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# SIZES OF QUOTIENT SPACES OF CERTAIN FUNCTION ALGEBRAS ON TOPOLOGICAL SEMIGROUPS

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## Introduction

Let $S$ be a locally compact topological semigroup, $C(S)$ the space of all bounded complex-valued continuous functions on $S$, $LWUC(S)$ the space of all left weakly uniformly continuous functions in $C(S)$ and $M_a(S)$ the convolution measure algebra of absolutely continuous bounded complex-valued Radon measures on $S$.

When $S$ is a closed subsemigroup of a locally compact topological group such that $S$ is neither compact nor discrete, we showed that the quotient space $C(S)/LWUC(S)$ is nonseparable in [9]. In this paper, we will extend this result to a more general class of topological semigroups.

For a locally compact topological group $G$, $M_a(G)$ can be identified with the usual group algebra, $L^1(G)$, of $G$–see e.g. Hewitt and Ross [13]. When $G$ is nondiscrete it is known that the quotient space $L^\infty(G)/C(G)$ and the radical of the Banach algebra $L^\infty(G)^*$ are nonseparable–see E.E. Granirer [10] and S.L. Gulick [12]. Motivated by these results, we will show that for a large class of nondiscrete topological semigroups $S$ we have $M_a(S)^*/C(S)$ and the radical of $M_a(S)^{**}$ nonseparable; the actual setting of our results being more general.

## Definitions and notations

Let $A$ and $B$ be any subsets of a semigroup $S$ and $x$ any element of $S$. We take $AB$, $A^{-1}B$, $x^{-1}B$ and $A^{-1}x$ to denote $\{ab\colon a \in A \text{ and } b \in B\}$, $\{y \in S\colon ay \in B \text{ for some } a \in A\}$, $\{x\}^{-1}B$ and $A^{-1}\{x\}$ (respectively). By symmetry the definitions of $BA^{-1}$, $Bx^{-1}$ and $xA^{-1}$ must be clear. By a *right cancellative semigroup* we mean a semigroup $S$ such that whenever $yx = zx$ then $y = z$, for all $x$, $y$ and $z$ in $S$.

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Throughout this paper, a semigroup, $S$, endowed with a Hausdorff topology with respect to which the semigroup operation $(x, y) \to xy$ is a jointly continuous mapping of $S \times S$ into $S$, is called a *topological semigroup*.

Let $S$ be a locally compact topological semigroup for the reminder of this section. For every function $f$ in $C(S)$ and $x$ in $S$, we define the functions ${}_xf$ and $f_x$ in $C(S)$ by

$$ {}_xf(y) := f(xy) \text{ and } f_x(y) := f(yx)(y \in S). $$

Let

$$ LWUC(S) := \{ f \in C(S) \text{: the map } x \to {}_xf \text{ of } S \text{ into } C(S) \text{ is weakly continuous} \}, $$

$$ WAP(S) := \{ f \in C(S) \text{: the set } \{ {}_xf \text{: } x \in S \} \text{ is relatively weakly compact} \}, $$

$$ AP(S) := \{ f \in C(S) \text{: the set } \{ {}_xf \text{: } x \in S \} \text{ is relatively norm compact} \}. $$

These spaces of functions have been studied widely - see e.g. [4] and [5]. If $A$ is a subset of $C(S)$ and $E$ of $S$ we write $A_{|E} := \{ f_{|E} \text{: } f \in A \}$ where $f_{|E}$ denotes the restriction of a function $f$ to $E$.

Let $M(S)$ be the set of all bounded complex-valued Radon measures on $S$. It is well known that $M(S)$ is a Banach algebra with respect to the usual total variation norm, $\| \, \|$,, and convolution multiplication given by

$$ \nu * \mu(E) := \int \mu(x^{-1}E) \mathrm{d}\nu(x) = \int \nu(Ex^{-1}) \mathrm{d}\mu(x), $$

for all Borel subsets $E$ of $S$ and measures $\nu$, $\mu$ in $M(S)$. For each $\mu$ in $M(S)$ and $x$ in $S$ we take $|\mu|$ to be the Radon measure arising from the total variation of $\mu$ and $\tilde{x}$ the point mass at $x$. Let $M_a(S) := \{ \mu \in M(S)$: the maps $x \to |\mu|(x^{-1}(C)$ and $x \to |\nu|(Cx^{-1})$ of $S$ into $\mathbb{R}$ are continuous, for every compact subset $C$ of $S \}$

The set $M_a(S)$ has been studied in many publications; for $S$, it plays a role analogous to that of $L^1(G)$ for a locally compact topological group $G$ –see e.g. [1], [2], [15] and [16]. In particular, we have the following result proved in [1] and [2]

THEOREM 1: *We have* $M_a(S)$ *an L-ideal of* $M(S)$ *(–that is:* $M_a(S)$ *is a norm-closed subalgebra of* $M(S)$ *such that for all* $\mu \in M(S)$ *and* $\nu \in M_a(S)$ *we have* $\nu^*\mu$, $\mu^*\nu \in M_a(S)$ *and if* $\mu \ll |\nu|$ *then* $\mu \in M_a(S)$*).*

For each $\mu \in M(S)$, let $\operatorname{supp}(\mu) := \{x \in S$: if $V$ is an open neighbourhood of $x$ then $|\mu|(V) > 0\}$.

Following A.C. and J.W. Baker we say $S$ is a *foundation semigroup* if $S$ coincides with the closure of $U\{\operatorname{supp}(\mu)\colon \mu \in M_a(S)\}$.

For ease of reference we quote the following result proved by Sleijpen [15].

THEOREM 2: *Let* $S$ *be a foundation semigroup with identity element* 1 *and* $S_1 := (x \in S\colon 1 \in \operatorname{int}(X^{-1}x \cap xX^{-1})$ *whenever* $X$ *is a neighbourhood of* $x\}$*. Then* $S_1$ *is dense in* $S$ *and if* $V$ *is an open neighbourhood in* $S$ *then* $Vv^{-1}$ *is a neighbourhood of* 1*, for all* $v \in V \cap S_1$*.*

## The main results

Our next theorem is a generalization, to a larger class of semigroups, of a result we proved before–see [9, Theorem 2.5}. The proof employed contains a mixture of techniques we employed in [9] and those used in the proof of Baker and Butcher [3, Theorem 3]. The proof we give is also much simpler compared with that in [9].

THEOREM 3: *Let* $S$ *be a normal, locally compact and right cancellative topological semigroup. Suppose* $S$ *is neither countably compact nor discrete and* $C^{-1}D$ *is compact for all compact subsets* $C$ *and* $D$ *of* $S$*. Then, for some closed subset* $\overline{X}$ *of* $S$ *we have that* $(C(S)\backslash LWUC(S))_{|\overline{X}}$ *contains a linear isometric copy of* $l^\infty$ *and so the quotient space* $C(S)/LWUC(S)$ *is nonseparable.*

PROOF. Since $S$ is nondiscrete, we can find a relatively compact infinite set $\{s_n\colon n \in \mathbb{N}\}$ in $S$. As $C := cl(\{s_n\colon n \in \mathbb{N}\})$ is compact, we can choose a sequence $\{t_n\}$ in $S$ with no cluster point and such that

$$t_{n+1} \notin \bigcup_{i=1}^{n} C^{-1}(Ct_i) \qquad \text{for all } n \in \mathbb{N} \tag{1}$$

Choose infinite subsequences $T_k := \{t_{k1}, t_{k2}, \dots\}$ of $T := \{t_1, t_2, \dots\}$ such that

$$\bigcup_{k=1}^{\infty} T_k = T$$

and

$$T_k \cap T_{k'} = \emptyset$$

if and only if $k \neq k'$.

Let $X_k := \{s_m t_{k_n}: m, n \in \mathbb{N}\}$, $X := \{s_m t_n: m, n \in \mathbb{N}\}$ and note that our construction of the $T_k$'s and $T$ imply

(a) $\overline{X}_k = \{ct_{k_n}: c \in C \text{ and } n \in \mathbb{N}\}$ (–see proof of [3, Theorem 3]),

(b) $\overline{X}_k \cap \overline{X}_{k'} = \emptyset$ if and only if $k \neq k'$,

(c) $\bigcup_{k=1}^{\infty} \overline{X}_k = \overline{X}$.

Next we define the functions $f_k: \overline{X}_k \to \mathbb{R}$ by

$$f_k(s_m t_{k_n}) := \begin{cases} 1 & \text{if} \quad m < n \\ -1 & \text{if} \quad m \geqslant n \end{cases}$$

$$f_k(ct_{k_n}) := -1 \quad \text{if } c \in C \setminus \{s_m: m \in \mathbb{N}\}.$$

then (as similarly shown in [3, page 105],) $f_k$ is continuous, for all $k$ in $\mathbb{N}$.

Corresponding to each element $\{d_{k'}\}$ in $l^\infty$, let $F_{(d_{k'})}$ be the function defined on $\overline{X}$ by

$$F_{(d_{k'})}(x) := d_l f_l(x)$$

if and only if $x \in \overline{X}_l$ for some $l \in \mathbb{N}$.

By items (b) and (c) we have $F_{(d_{k'})}$ well-defined as a function. From items (b) and (c) we have that each $\overline{X}_k$ is both closed and open in the space $\overline{X}$. Consequently $F_{(d_{k'})}$ is continuous, by the continuity of the $f_l$'s.

Now noting that

$$(*) \quad F_{(d_{k'})}(s_m t_{k_n}) = \begin{cases} d_k & \text{if } m < n \\ -d_k & \text{if } m \geqslant n, \end{cases}$$

[3, Theorem 5] and Tietze's Extension Theorem imply the existence of a function $\overline{F}_{(d_{k'})}$ in $C(S) \setminus LWUC(S)$ such that

$$\overline{F}_{(d_{k'})|X} = F_{(d_{k'})} \text{ and } \| \overline{F}_{(d_{k'})} \|_S = \| F_{(d_{k'})} \|_{\overline{X}} = \| \{d_{k'}\} \|_\infty.$$

Thus the (clearly) linear map $\{d_{k'}\} \to \overline{F}_{(d_{k'})|X}$ of $l^{\infty}$ into $(C(S)\backslash LWUC(s))_{|\overline{X}}$ is isometric.

Since $l^{\infty}$ is nonseparable, it follows that $C(S)\backslash LWUC(S)$ and hence the quotient space $C(S)/LWUC(S)$ is nonseparable.

For our next results, recall that the norm of $M_a(S)^*$ is given by

$$\|h\|_{M_a(S)^*} := \sup\{\,|h(\nu)|:\ \nu \in M_a(S) \text{ with } \|\nu\| = 1\}.$$

For a locally compact topological group $G$, $M_a(G)^*$ is simply $L^{\infty}(G)$.

THEOREM 4: *Let $S$ be a nondiscrete and right cancellative foundation semigroup with an identity element* 1. *Then the quotient spaces* $M_a(S)^*/C(S)$ and $M_a(S)^*/LWUC(S)$ *contain isometric linear copies of* $l^{\infty}$.

PROOF. Let $W$ be a compact neighbourhood of 1 and corresponding to each function $g$ in $C(S)$ let $G$ be the function in $C(W \times W)$ given by

$$G(x, y) := g(xy) \qquad \text{for all } x,\ y \in W.$$

Then a simple compactness argument shows that the set $\{G(x,.): x \in W\}$ is relatively (norm and hence) weakly compact in $C(W)$. (Here each $G(x,.)$ is given by $G(x,.)(y) := G(x, y)$ for all $x$, $y$ in $W$ and $C(X)$ denotes the space of all bounded complex-valued continuous functions on a topological space $X$.)

Since $S$ is not discrete, 1 is not isolated and so we can find a sequence $\{V_k\}$ of disjoint open neighbourhoods contained in $W$. Choose $v_k \in V_k \cap S_1$ and note that $V_k v_k^{-1}$ is a neighbourhood of 1, by Theorem 2. So there is a sequence $\{U_k\}$ of open neighbourhoods of 1 such that

$$U_k^2 \subset V_k v_k^{-1} \qquad \text{for all } k \text{ in } \mathbb{N}.$$

By [8, Lemma 4.2] we can choose sequences $\{C_{k_1}, C_{k_2}, \dots\}$ and $\{D_{k_1}, D_{k_2}, \dots\}$ of non-$M_a(S)$-negligible compact subsets of $U_k$ such that, for all $n$, $m$, $i$ and $j$ in $\mathbb{N}$,

$$C_{k_n} D_{k_m} \cap C_{k_i} D_{k_j} = \emptyset \quad \text{whenever } n < m \text{ and } i > j.$$

By right cancellation we have

$$C_{k_n} D_{k_m} v_k \cap C_{k_i} D_{k_j} v_k = \emptyset \quad \text{whenever } n < m \text{ and } i > j. \tag{1}$$

In the notation of [8, page 166], take $A = M_a(S)$ and set $d_a := d_{M_a(S)}$. We can choose sequences of points $\{c_{k_n}\}$ and $\{e_{k_n}\}$ such that

$$c_{k_n} \in d_a(C_{k_n}) \quad \text{and} \quad e_{k_n} \in d_a(D_{k_n} v_k). \tag{2}$$

Let

$$E_k := \bigcup_{j=1}^{\infty} \bigcup_{i<j} C_{k_i} D_{k_j} v_k \quad \text{and} \quad F_k := \bigcup_{j=1}^{\infty} \bigcup_{i>j} C_{k_i} D_{k_j} v_k .$$

We define the function $h_k$ on $S$ by

$$h_k := X_{E_k} - X_{F_k},$$

where $X_A$ denotes the characteristic function of a subset $A$ of $S$. Since $E_k$ and $F_k$ are disjoint $\sigma$-compact subsets of $S$, we also have that $h_k$ is a functional in $M_a(S)^*$ (where $h_k(\nu) := \int h_k(x)\mathrm{d}\nu(x)$, for all $\nu$ in $M_a(S)$).

We claim that, in the norm of $M_a(S)^*$,

$$\| h_k + g \|_{M_a(S)} \geqslant 1 \quad \text{for all } g \text{ in } C(S) \tag{3}$$

If not, then for some (real-valued) function $g$ in $C(S)$ we can find $\epsilon > 0$ such that

$$\| h_k + g \|_{M_a(S)} \leqslant 1 - \epsilon .$$

In particular, for positive measures $\nu_\alpha$, $\mu_\beta$ in $M_a(S)$ such that $\| \nu_\alpha \| = \| \mu_\beta \| = 1$, $\text{supp}(\nu_\alpha) \subseteq C_{k_n}$ and $\text{supp}(\mu_\beta) \subset D_{k_m} v_k$, we have

$$| h_k(\nu_\alpha^* \mu_\beta) + g(\nu_\alpha^* \mu_\beta) | \leqslant 1 - \epsilon . \tag{4}$$

Recalling our definition of $h_k$, (4) implies that

$$\begin{cases} \text{if } n < m \text{ then } |1 + g(\nu_\alpha^* \mu_\beta)| \leqslant 1 - \epsilon \\ \text{if } n > m \text{ then } |-1 + g(\nu_\alpha^* \mu_\beta)| \leqslant 1 - \epsilon . \end{cases}$$

Letting the net $(\nu_\alpha)$ converge in the weak*-topology to $\bar{c}_{k_n}$ and $(\mu_\beta)$ to $\bar{e}_{k_m}$ we thus get that (, since $g$ is continuous on $W$),

$$\begin{cases} \text{if } n < m \text{ then } |1 + g(c_{k_n} e_{k_m})| \leqslant 1 - \epsilon \\ \text{if } n > m \text{ then } |-1 + g(c_{k_n} e_{k_m})| \leqslant 1 - \epsilon . \end{cases}$$

It follows that

$$G(c_{k_n}, e_{k_m}) = g(c_{k_n} e_{k_m}) \begin{cases} < -\epsilon & \text{if } n < m \\ > \epsilon & \text{if } n > m \end{cases}$$

and so $G(x,.)$: $x \in W\}$ is not relatively weakly compact, by Grothendieck's Theorem [11]. This contradicts the observation at the beginning of our proof. By this conflict, claim (3) holds.

Since the $V_k$'s are pairwise disjoint and $C_{k_n} D_{k_m} v_k \subset V_k$ for all $n$, $m$ and $k$ in ℕ, we have that

$$\{t_k\} \to \sum_{k=1}^{\infty} t_k h_k + C(S)$$

defines a linear mapping of $l^\infty$ into $M_a(S)^*/C(S)$. Noting that $\| h_k \|_{M_a(S)} = 1$, item (3) implies that

$$\| \{t_k\} \|_\infty = \| \sum_{k=1}^{\infty} t_k h_k + C(S) \|_{M_a(S)^*/C(S)}$$

and so the mapping $\{t_k\} \to \sum_{k=1}^{\infty} t_k h_k + C(S)$ is isometric.

Similarly the mapping $\{t_k\} \to \sum_{k=1}^{\infty} t_k h_k + LWUC(S)$ of $l^\infty$ into $M_a(S)^*/LWUC(S)$ is linear and isometric. This completes our proof. (The idea of embedding $l^\infty$ used here is inspired by [6] and [9].)

The second dual of $M_a(S)$, namely $M_a(S)^{**}$, can be turned into a Banach algebra with Arens product $\odot$ defined as follows: For $\phi \in M_a(S)^{**}$, $h \in M_a(S)^*$ and $\nu \in M_a(S)$ we define $\nu \odot h$, $h \odot \nu$ and $h^\phi$ in $M_a(S)^*$ by

$$\nu \odot h(\mu) := h(\nu * \mu), \quad h \odot \nu(\mu) := h(\mu * \nu) \quad \text{and}$$

$$h^\phi(\mu) := \phi(\mu \odot h) \quad \text{for all } \mu \text{ in } M_a(S).$$

For all $\phi$, $\psi$ in $M_a(S)^{**}$ we have $\phi \odot \psi$ given by

$$\phi \odot \psi(h) := \phi(h^\psi).$$

$R_a(S)$, the radical of $M_a(S)^{**}$, is the intersection of all maximal modular left (or right) ideals (See [14] page 55).

Let $G$ be a locally compact topological group. When $G$ is nondiscrete and abelian, Civin and Yood [7] showed that $R_a(G)$ is infinite dimensional and later $S$. Gulick [12] showed that $R_a(G)$ is even nonseparable.

In [10], Granirer showed that $R_a(G)$ is nonseparable whenever $G$ is nondiscrete or $G$ is discrete and amenable. We generalize the former result to the semigroup situation.

THEOREM. *Let $S$ be a nondiscrete and right cancellative foundation semigroup with an identity element. Then there exists a subspace $P$ of $M_a(S)^*$ such that $P^*$ is a linear isometric copy of $(l^\infty)^*$ and the restriction of the radical of $M_a(S)^{**}$ to $P$ is $P^*$. In particular the radical of $M_a(S)^{**}$ is nonseparable.*

PROOF. (c.f. [10] for a related proof in the group case.) Let

$$A := \{\phi \in M_a(S)^{**}: \phi(f) = 0 \quad \text{for all } f \text{ in } LWUC(S).$$

For all $\psi \in M_a(S)^{**}$, $\nu \in M_a(S)$ and $h \in M_a(S)^*$, we have that $\nu \odot h \in$ LWUC(S), by the left handed version of [9, Lemma 4.1]; consequently $h^\phi(\nu) := \phi(\nu \odot h) = 0(\phi \in A)$ and so

$$\phi \odot \psi(h) := \psi(h^\phi) = 0(\phi \in A).$$

Thus $A$ is a right ideal of $M_a(S)^{**}$ such that

$$A \odot M_a(S)^{**} = \{0\}.$$

Hence $A \subset R_a(S)$–see e.g. Richart [14, Theorem 2.3.5(ii)].

Now by Theorem 4, there exists an isometric linear map $\Pi$ of $l^\infty$ $M_a(S)^*/LWUC(S)$. So for some closed subspace $P$ of $M_a(S)^*$, we have $\Pi(l^\infty)$ dense in $P/LWUC(S)$. The inverse map $\Pi^{-1}$ therefore extends to a unique isometric linear map $\tau: P/LWUC(S) \to l^\infty$. Hence the dual $\tau^*$: $(l^\infty)^* \to (P/LWUC(S))^*$ is an isometric linear map that is onto. But $A = LWUC(S)^\perp \subset M_a(S)^{**}$ can be identified with $(M_a(S)^*/LWUC(S))^*$. Hence each element of $(P/LWUC(S))^*$ can be identified with the restriction of some element of $A$ to $P$. This completes our proof.

The case for a cancellative discrete topological semigroup $S$ that is amenable can be similarly handled as in the equivalent group case -see [10, page 323]. To what extent one can drop the right cancellation requirement on $S$, in Theorems 4 and 5, remains an open problem.

We proved related results for other spaces of functions in [9]. In particular we showed that if $S$ is a $C$-distinguished topological semigroup such that $M_a(S)$ is nonzero and $S$ is not relatively neo-compact, then $WUC(S)/WAP(S)$ contains an isometric linear copy of $l^\infty$. (See [9] for definition of terms.) There seems to be some relationship between sizes of quotient spaces and the existence of continuous projections. We have the following conjecture.

CONJECTURE: *Let $S$ be as stated in the preceding paragraph. Then there does not exist bounded linear projections from $WUC(S)$ onto $WAP(S)$ or from $WUC(S)$ onto $AP(S)$.*

When $S = \mathbb{R}$ – the usual additive group of reals with usual topology then this conjecture is true–see e.g. [17]. we are indebted to Professor W.G. Bade for drawing our attention to the reference [17].

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