = y_1^{P \cdot a^1} y_2^{P \cdot a^2}$. A smooth variety over K, denoted $X(\Gamma)$, is obtained by patching $K^2(\sigma_1)$ and $K^2(\sigma_2)$ if and only if $\sigma_1 \cap \sigma_2 \neq \emptyset$ by the evident relation $z \in K^2(\sigma_1)$ is identified with $w \in K^2(\sigma_2)$ if and only if

$(\pi(\sigma_2)^{-1} o \pi(\sigma_1))(z) = w$ where this makes sense.

If $\sigma_1 = \langle c^1, c^2 \rangle$, $\sigma_2 = \langle a^1, a^2 \rangle$ and if A_1 denotes the matrix $\begin{pmatrix} c^1 \\ c^2 \end{pmatrix}$ and A_2 denotes the matrix $\begin{pmatrix} a^1 \\ a^2 \end{pmatrix}$; then A_1 determines $\pi(\sigma_1)$ and A_2^{-1} determines $\pi(\sigma_2)^{-1}$. Orientation is chosen so that $\det A_i = 1$ for $i = 1, 2$. Thus $A_1 A_2^{-1}$ determines $\pi(\sigma_2)^{-1} o \pi(\sigma_1)$.

Now $\sigma_1 \cap \sigma_2 \neq \emptyset$ so either i) $a^1 = c^2$ or ii) $a^2 = c^1$. We examine $A_1 A_2^{-1}$ in each case

$$\text{i) } A_1 A_2^{-1} = \begin{pmatrix} k & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{where } k = \det\begin{pmatrix} c^1 \\ a^2 \end{pmatrix} \geqslant 1.$$

$$\text{ii) } A_1 A_2^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & k' \end{pmatrix} \qquad \text{where } k' = \det\begin{pmatrix} a^1 \\ c^2 \end{pmatrix} \geqslant 1.$$

These compatibility relations precisely determine the charts needed to describe the preimage of $R \times R$ in $X(\Gamma)$. Index the cones used in the partition of $(\mathbb{R}_+^*)^2$ used to construct $X(\Gamma)$ so that $\sigma_1 = \langle (1, 0), a^1 \rangle$, $\sigma_2 = \sigma_2 = \langle a^1, a^2 \rangle, \ldots, \sigma_N = \langle a^{N-1}, (0, 1) \rangle$. If (y_1, y_2) are coordinates in $K^2(\sigma_i)$ and (y_1', y_2') coordinates in $K^2(\sigma_{i+1})$, then by the above we see that the monomial transformation $(\pi(\sigma_{i+1})^{-1} o \pi(\sigma_i))(y_1, y_2) = (y_1^k y_2, y_1^{-1}) = (y_1', y_2')$. Hence if we assume (y_1, y_2) lies in $R \times R \subset K^2(\sigma_i)$, then since $y_2' = y_1^{-1}$ in the overlap, we can shrink y_2' in $K^2(\sigma_{i+1})$ from lying in R to lying in P, the unique maximal ideal of R. As a result $\pi^{-1}(R \times R)$ may be covered by $R \times R$ in $K^2(\sigma_{2i})$ and $R \times P$ in $K^2(\sigma_{2i+1})$. We shall use this in order to examine the pullback of $Z(s)$ in $X(\Gamma)$.

Having constructed $X(\Gamma)$ and π: $X(\Gamma) \to K^2$ a proper birational modification of K^2; one can then resolve f via π by imposing a non-degeneracy condition on f_Γ, as in [5]:

NON-DEGENERACY CONDITION: For each closed face of Γ, the functions $(x_1 f_{x_1})_\Gamma$ and $(x_2 f_{x_2})_\Gamma$, consisting of the monomials in $x_1 f_{x_1}$ lying on Γ, have no common zeroes in $(K^*)^2$.

[6] discusses how this condition implies that at each point in $\pi^{-1}(0, 0)$ there are local coordinates in which f is written in normal crossing form. In particular, it follows from the assumption of the origin being an isolated singularity of f in $R \times R$ that the strict transform of f is nonsingular in $\pi^{-1}(R \times R) \subseteq X(\Gamma)$, and can only have simple roots on any component of the exceptional divisor inside $\pi^{-1}(R \times R)$. This observation implies the genericity of the non-degeneracy condition in the following sense:

PROPOSITION 1: *In fixed coordinates* (x_1, x_2) *and for a fixed polygon* Γ, *let* $\mathscr{F}_\Gamma = \{f \in K[x_1, x_2]\colon \Gamma = \Gamma_+(f)\}$. *Then* $\mathscr{G}_\Gamma = \{g \in \mathscr{F}_\Gamma\colon$ *g is non-degenerate with respect to* $\Gamma\}$ *is an inductive limit of Zariski open subsets of* $\mathscr{F}_\Gamma$.

PROOF: We refer to [5] for the proof of an entirely analogous statement.

The results in Sections 3 and 4 concern subsets of polynomials in $\mathscr{G}_\Gamma$. $\mathscr{G}_\Gamma$ is a useful class of polynomials for which to analyze $Z(s)$ because of the ease of constructing the resolution.

2. Relationships on the numerical data

The data needed in the next sections concerns the set of divisors in the exceptional locus $\pi^{-1}(0, 0)$ of π. Note that to each covector a, used in the refinement of the partition of $(\mathbb{R}_+^*)^2$ described in Section 1, one associates a divisor D_a. To each D_a correspond two integers $(m(a), |a|)$ and a ratio $p_a = -|a|/m(a)$, where $m(a) = \inf\{x \cdot a\colon x \in \Gamma\}$ and $|a| = a_1 + a_2$ if $a = (a_1, a_2)$. The set $\mathscr{A} = \{(m(a), |a|)\}_a$ is called the numerical data of the resolution.

From [3] one knows that the set $\{p_a\}_a$ is the set of possible poles for the meromorphic continuation of $Z(s)$ into $\mathrm{Re}(s) < 0$. In terms of Γ one may easily interpret p_a, as described in [10]. The support line to Γ in the direction determined by a is defined by $x \cdot a = m(a)$. The point of intersection with the diagonal $x = y = t$ is at the value $t_a = m(a)/|a|$. Thus $p_a = -1/t_a$. The graph of t_a versus a_2/a_1 for covectors (a_1, a_2) dual to faces is represented by Diagram one in the case where there is one face of Γ intersecting the diagonal and $b = (b_1, b_2)$ is dual to that face. The case where two faces of Γ intersect the diagonal is represented by Diagram two where $b = (b_1, b_2)$ and $b' = (b'_1, b'_2)$ denote the vectors dual to the two faces.

The following proposition summarizes the preceding discussion.

PROPOSITION 2: *Let b be a covector dual to any face of* Γ *not containing the intersection with the diagonal as a vertex. Let* $\sigma_1 = \langle a^1, b\rangle$, $\sigma_2 = \langle b, a^2\rangle$ *be two unit cones oriented so that* $\det\begin{pmatrix} a^1 \\ b \end{pmatrix} = \det\begin{pmatrix} b \\ a^2 \end{pmatrix} = 1$. *Then* $p_b \neq p_{a^i}$ *for* $i = 1, 2$.

Of more interest is the following:

PROPOSITION 3: *With the orientation of Proposition* 2, *if* a^1, b, *and* a^2 *are covectors not dual to any face of* Γ *and satisfying the property that there is a*

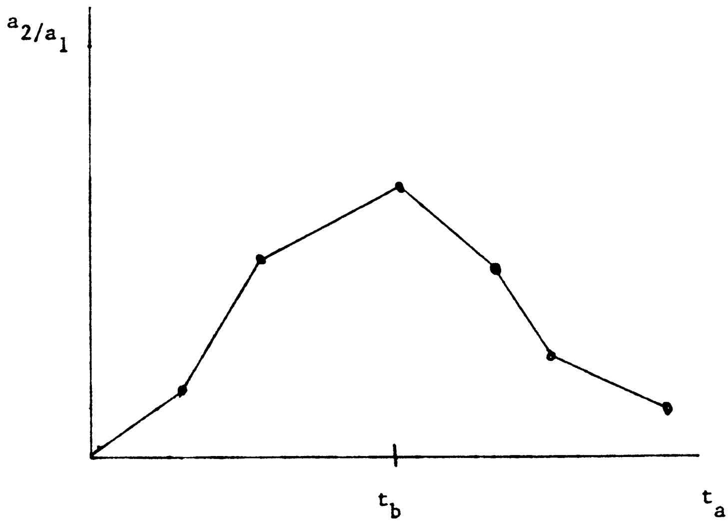

Diagram 1

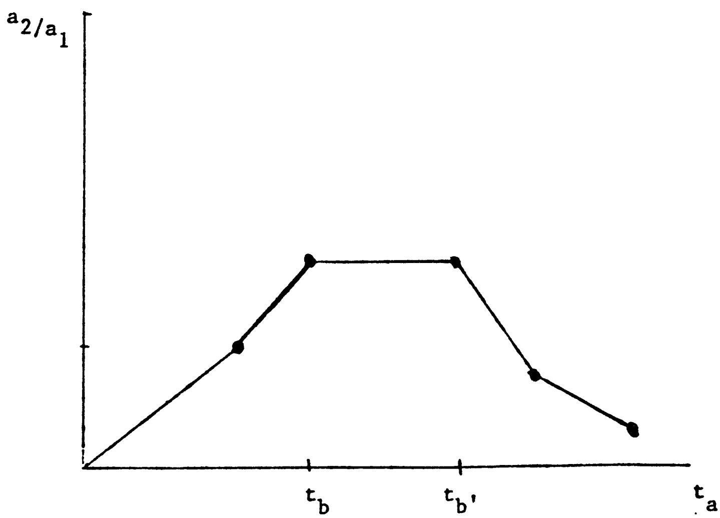

Diagram 2

unique vertex $P = (p_1, p_2)$ *with* $P = K_b = K_{a^1} = K_{a^2}$, *then*

$$\frac{|a^1| + |a^2|}{m(a^1) + m(a^2)} = \frac{|b|}{m(b)}.$$

PROOF: Consider the expression $m(b)|a^i| - |b|m(a^i)$. Let $\bar{1} = (1, 1)$. By the property defining P the difference equals

$$\begin{aligned}
&(b\cdot P)(a'\cdot\bar{1})-(a'\cdot P)(b\cdot\bar{1})\\
&=\det\begin{pmatrix} b\cdot P & b\cdot\bar{1}\\ a'\cdot P & a'\cdot\bar{1}\end{pmatrix}\\
&=\det\begin{pmatrix} b\\ a'\end{pmatrix}\det\begin{pmatrix} p_1 & 1\\ p_2 & 1\end{pmatrix}\\
&=\begin{cases} p_2-p_1 & \text{if}\quad i=1\\ p_1-p_2 & \text{if}\quad i=2.\end{cases}
\end{aligned}$$

The proposition is equivalent to showing $m(b)|a^1|-m(a^1)|b|=m(a^2)|b|-m(b)|a^2|$ which is immediate from the above formula.

COROLLARY 1: *In the situation of Proposition* 2, *if* $K_{a^1}=\{I\}$, $K_{a_u}2=\{J\}$, *where* $I=(I_1, I_2)$ *and* $J=(J_1, J_2)$ *are the vertices of the face* τ_b *dual to* b, *then*

$$\frac{|a^1|+|a^2|}{m(a^1)+m(a^2)}=\frac{|b|}{m(b)}+\frac{k|b|}{m(b)(m(a^1)+m(a^2))}$$

for some integer $k\geqslant 1$.

PROOF: Proposition 3 implies $m(b)|a^1|-m(a^1)|b|=I_2-I_1$ and $m(b)|a^2|-m(a^2)|b|=J_1-J_2$. Thus the numerator of $(|a^1|+|a^2|/m(a^1)+m(a^2))-|b|/m(b)$ is equal to $I_2-I_1+J_1-J_2$. Now $(I_2-J_2)/(I_1-J_1)$ is the slope of the face τ_b, so

$$\frac{I_2-J_2}{I_1-J_1}=\frac{-b_1}{b_2}.$$

Using the fact that b is primitive and that $I_2>J_2$ we have that $I_2-J_2=kb_1$ and $I_1-J_1=-kb_2$ for some positive integer k. Therefore $I_2-I_1+J_1-J_2=k(b_1+b_2)=k|b|$ which gives us the desired statement. q.e.d.

COROLLARY 2: *If* b *is a covector dual to a face* τ *then using the notation of Corollary* 1 *we have*

(*a*) *If* $b=(b_1, 1)$ *is dual to a face with a vertex* $P=(0, p_2)$, *then*

$$m(b)|a^1|-m(a^1)|b|=m(b).$$

(*b*) *If* $b=(1, b_2)$ *is dual to a face with a vertex* $P=(p_1, 0)$ *then*

$$m(b)|a^2|-m(a^2)|b|=m(b).$$

(*c*) *In all other cases we have*

$$m(b)\,|a'| - m(a')\,|b| < m(b)$$

PROOF: These statements follow from the expressions for $m(b)\,|a'| - m(a')\,|b|$ and the fact that $m(b) = b \cdot P$.

3. On the vanishing of possible poles for $Z(s)$

In Section 2 we described the set of possible poles for $Z(s)$ associated to any $f \in \mathcal{G}_\Gamma$. This section eliminates most of these ratios from candidacy in the set of poles for $Z(s)$. We think of $\mathrm{Res}_{s=p} Z(s)$ for a possible pole p as a sum of integrals along all divisors D_a in $\pi^{-1}(R \times R)$ where D_a has numerical data $(m(a), |a|)$ and $p = p_a = -|a|/m(a)$ (in the notation of section 1). The theorem we shall prove here says that when the covector a is dual to a vertex only of Γ, then the contribution to $\mathrm{Res}_{s=p} Z(s)$ along D_a is zero.

We let $q = \mathrm{card}(R/P)$, and recall that $|x| = q^{-\mathrm{ord}\, x}$ where ord denotes the usual order of a fixed element in $P - P^2$.

THEOREM 1: *Let p be a possible pole of* $Z(s)$ *in the sense that there is at least one divisor* D_a *with the numerical data* $(m(a), |a|)$ *such that* $p = -|a|/m(a)$. *Moreover assume* $p \neq -1$. *If b is a covector in a refinement of the partition dual to* Γ *such that* K_b *is only a vertex of* Γ *and* $p = p_b$, *then the contribution to* $\mathrm{Res}_{s=p} Z(s)$ *along divisor* D_b *with numerical data* $(m(b), |b|)$ *is zero.*

PROOF: We examine the pullback of $Z(s)$ in the charts containing the divisor D_b. Let a^1 and a^2 be the covectors oriented about b as described above so that $\sigma_1 = \langle a^1, b \rangle$ and $\sigma_2 = \langle b, a^2 \rangle$ are unit cones and $K_{a^1} = K_{a^2} = K_b$. Let $\pi(\sigma_1)$: $K^2(\sigma_1) \to K^2$ and $\pi(\sigma_2)$: $K^2(\sigma_2) \to K$ be the associated charts. In $K^2(\sigma_1)$ we have $(f \circ \pi(\sigma_1))(y_1, y_2) = y_1^{m(a^1)} y_2^{m(b)} f_{\sigma_1}(y_1, y_2)$ and in $K^2(\sigma_2)$ we have $(f \circ \pi(\sigma_2))(y_1', y_2') = y_1'^{m(b)} y_2'^{m(a_2)} f_{\sigma_2}(y_1', y_2')$ where f_{σ_1} and f_{σ_2} denote the strict transforms of f in each chart. Thus the integrals $Z_1(s)$ and $Z_2(s)$ will contribute to the pole at $s = -|b|/m(b)$ where

$$Z_1(s) = \int_R \int_R |y_1|^{m(a^1)s + |a^1| - 1} |y_2|^{m(b)s + |b| - 1} |f_{\sigma_1}|^s |\mathrm{d}y_1|\, |\mathrm{d}y_2|$$

and

$$Z_2(s) = \int_P \int_R |y_1'|^{m(b)s + |b| - 1} |y_2'|^{m(a^2)s + |a^2| - 1} |f_{\sigma_2}|^s |\mathrm{d}y_1'|\, |\mathrm{d}y_2'|.$$

We now examine f_{σ_1} and f_{σ_2} more closely. Under the change of coordinates given by $\pi(\sigma_1)$ a monomial corresponding to the point $p=(p_1, p_2)$, i.e. $x_1^{p_1}x_2^{p_2}$; is transformed to the monomial $y_1^{a^1\cdot p}y_2^{b\cdot p}$. f_{σ_1} is obtained by factoring $y_1^{m(a^1)}y_2^{m(b)}$ from each monomial. Since b is not dual to a face of the polygon, $m(b)$ is determined by a unique vertex of the polygon, and this vertex also determines the value of $m(a^1)$. For all other points of the polygon we have $b\cdot p > m(b)$; therefore f_{σ_1} has the form $c+y_2g(y_1, y_2)$, where c is the coefficient of the monomial term corresponding to the vertex of the polygon that comprises K_b. Under the change of coordinates given by $\pi(\sigma_2)$ a monomial $x_1^{p_1}x_2^{p_2}$ is transformed to the monomial $y_1'^{b\cdot p}y_2'^{a^2\cdot p}$. Thus a similar argument shows that f_{σ_2} has the form $c+y_1'h(y_1', y_2')$. Thus if ord $c=e$ we see that $|f_{\sigma_1}|$ is constant on $R\times P^{e+1}$ and $|f_{\sigma_2}|$ is constant on $P^{e+1}\times P$.

We split the domain of integration for $Z_1(s)$ into $R\times(R-P^{e+1})$ and $R\times P^{e+1}$. First consider the domain of integration $R\times(R-P^{e+1})$. We can write this as a disjoint union of U_i where each U_i is a coset of the form $R\times(a_i+P^{e+1})$. We have that $|y_2|$ is constant on each coset. Consider separately the cases where $f_{\sigma_1}\neq 0$ on U_i and $f_{\sigma_1}=0$ on U_i.

If $f_{\sigma_1}\neq 0$ on U_i we can write U_i as a disjoint union of cosets U_{ij} modulo $P^k\times P^k$ for k sufficiently large such that $|f_{\sigma_1}|$ is also constant on these cosets. Then suppose a given coset is $(c_i+P^k)\times(d_j+P^k)$ and $|f_{\sigma_1}|=q^{-l}$ on this coset. Then the integral over this coset will contribute a term of the form

$$Cq^{-(\operatorname{ord} a_i)(m(b)s+|b|-1)}q^{-(\operatorname{ord} c_i)(m(a^1)s+|a^1|-1)}q^{-ls} \tag{1}$$

for some positive constant C if $c_i\notin P^k$ and

$$Cq^{-(\operatorname{ord} a_i)(m(b)s+|b|-1)}q^{-ls}\frac{q^{-(m(a^1)s+|a^1|)k}}{1-q^{-(m(a^1)s+|a^1|)}} \tag{2}$$

for some positive constant C if $c_i\in P^k$.

Now consider the case where $f_{\sigma_1}=0$ on U_i. Because (0, 0) is the only singularity of f in $R\times R$, we must have at each point (y_1, y_2) in U_i either $\partial f_{\sigma_1}/\partial y_1(y_1, y_2)\neq 0$ or $\partial f_{\sigma_1}/\partial y_2(y_1, y_2)\neq 0$. Refine U_i into smaller cosets U_{ij} so that on each U_{ij} one of $|\partial f_{\sigma_1}/\partial y_1|$ or $|\partial f_{\sigma_1}/\partial y_2|$ is a non-zero constant. Suppose $|\partial f_{\sigma_1}/\partial y_2|=q^{-m}$ on U_{ij}. Write U_{ij} as a disjoint union of cosets modulo $P^l\times P^l$ for $l>m$. Let D be one of these cosets. Then the change of coordinates given by $\tilde{y}_1=y_1$, $\tilde{y}_2=f_{\sigma_1}(y_1, y_2)$ maps D homeomorphically to its image, cf. [4], Lemma 5. Moreover for large enough k this image can be written as a disjoint union of cosets modulo $P^k\times P^k$. Then $|dy_1||dy_2|=q^m|d\tilde{y}_1||d\tilde{y}_2|$ so an integral over one of these cosets has the form

$$q^m q^{-(\operatorname{ord} a_l)(m(b)s+|b|-1)}$$
$$\times \int_{d_l+P^k}\int_{c_l+P^k} |\tilde{y}_1|^{m(a^1)s+|a^1|-1} |\tilde{y}_2|^s |\mathrm{d}\tilde{y}_1|\,|\mathrm{d}\tilde{y}_2|$$

which contributes a term of the form (1) if $c_l \notin P^k$ and $d_l \notin P^k$, where $l = \operatorname{ord} d_l$; a term of the form (2) if $c_l \in P^k$ and $d_l \notin P^k$; a term of the form

$$Cq^{-(\operatorname{ord} a_l)(m(b)s+|b|-1)} q^{-(\operatorname{ord} c_l)(m(a^1)s+|a^1|-1)} \frac{q^{-(s+1)k}}{1-q^{-(s+1)}} \tag{3}$$

for a positive constant C if $c_l \notin P^k$ and $d_l \in P^k$ and a term of the form

$$Cq^{-(\operatorname{ord} a_l)(m(b)s+|b|-1)} \frac{q^{-(s+1)k} q^{-(m(a^1)s+|a^1|)k}}{\left(1-q^{-(s+1)}\right)\left(1-q^{-(m(a^1)s+|a^1|)}\right)} \tag{4}$$

for a positive constant C if $c_l \in P^k$ and $d_l \in P^k$.

By our assumptions $|b|/m(b) \neq 1$ and by Proposition 2 $|b|/m(b) \neq |a^1|/m(a^1)$. Therefore the terms in (1), (2), (3) and (4) do not contribute to the pole at $-|b|/m(b)$. By splitting the domain of integration for $Z_2(s)$ into $P^{e+1} \times P$ and $(R - P^{e+1}) \times P$ and applying a similar argument we see that the integral over $(R - P^{e+1}) \times P$ does not contribute to the residue of the pole at $s = -|b|/m(b)$.

Therefore the contributions to the residue at $s = -|b|/m(b)$ come from the integrals

$$q^{-es} \int_{P^{e+1}}\int_R |y_1|^{m(a^1)s+|a^1|-1} |y_2|^{m(b)s+|b|-1} |\mathrm{d}y_1|\,|\mathrm{d}y_2|$$

and

$$q^{-es} \int_P\int_{P^{e+1}} |y_1|^{m(b)s+|b|-1} |y_2|^{m(a^2)s+|a^2|-1} |\mathrm{d}y_1|\,|\mathrm{d}y_2|.$$

Evaluating these integrals gives us $(1-q^{-1})^2 q^{-es}$ times

$$\frac{q^{-(m(b)s+|b|)(e+1)}}{\left(1-q^{-(m(a^1)s+|a^1|)}\right)\left(1-q^{-(m(b)s+|b|)}\right)}$$
$$+\frac{q^{-(m(b)s+|b|)}q^{-(m(a^2)s+|a^2|)}}{\left(1-q^{-(m(b)s+|b|)}\right)\left(1-q^{-(m(a^2)s+|a^2|)}\right)}$$

which gives a contribution to the residue of $(1-q^{-1})^2 q^{-es}$ times

$$\frac{1}{1-q^{-(m(a^1)s+|a^1|)}}+\frac{q^{-(m(a^2)s+|a^2|)}}{1-q^{-(m(a^2)s+|a^2|)}}$$

for $s=-|b|/m(b)$. But this expression will be zero if and only if

$$\frac{|a^1|+|a^2|}{m(a^1)+m(a^2)}=\frac{|b|}{m(b)}$$

which is true by Proposition 3. q.e.d.

4. The existence of poles of $Z(s)$

We first obtain a result analogous to the one obtained in [10] finding the largest root of the Bernstein-Sato polynomial for a real-valued function.

THEOREM 2: *Let b be a vector dual to a face of the Newton polygon that intersects the diagonal $x=y$. Assume $|b|/m(b)\neq 1$. Then $Z(s)$ has a simple pole at $-|b|/m(b)$ and this is the pole closest to the origin.*

PROOF: As before let $b=(b_1, b_2)$ and let $a^1=(a_1^1, a_2^1)$ and $a^2=(a_1^2, a_2^2)$ denote the vectors such that a^1 and b form the basis for the unit cone σ_1 and a^2 and b form the basis for the unit cone σ_2 where we assume $\det\begin{pmatrix} a^1 \\ b \end{pmatrix}=\det\begin{pmatrix} b \\ a^2 \end{pmatrix}=1$. Then in the charts $K^2(\sigma_1)$ and $K^2(\sigma_2)$ the pullbacks of $Z(s)$ are

$$Z_1(s)=\int_R\int_R |y_1|^{m(a^1)s+|a^1|-1}|y_2|^{m(b)s+|b|-1}|f_{\sigma_1}|^s|dy_1||dy_2|$$

and

$$Z_2(s)=\int_P\int_R |y_1|^{m(b)s+|b|-1}|y_2|^{m(a^2)s+|a^2|-1}|f_{\sigma_2}|^s|dy_1||dy_2|$$

where $f_{\sigma_1}(0,0)\neq 0$ and $f_{\sigma_2}(0,0)\neq 0$.

Consider $Z_1(s)$. We can decompose $R\times R$ into cosets of $P^k\times P^k$ for k sufficiently large so that in each coset at least one of the following is true:

a) $|f_{\sigma_1}|$ is a non-zero constant
b) $|y_2|$ is a non-zero constant
c) $|y_1|$ is a non-zero constant.

Consider first of all a coset $(c_i+P^k)\times(a_i+P^k)$ of type (a). The integral over this coset has the form

$$q^{-ls}\int_{a_\iota+P^k}\int_{c_\iota+P^k}|y_1|^{m(a^1)s+|a^1|-1}|y_2|^{m(b)s+|b|-1}|\mathrm{d}y_1||\mathrm{d}y_2|$$

where $|f_{\sigma_1}|=q^{-l}$. In the following, terms of type (1), (2), (3) or (4) refer to those appearing in Theorem 1. This integral contributes a term of type (1) if $a_\iota \notin P^k$ and $c_\iota \notin P^k$, a term of type (2) if $c_\iota \in P^k$ and $a_\iota \notin P^k$, a term of type

$$Cq^{-ls}q^{-(\operatorname{ord} c_\iota)(m(a^1)s+|a^1|-1)}\frac{q^{-(m(b)s+|b|)k}}{1-q^{-(m(b)s+|b|)}} \tag{5}$$

for a positive constant C if $c_\iota \notin P^k$ and $a_\iota \in P^k$; and a term of type

$$Cq^{-ls}\frac{q^{-(m(a^1)s+|a^1|)k}q^{-(m(b)s+|b|)k}}{\left(1-q^{-(m(a^1)s+|a^1|)}\right)\left(1-q^{-(m(b)s+|b|)}\right)} \tag{6}$$

for a positive constant C if $c_\iota \in P^k$ and $a_\iota \in P^k$. If we consider a coset of type (b) then using an argument similar to that given in Theorem 1 we will get terms of the form (1), (2), (3), or (4). Similarly, if we integrate over cosets of type (c) we get terms entirely similar to (1), (2), (3) and (4) where we replace a^1 by a^2. We can similarly determine the form of all contributions to $Z_2(s)$.

First of all suppose the diagonal intersects the Newton polygon in a vertex (d, d) of the polygon. Let b^1 and b^2 denote the covectors dual to the two faces having (d, d) as a common vertex, where we assume $\det\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} > 0$. Let (I_1, I_2) denote the other vertex of the face dual to b^1 and (J_1, J_2) denote the other vertex of the face dual to b^2. First consider the case where $b=b^1$, and let a^1, a^2 be the vectors associated to b as above. Then as the proof of Corollary 1 to Proposition 3 shows we have $m(b)|a^1|-m(a^1)|b|=0$ and $m(b)|a^2|-m(a^2)|b|=I_1-I_2>0$. Similarly if $b=b^2$ and a^1 and a^2 are again associated to b as above we have $m(b)|a^1|-m(a^1)|b|=J_2-J_1>0$ and $m(b)|a^2|-m(a^2)|b|=0$.

If the diagonal does not intersect the Newton polygon in a vertex let (I_1, I_2) and (J_1, J_2) denote the vertices of the face intersecting the diagonal, where we assume $I_1<J_1$. Then the proof of Corollary 1 to Proposition 3 shows that if b, a^1, a^2 are as above we have $m(b)|a^1|-m(a^1)|b|=I_2-I_1>0$ and $m(b)|a^2|-m(a^2)|b|=J_1-J_2>0$.

Thus in all cases the ratios $m(b)|a^i|-m(a^i)|b|\geqslant 0$ and we also have $m(b)-|b|\geqslant 0$. Hence all of the terms of the form (1)-(6) give contributions to the residue at $s=-|b|/m(b)$ which are greater than or equal to zero. Since there must be terms of type (5) we see that the total contribution to the residue at $s = -|b|/m(b)$ is strictly positive. q.e.d.

REMARK: An evident extension of this result to polyhedra ($n > 2$) can also be easily inferred from the proof. The essential point for the case in which the polyhedra has a unique face τ intersecting the diagonal is the following. Let b be the covector dual to τ and let $\sigma = \langle a^1 = b, a^2, \ldots, a^n \rangle$ be a cone containing b as one of its spanning vectors. Set $s_0 = -|b|/m(b)$. Then the values $m(a^i)s_0 + |a^i|$ are all strictly positive for $i \neq 1$. As such, the contribution to the residue of $Z(s)$ at $s = s_0$ in $K^n(\sigma)$ along the divisor with numerical data $(m(b), |b|)$ is always positive. Summing up over all the cones containing b gives a positive value to the residue at s_0.

In order to extend this result to smaller poles, that is values $-|b|/m(b)$ where b is dual to a face of Γ not intersecting the diagonal, it appears necessary to impose additional technical conditions both on the polygon Γ and functions $f \in \mathcal{G}_\Gamma$. These are:

CONDITION 1: No two support lines containing faces of Γ should intersect on the diagonal $x = y$.

CONDITION 2: The coefficients of the principal part f are all in $R - P$.

CONDITION 3: Let $\sigma = \langle b, a \rangle$ where b is dual to a face. Let D_b be the divisor in the exceptional locus corresponding to b. In the chart $K^2(\sigma)$, if the strict transform f_σ of f has the property $\deg(f_\sigma \mid_{D_b}) = s$, then there must be s numbers $\lambda_1, \ldots, \lambda_s$ in R satisfying the property that $\lambda_i \not\equiv \lambda_j \bmod P$ for $i \neq j$ and $(f_\sigma \mid_{D_b})(\lambda_i) = 0$ for all i.

REMARK: Conditions (2) and (3) are actually independent of the choice of resolution (or equivalently the refinement into unit cones of the original partition dual to the polygon). To see this fix a covector $b = b_i$ dual to the face τ_i with vertices $I = (I_1, I_2)$, $J = (J_1, J_2)$ such that $I_2 > J_2$.

The conditions $\det\begin{pmatrix} x \\ b \end{pmatrix} = 1$, $x = (x_1, x_2)$ with $x_1, x_2 \geqslant 0$, determine a ray on the first quadrant, all integral points of which correspond to possible covectors with which b could be paired to form a unit cone refining the original partition. Of course not all covectors lead to valid choices as some might have first meet loci on Γ in vertices other than I.

Since $x_2 x_1^{-1} - b_2 b_1^{-1} = -(x_1 b_1)^{-1}$, as x_1 increases the direction of (x_1, x_2) approaches that of b. Thus there is a smallest positive integer ξ_1 such that

i) $\xi_2 = \xi_1\left(\dfrac{b_2}{b_1}\right) - \dfrac{1}{b_1} \in \mathbb{N}$

ii) If $(\xi_1, \xi_2) = a^0$, the first meet locus $K_{a^0} = \{I\}$.

Observe that if $a=(a_1, a_2)$ is any integral point with $b_2/b_1 > a_2/a_1 > \xi_2/\xi_1$ and $\det\binom{a}{b}=1$, there is some positive integer r such that $a=a^0+r\cdot b$.

Let $f_{\tau_l}=\sum_{K\in\tau_l} a_K x_1^{k_1} x_2^{k_2}$ be the principal part of f for the face τ_l. Let $\sigma=\langle a, b\rangle$ be a unit cone in a refinement of the original partition with $\det\binom{a}{b}=1$. Then one sees that

$$\left(f_{\tau_l}\circ\pi(\sigma)\right)(y_1, y_2)=y_1^{m(a)}y_2^{m(b)}\sum_{K\in\tau_l} a_K y_1^{K\cdot a-M(a)}.$$

The right hand side is the part of the strict transform f_σ upon which conditions (2) and (3) are imposed.

Now observe that $m(a)=m(a^0)+rm(b)$ and $K\cdot a=K\cdot a^0+rm(b)$ since $K\in\tau_l$. Thus, $K\cdot a-m(a)=K\cdot a^0-m(a^0)$ is independent of r and depends only on b since a^0 only depends on b. Identical reasoning is used for the conditions $\det\binom{b}{x}=1$, $x=(x_1, x_2)$ with $x_1, x_2\geqslant 0$ and then applied to cones $\sigma=\langle b, a\rangle$ with $\det\binom{b}{a}=1$.

This shows that conditions (2) and (3) are independent of the refinement and so, the resolution π: $X(\Gamma)\to K^2$.

As seen next, conditions (1)–(3) imply that the residue of $Z(s)$ at $s=-|b|/m(b)$ cannot be zero. On the other hand lifting any of these conditions seems to produce an expression for the residue for which showing it is non-zero is difficult.

THEOREM 3: *Let b be a covector dual to a face of a polygon Γ satisfying condition* (1) *and such that if $b=(b_1, 1)$ (resp. $b=(1, b_2)$) then neither vertex of the face dual to b has the form* $(0, p_2)$ *(resp.* $(p_1, 0)$*). Then for any function $f\in\mathcal{G}_\Gamma$ satisfying conditions* (2) *and* (3)*, the ratio $s=-|b|/m(b)$ is a pole of $Z(s)$.*

PROOF: As before let a^1 and a^2 denote the vectors such that a^1 and b form the basis for the unit cone which we shall denote by σ_1, and a^2 and b form the basis for the unit cone which we shall denote by σ_2. As before we have that the contribution to the pole at $-|b|/m(b)$ will come from the pullback of the integral for $Z(s)$ in the charts $K^2(\sigma_1)$ and $K^2(\sigma_2)$. In $K^2(\sigma_1)$ we have

$$(f\circ\pi(\sigma_1))(y_1, y_2)=y_1^{m(a^1)}y_2^{m(b)}f_{\sigma_1}(y_1, y_2)$$

and in $K^2(\sigma_2)$ we have

$$(f\circ\pi(\sigma_2))(y_1', y_2')=y_1'^{m(b)}y_2'^{m(a^2)}f_{\sigma_2}(y_1', y_2'),$$

where f_{σ_1} and f_{σ_2} denote the strict transforms of f in each chart. Thus the integrals $Z_1(s)$ and $Z_2(s)$ will contribute to the pole at $s = -|b|/m(b)$ where

$$Z_1(s) = \int_R \int_R |y_1|^{m(a^1)s+|a^1|-1} |y_2|^{m(b)s+|b|-1} |f_{\sigma_1}|^s |dy_1| |dy_2|$$

and

$$Z_2(s) = \int_P \int_R |y_1'|^{m(b)s+|b|-1} |y_2'|^{m(a^2)s+|a^2|-1} |f_{\sigma_2}|^s |dy_1'| |dy_2'|$$

We examine the form of f_{σ_1} in more detail. Since b is a covector dual to a face we have that for any point P on the face $b \cdot P = m(b)$. The values of $m(a^1)$ and $m(a^2)$ are determined by opposite vertices of this face. We again recall that a monomial of the form $x_1^{p_1}x_2^{p_2}$ is transformed into $y_1^{a^1 \cdot P} y_2^{b \cdot P}$ in $K^2(\sigma_1)$; and f_{σ_1} is obtained by factoring $y_1^{m(a^1)}y_2^{m(b)}$ from each monomial. Therefore f_{σ_1} has the form $c + p(y_1) + y_2 h(y_1, y_2)$ where c is the coefficient of the monomial term corresponding to the vertex of the polygon that determines $m(b)$ and $m(a^1)$, $p(y_1)$ comes from the other monomials on the face for which b is the covector, and $y_2 h(y_1, y_2)$ comes from all the other monomials. Similarly f_{σ_2} has the form $d + p'(y_2') + y_1' h(y_1', y_2')$ where d is the coefficient of the monomial term that determines $m(b)$ and $m(a^2)$.

We first consider the integral for $Z_1(s)$. By an argument similar to that given in Theorem 1 we can reduce the domain of integration to $R \times P$ without affecting the residue. We then split this domain of integration into $(R-P) \times P$ and $P \times P$. On $P \times P$ we have that $|f_{\sigma_1}| = 1$, hence the integral over this domain is

$$(1-q^{-1})^2 \frac{q^{-(m(a^1)s+|a^1|)} q^{-(m(b)s+|b|)}}{\left(1-q^{-(m(a^1)s+|a^1|)}\right)\left(1-q^{-(m(b)s+|b|)}\right)}$$

which gives a contribution to the residue of

$$(1-q^{-1})^2 \frac{q^{-(m(a^1)s+|a^1|)}}{1-q^{-(m(a^1)s+|a^1|)}}. \tag{7}$$

In order to consider the domain of integration $(R-P) \times P$ we observe that condition 3 allows us to write $c + p(y_1) = \prod_{i=1}^r (a_i y_1 - b_i)$. By our assumptions we have a_i and b_i are in $R - P$ for all i by condition 2 and if we let $\lambda_i = b_i a_i^{-1}$ for $1 \leqslant i \leqslant r$, we have that the λ_i are distinct modulo P by condition 3. We write $R - P$ as a disjoint union of cosets modulo P so that the λ_i are in distinct cosets. Let $\{\lambda_1, \ldots, \lambda_r, \theta_1, \ldots, \theta_t\}$ denote a complete set of coset representatives modulo P in $R - P$.

We first consider a domain of integration of the form $(\theta_i + P) \times P$. Then $|y_1| = 1$ on $\theta_i + P$ and by replacing y_1 by $\theta_i + y_1$ we get an integral of the form

$$\int_P \int_P |y_2|^{m(b)s + |b| - 1}$$

$$\times \left| \prod_{j=1}^{r} \left(a_j(\theta_i + y_1) - b_j \right) + y_2 h(\theta_i + y_1, y_2) \right|^s |dy_1| \, |dy_2|.$$

Observing that $a_j \theta_i - b_j \notin P$ for any j we have that the integral over this domain is equal to

$$\int_P \int_P |y_2|^{m(b)s + |b| - 1} |dy_1| \, |dy_2|,$$

which gives a contribution to the residue of

$$(1 - q^{-1}) q^{-1}. \tag{8}$$

Now consider a domain of integration of the form $(\lambda_i + P) \times P$. Then $|y_1| = 1$ on this coset so by replacing y_1 by $\lambda_i + y_1$ we get an integral of the form

$$I(q^{-s}) = \int_P \int_P |y_2|^{m(b)s + |b| - 1}$$

$$\times \left| a_i y_1 \prod_{\substack{j=1 \\ j \neq i}}^{r} \left(a_j(\lambda_i + y_1) - b_j \right) + y_2 h(\lambda_i + y_1, y_2) \right|^s$$

$$\times |dy_1| \, |dy_2|.$$

We wish to calculate $\lim_{s \to -|b|/m(b)} (1 - q^{-(m(b)s + |b|)}) I(q^{-s})$. Let

$$I_n(q^{-s}) = \int_P \int_{P - P^n} |y_2|^{m(b)s + |b| - 1}$$

$$\times \left| a_i y_1 \prod_{j=1 \, j \neq i}^{r} \left(a_j(\lambda_i + y_1) - b_j \right) + y_2 h(\lambda_i + y_1, y_2) \right|^s$$

$$\times |dy_1| \, |dy_2|.$$

Since $(1 - q^{-(m(b)s + |b|)}) I_n(q^{-s})$ converges uniformly to $(1 -$

$q^{-(m(b)s+|b|)}I(q^{-s})$ as $n \to \infty$ in a neighborhood of $-|b|/m(b)$; we have that the contribution to the residue is equal to $\lim_{n\to\infty} \lim_{S\to -|b|/m(b)}(1 - q^{-(m(b)s+|b|)})I_n(q^{-s})$. By reasoning similar to before, we can reduce the domain of integration from $(P - P^n)\times P$ to $(P - P^n)\times P^n$ without affecting the contribution to the residue at $s = -|b|/m(b)$ of $I_n(q^{-s})$. Then, again observing that $a_j\lambda_i - b_j \notin P$ for any $j \neq i$ we have that the integral over this domain is equal to

$$\int_{P^n}\int_{P-P^n} |y_2|^{m(b)s+|b|-1}|y_1|^s|\mathrm{d}y_1||\mathrm{d}y_2|.$$

So $I_n(q^{-s})$ gives a residue contribution of

$$(1 - q^{-1})^2 \frac{q^{-(s+1)} - q^{-(s+1)n}}{1 - q^{-(s+1)}}.$$

So taking the limit as $n \to \infty$ we see that the contribution to the residue at $s = -|b|/m(b)$ from $I(q^{-s})$ is

$$(1 - q^{-1})^2 \frac{q^{-(s+1)}}{1 - q^{-(s+1)}}. \tag{9}$$

Now consider the integral $Z_2(s)$. Again by an argument similar to that given in Theorem 1 we can reduce the domain of integration to $P\times P$ without affecting the residue at $s = -|b|/m(b)$. Then $|f_{\sigma_2}| = 1$ on this domain so the integral over this domain is

$$(1 - q^{-1})^2 \frac{q^{-(m(a^2)s+|a^2|)}q^{-(m(b)s+|b|)}}{\left(1 - q^{-(m(a^2)s+|a^2|)}\right)\left(1 - q^{-(m(b)s+|b|)}\right)}$$

which gives a contribution to the residue of

$$(1 - q^{-1})^2 \frac{q^{-(m(a^2)s+|a^2|)}}{\left(1 - q^{-(m(a^2)s+|a^2|)}\right)} \tag{10}$$

for $s = -|b|/m(b)$.

Combining the contributions to the residue from (7), (8), (9) and (10) gives

$$(1 - q^{-1})^2\left(\frac{q^{\alpha_1}}{1 - q^{\alpha_1}} + \frac{q^{\alpha_2}}{1 - q^{\alpha_2}} + \frac{rq^{\alpha_3}}{1 - q^{\alpha_3}} + \frac{tq^{-1}}{1 - q^{-1}}\right)$$

where $\alpha_i = -(m(a')s + |a'|)$ for $i = 1, 2$; $\alpha_3 = -(s+1)$, and $s = -|b|/m(b)$.

If $q = p^f$, then we consider the above expression in the fully ramified extension of Q_p obtained by adjoining π where $\pi^{m(b)} = p$. We then consider the expansion of the above expression in integral powers of π. Let $\gamma_j = m(b)\alpha_j$ for $1 \leqslant j \leqslant 3$. We have

$$\frac{q^{\alpha_j}}{1 - q^{\alpha_j}} = \pi^{\gamma_j f} + \pi^{2\gamma_j f} + \dots$$

if $\alpha_j > 0$, and

$$\frac{q^{\alpha_j}}{1 - q^{\alpha_j}} = -1 - \pi^{-\gamma_j f} - \pi^{-2\gamma_j f} - \dots$$

if $\alpha_j < 0$. We observe that $\alpha_3 = |b| - m(b)$. Thus $\alpha_3 \geqslant 0$ only in the cases excluded in the statement of the theorem or the case covered by Theorem 2. Therefore we may assume $\alpha_3 < 0$. Also we observe that the proof of Corollary 1 to Proposition 3 shows that α_1 and α_2 have opposite sign. We write r in terms of its p-adic expansion $r = a_0 + a_1 p + \dots + a_{(f-1)} p^{(f-1)}$ where $0 \leqslant a_i \leqslant (p-1)$ and recall that $r + t = q - 1$.

Thus in the case where $\alpha_1 < 0$ and $\alpha_2 > 0$ the expression for the residue becomes

$$\begin{aligned}
&-(1 + \pi^{-\gamma_1 f} + \pi^{-2\gamma_1 f} + \dots) \\
&+ \pi^{\gamma_2 f} + \pi^{2\gamma_2 f} + \dots \\
&- \sum_{i=0}^{f-1} a_i \pi^{im(b)} (1 + \pi^{-\gamma_3 f} + \pi^{-2\gamma_3 f} + \dots) \\
&+ \left(1 + \sum_{i=0}^{f-1} a_i \pi^{im(b)} - \pi^{m(b)f}\right)(1 + \pi^{m(b)f} + \pi^{2m(b)f} + \dots).
\end{aligned}$$

By cancelling leading terms the above expression becomes

$$\begin{aligned}
&-(\pi^{-\gamma_1 f} + \pi^{-2\gamma_1 f} + \dots) \\
&+ \pi^{\gamma_2 f} + \pi^{2\gamma_2 f} + \dots \\
&- \sum_{i=0}^{f-1} a_i \pi^{im(b)} (\pi^{-\gamma_3 f} + \pi^{-2\gamma_3 f} + \dots) \\
&+ \sum_{i=0}^{f-1} a_i \pi^{im(b)} (\pi^{m(b)f} + \pi^{2m(b)f} + \dots).
\end{aligned}$$

By Corollary 1 to Proposition 3 we have that $-\gamma_3 = \gamma_1 + \gamma_2 + m(b) + (k-1)|b|$ for $k \geqslant 1$. The assumptions of the theorem assure us that by Corollary 2 to Proposition 3 we have $\gamma_1 > -m(b)$ and hence $-\gamma_3 > \gamma_2$. Corollary 1 to Proposition 3 also assures us that $\gamma_1 \neq -\gamma_2$ hence the lowest order term in the above expression is either $-\pi^{-\gamma_1 f}$ or $\pi^{\gamma_2 f}$. Therefore the residue is non-zero at $s = -|b|/m(b)$. The proof in the case where $\alpha_1 > 0$ and $\alpha_2 < 0$ is entirely similar. q.e.d.

We give an example to illustrate the possibility of an exceptional case where the residue is zero excluded by the above theorem. Let $f(x_1, x_2) = x_2^6 + x_1x_2^4 + x_1^3x_2^2 + x_1^6$. Then the vector $b = (2, 1)$ is dual to the face with vertice (0, 6). Take $a^1 = (1, 0)$ and $a^2 = (1, 1)$. Then we have a possible pole at $-1/2$ but in the above expression for the residue we have $r = 1$, $-\gamma_1 = m(b) = 6$, $-\gamma_3 = \gamma_2 = 3$ which gives a residue of zero.

5. Examples

Examples of curves satisfying the conditions in Theorem 2 can be easily found. We illustrate the preceding theory with one such class of examples.

Let $\{(p_i, q_i)\}_{i=1}^N$ be a set of pairs of relatively prime positive integers such that

$$\text{i)} \quad \frac{-q_N}{p_N} \leqslant \frac{-q_{N-1}}{p_{N-1}} \leqslant \dots \leqslant \frac{-q_1}{p_1}$$

$$\text{ii)} \quad q_N \geqslant q_{N-1} \geqslant \dots \geqslant q_1 \quad \text{and} \quad p_N \geqslant p_{N-1} \geqslant \dots \geqslant p_1.$$

Let $\{(\alpha_i, \beta_i)\}_{i=1}^N$ be a set of pairs of elements each belonging to $R - \{0\}$. Set $f_\Gamma(x_1, x_2) = \prod_{i=1}^N(\alpha_i x_1^{p_i} + \beta_i x_2^{q_i})$. Let $H(x_1, x_2)$ be any polynomial such that the polygon for H lies completely above the polygon Γ for f_Γ. Set $f = f_\Gamma + H$. We consider the curve defined by f.

The polygon Γ can be easily described for the above class of curves. Each distinct ratio $-q_i/p_i$ is associated to a face τ_i of the polygon having that ratio as its slope, and these comprise all of the faces of the polygon. The primitive covector $b^i = (q_i, p_i)$ is dual to the face τ_i. The number of integral points on each face τ_i is one more than the number of pairs (p_k, q_k) which are equal to the pair (p_i, q_i). The b^i determine a partition of $(\mathbb{R}_+^*)$ which we refine as indicated in Section 1.

Consider a fixed b^i and let a^1 and a^2 denote the vectors such that $\sigma_1 = \langle a^1, b^i \rangle$ and $\sigma_2 = \langle b^i, a^2 \rangle$ form unit cones for the partition where we assume $\det\begin{pmatrix} a^1 \\ b^i \end{pmatrix} = \det\begin{pmatrix} b^i \\ a^2 \end{pmatrix} = 1$. We consider the pullback of f_Γ in $K^2(\sigma_1)$. Let $i = i_1, \dots, i_t$ be such that $(p_{i_l}, q_{i_l}) = (p_i, q_i)$ for $1 \leqslant l \leqslant t$. Let

$j_1, \ldots, j_u$ denote those j such that $\det\begin{pmatrix} b^{j_l} \\ b^i \end{pmatrix} > 0$ for $1 \leqslant l \leqslant u$ and let $j_{u+1}, \ldots, j_v$ denote those j such that $\det\begin{pmatrix} b^{j_l} \\ b^i \end{pmatrix} < 0$ for $u < l \leqslant v$. Then

$$(f_\Gamma o\pi(\sigma_1))(y_1, y_2) = y_1^{m(a^1)} y_2^{m(b^i)} \times \prod_{l=1}^{t} (\alpha_{\iota_l} y_1 + \beta_{\iota_l})(c + y_2 h_{\sigma_1}(y_1, y_2))$$

where $c = \prod_{l=1}^{u} \alpha_{j_l} \prod_{l=u+1}^{v} \beta_{j_l}$. Similarly one can show that in $K^2(\sigma_2)$ we have

$$(f_\Gamma o\pi(\sigma_2))(y_1', y_2') = y_1'^{m(b^i)} y_2'^{m(a^2)} \times \prod_{l=1}^{t} (\alpha_{\iota_l} + \beta_{\iota_l} y_2')(c + y_1' h_{\sigma_2}(y_1', y_2'))$$

In order to consider a possible pole of $Z(s)$ at $s = -|b^i|/m(b^i)$ we impose Condition 1 of Theorem 2, i.e. that $-|b^i|/m(b^i) \neq -|b^j|/m(b^j)$, thus localizing the calculation of the residue to the divisor D_{b^i}. In the case where $t = 1$ we need to consider the contribution from the integrals

$$Z_1(s) = \int_R \int_R |y_1|^{m(a^1)s + |a^1| - 1} |y_2|^{m(b^i)s + |b^i| - 1} |\alpha_\iota y_1 + \beta_\iota|^s |c + y_2 h_{\sigma_1}(y_1, y_2)|^s |\mathrm{d}y_1| |\mathrm{d}y_2|$$

and

$$Z_2(s) = \int_P \int_R |y_1'|^{m(b^i)s + |b^i| - 1} |y_2'|^{m(a^2)s + |a^2| - 1} |\alpha_\iota + \beta_\iota y_1'|^s |c + y_1' h_{\sigma_2}(y_1', y_2')|^s |\mathrm{d}y_1'| |\mathrm{d}y_2'|$$

at $s = -|b^i|/m(b^i)$. These integrals have the exact form as those for irreducible curves which are shown to give a non-zero residue in Theorem 2 in [7].

In the cases where $t > 1$ it is necessary to examine

$$Z_1(s) = \int_R \int_R |y_1|^{m(a^1)s + |a^1| - 1} |y_2|^{m(b^i)s + |b^i| - 1} |p_1(y_1)|^s |c + y_2 h_{\sigma_1}(y_1, y_2)|^s |\mathrm{d}y_1| |\mathrm{d}y_2|$$

and

$$Z_2(s)=\int_P\int_R |y_1'|^{m(b_l)s+|b_l|-1}|y_2'|^{m(a^2)s+|a^2|-1}|p_2(y_1')|^s|c$$

$$+y_1'h_{\sigma_2}(y_1',y_2')|^s|dy_1'|\,|dy_2'|$$

where $p_1(y_1)$ and $p_2(y_1')$ are polynomials of degree >1 where the roots of $p_1(y_1)$ are $-\beta_{\iota_l}/\alpha_{\iota_l}$ and the roots of $p_2(y_1')$ are $-\alpha_{\iota_l}/\beta_{\iota_l}$ for $1\leqslant l\leqslant t$. The imposition of Condition 2 is the statement that the α_{ι_l} and β_{ι_l} should be in $R-P$ for $1\leqslant l\leqslant t$ and the imposition of Condition 3 is the statement that $\alpha_{\iota_k}/\beta_{\iota_k}\not\equiv\alpha_{\iota_l}/\beta_{\iota_l}$ modulo P for $k\neq l$ $1\leqslant k\leqslant t$, $1\leqslant l\leqslant t$. Under these restrictions Theorem 2 shows that the residue is non-zero at $s=-|b'|/m(b')$.

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Ben Lichtin
Harvard University
Cambridge, MA 02138
USA

Current address:
University of Rochester
Rochester, N.Y. 14620
U.S.A.

and

Diane Meuser
Boston University
Boston, MA 02215
USA

Current address:
Harvard University
Cambridge, MA 02138