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CURVES OF g_d^1's

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Let C be a smooth complete (irreducible) curve of odd genus $2n+1$. We will assume that C is general enough in the sense of moduli to have the following properties. By the Brill-Noether property [1] we may assume that there is a complete curve $X \subset P_{n+2}$ consisting of all linear systems of degree $n+2$ and dimension 1. Furthermore by this property we may assume that none of these linear systems has any base points. By the Petri property [3], we may assume that the curve X is smooth. By the connectedness theorem [2] of Fulton-Lazarfeld X is also connected. We intended to compute the genus of the irreducible curve X in the Picard variety P_{n+2}.

Consider the locus S in the symmetric product $C^{(n+2)}$ of the effective divisors in the linear systems of X. Then S is a smooth surface which is a locally trivial $\mathbb{P}^1$-bundle over X. Let c be a fixed point of C. Consider the set $Y = S \cap C^{(n+1)} + c$. Then Y is a divisor on S which intersects any fiber over X in one point. Hence Y projects isomorphically onto X. We will compute the genus of Y as its equations as a subvariety of $C^{(n+1)}$ are more tractable than those of X in P_{n+2}.

Let E be any effective divisor on C of degree $n+1$. Then by the above properties we have

$$\dim \Gamma(C, \mathcal{O}_C(E)) = 1. \tag{1.1}$$

From the short exact sequence $0 \to \mathcal{O}_C(E) \to \mathcal{O}_C(E+c) \to \mathcal{O}_C(E+c)|_c \to 0$ we have the long exact sequence

$$0 \to \Gamma(C, \mathcal{O}_C(E)) \to \Gamma(C, \mathcal{O}_C(E+c)) \to \Gamma(C, \mathcal{O}_C(E+c)_{\ c}) \xrightarrow{\delta_E} H^1(C, \mathcal{O}_C(E)). \tag{1.2}$$

Thus as $\Gamma(C, \mathcal{O}_C(E+c)|_c)$ is one dimensional

$$E \text{ is in } Y \Leftrightarrow \dim \Gamma(C, \mathcal{O}_C(E+c)) = 2 \Leftrightarrow \delta_E = 0. \tag{1.3}$$

Next we will work out the variational for this calculation to get a global version of the last equation for Y.

Let $D \subset C \times C^{(n+1)}$ be the universal effective divisor of degree $n+1$. By 1.1, we have the natural isomorphism

$$\mathcal{O}_{C^{(n+1)}} \xrightarrow{\sim} \pi_{C^{(n+1)}*}\mathcal{O}_{C \times C^{(n+1)}}(D). \tag{2.1}$$

Furthermore as $\dim H^1(C, \mathcal{O}_C(E)) = 1-(n+1)+(2n+1)-1 = n$ for any choice of E in $C^{(n+1)}$, the sheaf

$$\mathscr{F} \equiv R^1\pi_{C^{(n+1)}*}\mathcal{O}_{C \times C^{(n+1)}}(D) \quad \text{is locally free of rank } n$$

$$\text{and} \quad \mathscr{F}_E \xrightarrow{\sim} H^1(C, \mathcal{O}_C(E) \quad \text{for all } E. \tag{2.2}$$

Let K be the divisor $D + c \times C^{(n+1)}$. The short exact sequence

$$0 \to \mathcal{O}_{C \times C^{(n+1)}}(D) \to \mathcal{O}_{C \times C^{(n+1)}}(K) \to \mathcal{O}_{C \times C^{(n+1)}}(K)\big|_{c \times C^{(n+1)}} \to 0$$

gives the long exact sequence

$$0 \to \mathcal{O}_{C^{(n+1)}} \to \pi_{C^{(n+1)}*}\mathcal{O}_{C \times C^{(n+1)}}(K) \to \mathscr{L} \xrightarrow{\delta} \mathscr{F} \tag{2.3}$$

where $\mathscr{L}$ is the invertible sheaf $\pi_{C^{(n+1)}*}\mathcal{O}_{C \times C^{(n+1)}}(K)|_{c \times C^{(n+1)}}$ and δ_E is isomorphic to δ evaluated at E. The equations for Y as a closed subscheme of $C^{(n+1)}$ is that Y is the scheme of seroes of δ.

By definition the scheme of zeroes of δ is the closed subscheme of $C^{(n+1)}$ whose ideal is the image of the homomorphism $\delta' : \mathscr{L} \otimes \mathscr{F}^{\check{}} \to \mathcal{O}_{C^{(n+1)}}$ which is associated to δ. Now Y is a smooth curve of codimension n in $C^{(n+1)}$. Thus we may compute the class $[Y]$ of Y in the Chow ring of $C^{(n+1)}$ by the rule

$$[Y] = c_n(\mathscr{F} \otimes \mathscr{L}^{\otimes -1}) \tag{3.1}$$

where $c_i(\mathscr{W})$ denotes the i-th Chern class of a coherent sheaf $\mathscr{W}$ [4]. Also we have an exact sequence, $0 \to \mathscr{L} \otimes \mathscr{F}^{\check{}}|_Y \to \Omega_{C^{(n+1)}}|_Y \to \Omega_Y \to 0$. Consequently we have an isomorphism $\Omega_Y = \Lambda^{n+1}\Omega_{C^{(n+1)}} \otimes \Lambda^n(\mathscr{F} \otimes \mathscr{L}^{\otimes -1})|_Y$. Thus if K_Y is the canonical class $c_1(\Omega_Y)$ of Y and we regrad it as a cycle class on $C^{(n+1)}$, we have the relation

$$K_Y = [Y] \cdot \left[c_1(\Omega_{C^{(n+1)}}) + c_1(\mathscr{F} \otimes \mathscr{L}^{\otimes -1})\right]. \tag{3.2}$$

By routine methods one may verify the following two expressions for the Chern classes of $\mathscr{F} \otimes \mathscr{L}^{\otimes -1}$, which appear in the above formulas:

$$c_1(\mathscr{F} \otimes \mathscr{L}^{\otimes -1}) = c_1(\mathscr{F}) - n \cdot c_1(\mathscr{L}) \text{ and} \tag{3.3}$$

$$c_n(\mathscr{F} \otimes \mathscr{L}^{\otimes -1}) = \sum_{i=0}^{n} (-1)^i c_{n-i}(\mathscr{F}) c_1(\mathscr{L})^i. \tag{3.4}$$

We can determine the invertible sheaf $\mathscr{L}$ more exactly. As $c \times C^{(n+1)}$ has trivial self-intersection, $K \cdot c \times C^{(n+1)} \sim D \cdot c \times C^{(n+1)} = c \times \{c + C^{(n)}\}$. From its definition we have

$$\mathscr{L} \approx \mathcal{O}_{C^{(n+1)}}(c + C^{(n)}). \tag{4.1}$$

Furthermore we also have shown that

$$\mathscr{L} \approx \pi_{C^{(n+1)}*}\left(\mathcal{O}_{C \times C^{(n+1)}}(D)\Big|_{c \times C^{(n)}}\right). \tag{4.2}$$

We will also denote by h the class of the cycle $c \times C^{(n)}$ on $C^{(n+1)}$. Thus

$$h = c_1(\mathscr{L}). \tag{4.3}$$

We have the short exact sequence

$$0 \to \mathcal{O}_{C \times C^{(n+1)}} \to \mathcal{O}_{C \times C^{(n+1)}}(D) \to \mathcal{O}_{C \times C^{(n+1)}}(D)\big|_D \to 0.$$

By 2.1 the direct images under $\pi_{C^{(n+1)}}$ of the first arrow is an isomorphism. Thus we have an exact sequence

$$0 \to \Omega^{\wedge}_{C^{(n+1)}} \to H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_{C^{(n+1)}} \to \mathscr{F} \to 0 \tag{5.1}$$

as $\Omega^{\wedge}_{C^{(n+1)}} = \pi_{C^{(n+1)}*}\mathcal{O}_{C \times C^{(n+1)}}(D)|_D$ by [5]. Therefore we have a relation between Chern polynomials

$$c_t(\mathscr{F}) \cdot c_t\left(\Omega^{\wedge}_{C^{(n+1)}}\right) = 1. \tag{5.2}$$

In particular $c_1(\mathscr{F}) + c_1(\Omega^{\wedge}_{C^{(n+1)}}) = 0$. In other words

$$c_1(\mathscr{F}) = c_1(\Omega_{C^{(n+1)}}). \tag{5.3}$$

This gives one relation satisfied by the Chern classes of $\mathscr{F}$.

We will use another such relation. First we will review some facts which are true for any smooth complete curve C of genus g with a marked point c. For any integer d the d-th Picard variety P_d of C is identified with the Jacobian $J \equiv P_0$ of C by translating by an appropriate multiple of the class of c. For each integer i between zero and g we have the subvariety W_i of J which is identified with the variety of divisor classes of degree $g - i$ which contain effective divisors. Thus W_i has codimension i in J. The theta divisor θ is W_1 and the W_i satisfy Poincare's relation.

$$W_i \text{ is numerical equivalent to } \frac{1}{i!}\theta^i. \tag{6.1}$$

A family $\mathscr{N}$ of invertible sheaves on C parametrized by a variety X is an invertible sheaf $\mathscr{N}$ on $C \times X$. The $\deg(\mathscr{N})$ of the family is equal $\deg(\mathscr{N}|_{C\times x})$ for each point x of X. Also we have the classifying morphism $f_{\mathscr{N}}: X \to J$ which sends x to the isomorphism class of the invertible sheaf $\mathscr{N}|_{C\times x}(-\deg(\mathscr{N})c)$ of degree zero on C. The family $\mathscr{N}$ is normalized if $\mathscr{N}|_{c\times X}$ is a trivial sheaf on X. With these definitions we have

THEOREM 6.2: *If $\mathscr{N}$ is a normal family parameterized by a smooth quasi-projective variety X, then*

$$\sum_{i=0}^{g} f_{\mathscr{N}}^{-1}(W_i)t^i = c_t(R^1\pi_{X*}\mathscr{N})/c_t(\pi_{X*}\mathscr{N}).$$

PROOF: If $\deg(\mathscr{N}) = 0$, then $\pi_{X*}\mathscr{N} = 0$ and $c_t(\pi_{X*}\mathscr{N}) = 1$ and the formation of $\mathscr{N}$ commutes with base extension. Thus in this case we need only verify the relation for the universal normalized family parameterized by J. Indeed the relation is just Mattuck's calculation [6] of the Chern classes of Picard handles. The general case follows by degree shifting using the long exact sequence of π_X for the short exact sequence

$$0 \to \mathscr{N}(-c\times X) \to \mathscr{N} \to \mathscr{O}_{c\times X} \to 0.$$

Q.E.D.

We will apply the above theorem to the family $\mathscr{O}_{C\times C^{(n+1)}}(D)$ of invertible sheaves of degree $n+1$ on C. The classifying morphism $f: C^{(n+1)} \to J$ sends $(c_1 + \ldots + c_{n+1})$ to the class of $c_1 + \ldots c_{n+1} - (n+1)c$. The normalized version of this family is $\mathscr{N} \equiv \mathscr{O}_{C\times C^{(n+1)}}(D) \otimes \pi^*_{C^{(n+1)}}\mathscr{L}^{\otimes -1}$ by 4.2. By the projection formula, 2.1 and 2.2, $\pi_{C^{(n+1)}*}\mathscr{N} \approx \mathscr{L}^{\otimes -1}$ and $R^1\pi_{C^{(n+1)}*}\mathscr{N} = \mathscr{F} \otimes \mathscr{L}^{\otimes -1}$. Thus from (6.2) we have the relation

$$\sum_{0 \leqslant i \leqslant 2n+1} W_i' t^i = c_t(\mathscr{F} \otimes \mathscr{L}^{\otimes -1})/c_t(\mathscr{L}^{\otimes -1}) \tag{7.1}$$

where W_t' denotes $f^{-1}(W_i)$. Thus from 4.3., we have

$$c_t(\mathscr{F} \otimes \mathscr{L}^{\otimes -1}) = (1 - ht)\Bigg(\sum_{0 \leqslant i \leqslant 2n+2} W_i' t^i\Bigg).$$

In particular we have

$$c_1(\mathscr{F} \otimes \mathscr{L}^{\otimes -1}) = W_1' - h \quad \text{and} \tag{7.2}$$

$$c_n(\mathscr{F} \otimes \mathscr{L}^{\otimes -1}) = W_n' - h \cdot W_{n-1}' \qquad \text{if} \quad n \geqslant 1. \tag{7.3}$$

With these calculations we can easily finish our job. In fact by 3.2 and 7.3, we have

$$[Y] = W'_n - h \cdot W'_{n-1} \qquad \text{if} \quad n \geqslant 1. \tag{8.1}$$

By 3.2 to find K_Y we need only intersect this cycle with $c_1(\Omega_{C^{(n+1)}}) + c_1(\mathscr{F} \otimes \mathscr{L}^{\otimes -1})$ which is $c_1(\mathscr{F}) + c_1(\mathscr{F} \otimes \mathscr{L}^{\otimes -1})$ by 5.3 or, rather, $2c_1(\mathscr{F} \otimes \mathscr{L}^{\otimes -1}) + nh$ by 3.3 and 4.3. Thus by 7.2 we get

$$\begin{aligned} K_Y &= (W'_n - h \cdot W'_{n-1})(2W'_1 + (n-2)h) \\ &= 2W'_1 W'_n + [(n-2)W'_n - 2W'_1 W'_{n-1}]\, h - (n-2) W'_{n-1} h^2 \\ &\text{if} \quad n \geqslant 1. \end{aligned} \tag{8.2}$$

$$\begin{aligned} f_* K_Y = {} & 2W_1 W_n f_* C^{(n+1)} + [(n-2)W_n - 2W_1 W_{n-1}]\, f_* h \\ & - (n-2) W_{n-1} f_*(h^2) \end{aligned} \tag{8.3}$$

where $n \geqslant 1$. Now $f_* C^{(n+1)} = W_n$, $f_* h = W_{n+1}$ and $f_*(h^2) = W_{n+2}$. Thus we have

$$\begin{aligned} f_* K_Y = {} & 2W_1 W_n^2 + [(n-2)W_n - 2W_1 W_{n-1}]\, W_{n+1} \\ & - (n-2) W_{n-1} W_{n+2}. \end{aligned}$$

Using Poincaré's relation 6.1 several times we can count the number of points (i.e. multiples of W_{2n+1}) in $f_* K_Y = K_X$. This gives

$$\begin{aligned} \deg K_X &= (2n+1)! \left\{ \frac{2}{(n!)^2} + \left[\frac{(n-2)}{n!} - \frac{2}{(n-1)!} \right] \frac{1}{(n+1)!} \right. \\ & \qquad \left. - \frac{(n-2)}{(n-1)!(n+2)!} \right\} \\ &= \frac{(2n+1)!}{n!(n+2)!} \{ n^2 + n + 2 \}. \end{aligned} \tag{8.4}$$

Using the relation 2 genus $(X) - 2 = \deg K_X$, we have genus $(X) = 3$ if $n = 1$ (in fact in this case $X = -C + K_G$ is isomorphic to C) and genus $(X) = 11$ if $n = 2$ (in this case X is double covering as it is invariant under the involution $x \to K_C - x$).

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