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ABSOLUTELY EXTREMAL POINTS IN MINIMAL FLOWS

S. Glasner

Abstract

A point of a minimal flow is called absolutely extremal if it is an extreme point in every affine embedding. We show that distal points are absolutely extremal and give an example of a weakly mixing minimal flow all of whose points are absolutely extremal.

§1. Introduction

Let (X, T) be a metric flow i.e. X is compact metric and T a homeomorphism of X onto itself. A flow (Q, T) is called *affine* if Q is a convex compact subset of a locally convex linear space and T is an affine homeomorphism of Q . We say that a map $\varphi: (X, T) \rightarrow (Q, T)$ is an *affine embedding* if φ is continuous, one to one, equivariant and $\overline{c\phi(\varphi(X))} = Q$. Thus for any embedding, the set $ex(Q)$ of extreme points of Q is contained in $\varphi(X)$. We often identify X with $\varphi(X)$. If (X, T) is a minimal flow (every orbit is dense), then $(\varphi(X), T)$ is minimal and we have $\overline{ex(Q)} = \varphi(X)$. Call a point $x_0 \in X$ *absolutely extremal* if for every embedding $\varphi: X \rightarrow Q$, $\varphi(x_0)$ is an extreme point of Q . (X, T) is an absolutely extremal flow if every point of X is absolutely extremal.

Using this terminology a theorem of I. Namioka [3] asserts that a distal minimal flow is absolutely extremal. We simplify the proof of this theorem and generalize it in showing that every distal point of a minimal flow is absolutely extremal. We give examples of almost automorphic absolutely extremal flows (X, T) and (Y, T) such that in the product flow $(X \times Y, T)$ which is minimal and almost automorphic, there are points which are not absolutely extremal. Finally using a result of del Junco and Keane we demonstrate the existence of a weakly mixing minimal absolutely extremal flow.

§2. Distal points are absolutely extremal

For a flow (X, T) denote by $\mathcal{P}(X)$ the space of probability measures on X equipped with the weak* topology. The homeomorphism T of X

induces an affine homeomorphism of $\mathcal{P}(X)$ under which $(\mathcal{P}(X), T)$ is an affine flow. For $x \in X$, let δ_x be the point mass at x . If $\varphi: X \rightarrow Q$ is an affine embedding then the barycenter map $\beta: \mathcal{P}(X) \rightarrow Q$ is defined by

$$\beta(\lambda) = \int_X \varphi(x) d\lambda(x) \quad (\lambda \in \mathcal{P}(X)).$$

The map β is a continuous affine homomorphism (i.e. an equivariant map) of $\mathcal{P}(X)$ onto Q . A point $q \in Q$ is extremal iff $\beta^{-1}(q) = \{\delta_x\}$ for some $x \in X$ (with $\varphi(x) = q$).

LEMMA 2.1: *Let (X, T) be a metric flow, ν a probability measure on X and assume that in $\mathcal{P}(X)$, $\lim T^{n_i} \nu = \delta_{x_0}$ for some sequence $\{n_i\}$ and a point $x_0 \in X$. Then there exist an F_σ subset A of X and a subsequence $\{n'_i\}$ such that $\nu(A) = 1$ and for every $x \in A$, $\lim T^{n'_i} x = x_0$. In particular every two points of A are proximal.*

PROOF: Let $n'_i \in \{n_j\}$ satisfy $T^{n'_i} \nu(B_{1/i}(x_0)) > 1 - 2^{-(i+1)}$ ($i = 1, 2, \dots$), where $B_r(x_0)$ is the closed ball of radius r around x_0 . Then

$$\nu \left(\bigcap_{i=k}^{\infty} T^{-n'_i} B_{1/i}(x_0) \right) > 1 - 2^{-k}$$

and thus

$$A = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} T^{-n'_i} B_{1/i}(x_0)$$

is the required set.

COROLLARY 2.2: *If (X, T) is a distal flow and $\lim T^{n_i} \nu = \delta_{x_0}$ for some $x_0 \in X$ and $\nu \in \mathcal{P}(X)$, then $\nu = \delta_x$ for some $x \in X$.*

THEOREM 2.3: *Let (X, T) be a metric minimal flow, $x_0 \in X$ a distal point then x_0 is an absolutely extremal point.*

PROOF: Let $\varphi: X \rightarrow Q$ be an affine embedding; we identify X with $\varphi(X)$. Let $\beta: \mathcal{P}(X) \rightarrow Q$ be the barycenter map. If x_0 is not an extreme point of Q then there exists a measure $\nu \in \mathcal{P}(X)$ with $\beta(\nu) = x_0$ and $\nu \neq \delta_{x_0}$. Taking $1/2(\delta_{x_0} + \nu)$ instead of ν we can assume ν has an atom at x_0 . Choose $\bar{x} \in X$ which is extreme in Q ; then by minimality there exists a sequence $\{n_i\}$ such that $\lim T^{n_i} x_0 = \bar{x}$. We can also assume that

$\lim T^{n_i} \nu = \bar{\nu}$ exists and then

$$\beta(\bar{\nu}) = \lim \beta(T^{n_i} \nu) = \lim T^{n_i} \beta(\nu) = \lim T^{n_i} x_0 = \bar{x}.$$

Thus $\bar{\nu} = \delta_{\bar{x}}$ and $\lim T^{n_i} \nu = \delta_{\bar{x}}$. By lemma 2.1. there exist an F_σ subset A of X with $\nu(A) = 1$ - hence $x_0 \in A$ - and a subsequence $\{n'_i\}$ such that $\lim T^{n'_i} x = x_0$ for every $x \in A$. In particular every point of A is proximal to x_0 . Since x_0 is distal $A = \{x_0\}$ and $\nu = \delta_{x_0}$, a contradiction. Thus x_0 is extreme and the proof is completed.

§3. Almost automorphic examples

Let $I = [0, 1)$, α an irrational number and (I, R_α) the rotation by α on I . Let $x_0(n) = \text{sgn} \cos(2\pi n\alpha)$; consider x_0 as an element of $\{1, -1\}^{\mathbb{Z}} = \Omega$ and let T be the shift on Ω . Set $X = \bar{\mathcal{O}}(x_0)$, the orbit closure of x_0 in Ω ; then (X, T) is an almost automorphic minimal flow. There exists a homomorphism $\pi : (X, T) \rightarrow (I, R_\alpha)$ such that for $\xi \in I \setminus E$, where $E = \{1/4 + n\alpha, 3/4 + n\alpha : n \in \mathbb{Z}\}$, $\pi^{-1}(\xi) = \{\text{sgn} \cos(2\pi(n\alpha + \xi))\}$ and for $\xi = 1/4 + k\alpha$, $\pi^{-1}(\xi) = \{x^+, x^-\}$ where for $n \neq -k$, $x^+(n) = x^-(n) = \text{sgn} \cos(2\pi(n+k)\alpha)$ and $x^+(-k) = 1$, $x^-(-k) = -1$. Similar situation exists for $\pi^{-1}(\xi)$ where $\xi = 3/4 + n\alpha$. Since every point of $\pi^{-1}(I \setminus E)$ is distal we conclude, by theorem 2.3 that these points are absolutely extremal. However it is clear that the points of the form x^\pm are also absolutely extremal. For if, say $\beta(\nu) = x^+$ in some affine embedding then by the proof of theorem 2.3. ν is supported by the proximal cell of x^+ (i.e. the set $\{y \in X : y \text{ is proximal to } x^+\}$) which in our case is the set $\{x^+, x^-\}$. Clearly then $\nu = \delta_{x^+}$ and x is extremal. We have shown that every point of X is absolutely extremal. Next let Y be the orbit closure of y_0 in Ω where $y_0(n) = \text{sgn} \cos(2\pi n\gamma)$ and γ is an irrational number independent of α . Put $z_0(n) = x_0(n) + \frac{1}{10}y_0(n)$ and set $Z = \bar{\mathcal{O}}(z_0) \subset \mathbb{R}^{\mathbb{Z}}$. Then (X, T) and (Y, T) are disjoint minimal flows and the minimal flow $(X \times Y, T)$ is isomorphic to (Z, T) . Now $\mathbb{R}^{\mathbb{Z}}$ with its product topology is a locally convex linear space and, denoting $Q = \overline{\text{co}}(Z) \subset \mathbb{R}^{\mathbb{Z}}$, we see that the inclusion map of Z into Q is an embedding of (Z, T) into the affine flow (Q, T) . Let x^\pm, y^\pm be the points in X and Y respectively, which lie over $1/4 \in I$. Then e.g.

$$x^+ + \frac{1}{10}y^- = \frac{10}{11}(x^+ + \frac{1}{10}y^+) + \frac{1}{11}(x^- + \frac{1}{10}y^-)$$

and we conclude that although x^+ and y^- are absolutely extremal, the point (x^+, y^-) of the minimal flow $X \times Y$ is not absolutely extremal. Thus the property of being an absolutely extremal point is not preserved under products.

Taking $x_0(n) = f(\cos 2\pi n\alpha)$ where f is continuous on $[-1, 1]$ except for at zero where it has say, $[0, 1]$ as a limit set, the embedding

$X = \overline{\mathcal{O}}(x_0) \rightarrow \overline{co}(X) \subset \mathbb{R}^Z$ will yield a continuum of non absolutely extremal points of X .

If (X, T) is an almost automorphic minimal flow, there exist an almost periodic flow (Y, T) and an almost one to one homomorphism $\pi: X \rightarrow Y$. Clearly every pair $(x, x') \in X \times X$ with $\pi(x) = \pi(x')$ has the property that the only minimal set in $\mathcal{O}(x, x')$ is the diagonal Δ . For a general minimal flow (X, T) put

$$L = \{(x, x') \in X \times X: \Delta \text{ is the unique minimal set in } \overline{\mathcal{O}}(x, x')\}.$$

L is an invariant equivalence relation but not necessarily closed. We shall use the following lemma in the next section.

LEMMA 3.1: *Let (X, T) be a minimal flow and $\varphi: X \rightarrow Q$ an affine embedding. Suppose $\beta(\nu) = \varphi(x)$ for some $\nu \in \mathcal{P}(X)$ and $x \in X$. If y is an atom of ν then $(x, y) \in L$.*

PROOF: Since $\beta(1/2(\nu + \delta_x)) = \varphi(x)$ we can assume that x itself is an atom of ν . Write $\nu = a\delta_x + b\delta_y + (1 - (a + b))\theta$ where $0 < a, b < 1$ and $\theta \in \mathcal{P}(X)$. If $M \subset \overline{\mathcal{O}}(x, y)$ is a minimal set then, since X is minimal, there is a point $(z, w) \in M$ with $\varphi(z) \in ex(Q)$. Let $\{n_i\}$ be a sequence with $\lim T^{n_i}(x, y) = (z, w)$ and we can assume that $\lim T^{n_i}\theta = \tilde{\theta}$ and $\lim T^{n_i}\nu = \tilde{\nu}$ exist. Then

$$\beta(\tilde{\nu}) = a\beta(\delta_z) + b\beta(\delta_w) + (1 - (a + b))\beta(\tilde{\theta}) = \varphi(z),$$

whence $\delta_z = \delta_w = \tilde{\theta}$. Thus $M = \Delta$ and $(x, y) \in L$.

§4. A weakly mixing example

In this section we demonstrate the existence of a minimal absolutely extremal weakly mixing flow. Recall that a minimal flow (X, T) is P.O.D. if it is weakly mixing and for every $x, y \in X, x \neq y$ there exists some $n \neq 0$ with $T^n y$ and x proximal. Every P.O.D flow is prime [1]. Let (X, T) be P.O.D. and suppose $\varphi: X \rightarrow Q$ is an affine embedding of X and assume $X \subset Q$. Let $\bar{\nu} \in \mathcal{P}(X)$ and $x_0 \in X$ with $\beta(\bar{\nu}) = x_0$. If y is an atom of $\bar{\nu}$ then by lemma 3.1. $(x_0, y) \in L$. However it is easy to see that in a P.O.D. flow $L = \Delta$, so that $x_0 = y$ and $\bar{\nu}$ can have an atom only at x_0 . If $\bar{\nu} \neq \delta_{x_0}$ then there exists a measure $\nu \in \mathcal{P}(X)$ which is continuous (has no atoms) and for which $\beta(\nu) = x_0$.

Suppose further now that (X, T) is also strictly ergodic with an invariant measure μ and that for every $x \in X$ the set,

$$F_x = \{y \in X: (x, y) \text{ is not generic for } \mu \times \mu\},$$

is countable. We show that every point of X is absolutely extremal.

That such a flow exists is a result of A. del Junco and M. Keane [4]. They showed that the Chacon transformation which is P.O.D. [2] and strictly ergodic, has also the latter property. Set

$$F = \{(x, y) : (x, y) \text{ is not generic for } \mu \times \mu\},$$

then

$$\nu \times \nu(F) = \int 1_F d\nu \times \nu = \int \int 1_{F_x}(y) d\nu(y) d\nu(x) = 0,$$

since ν is continuous and for every x , F_x is countable.

If $f(x, y)$ is a continuous function on $X \times X$ then for $(x, y) \notin F$ we have

$$\frac{1}{2N+1} \sum_{j=-N}^N f(T^j x, T^j y) \rightarrow \int f d\mu \times \mu.$$

In particular this convergence holds $\nu \times \nu$ a.e. and integrating we get

$$\frac{1}{2N+1} \sum_{j=-N}^N \int \int f(T^j x, T^j y) d\nu(x) d\nu(y) \rightarrow \int f d\mu \times \mu.$$

Now let g be a continuous affine function on X (i.e. g is the restriction of an affine function on Q). Then for $f(x, y) = g(x)g(y)$ we have

$$\begin{aligned} \left(\int g d\mu \right)^2 &\leftarrow \frac{1}{2N+1} \sum_{j=-N}^N \int \int g(T^j x) g(T^j y) d\nu(x) d\nu(y) \\ &= \frac{1}{2N+1} \sum_{j=-N}^N \left(\int g(T^j x) d\nu(x) \right)^2. \end{aligned}$$

Since g is affine so is $g \circ T^j$ and recalling our assumption that $\beta(\nu) = x_0$ we see that $\int g(T^j x) d\nu(x) = g(T^j x_0)$. Thus the right hand side of the above equation equals $1/(2N+1) \sum_{j=-N}^N (g(T^j x_0))^2$. By strict ergodicity this tends to $\int g^2 d\mu$ so that

$$\int g^2 d\mu = \left(\int g d\mu \right)^2.$$

Choosing $g \neq 0$ with $\int g d\mu = 0$ we get a contradiction. Thus x_0 is extremal in Q and the proof is completed.

PROBLEMS:

- (1) Is there a minimal flow no point of which is absolutely extremal?
- (2) Is every minimal flow with $L = \Delta$, absolutely extremal?
- (3) Is the homomorphic image of an absolutely extremal point an absolutely extremal point?

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Added in proof: This problem has been solved by D. Maon and S. Glasner; see “On absolutely extremal points”, to appear in this journal.