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### CURVES OF $g_d^1$ 's

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Let  $C$  be a smooth complete (irreducible) curve of odd genus  $2n + 1$ . We will assume that  $C$  is general enough in the sense of moduli to have the following properties. By the Brill-Noether property [1] we may assume that there is a complete curve  $X \subset P_{n+2}$  consisting of all linear systems of degree  $n + 2$  and dimension 1. Furthermore by this property we may assume that none of these linear systems has any base points. By the Petri property [3], we may assume that the curve  $X$  is smooth. By the connectedness theorem [2] of Fulton-Lazarfeld  $X$  is also connected. We intended to compute the genus of the irreducible curve  $X$  in the Picard variety  $P_{n+2}$ .

Consider the locus  $S$  in the symmetric product  $C^{(n+2)}$  of the effective divisors in the linear systems of  $X$ . Then  $S$  is a smooth surface which is a locally trivial  $\mathbb{P}^1$ -bundle over  $X$ . Let  $c$  be a fixed point of  $C$ . Consider the set  $Y = S \cap C^{(n+1)} + c$ . Then  $Y$  is a divisor on  $S$  which intersects any fiber over  $X$  in one point. Hence  $Y$  projects isomorphically onto  $X$ . We will compute the genus of  $Y$  as its equations as a subvariety of  $C^{(n+1)}$  are more tractable than those of  $X$  in  $P_{n+2}$ .

Let  $E$  be any effective divisor on  $C$  of degree  $n + 1$ . Then by the above properties we have

$$\dim \Gamma(C, \mathcal{O}_C(E)) = 1. \tag{1.1}$$

From the short exact sequence  $0 \rightarrow \mathcal{O}_C(E) \rightarrow \mathcal{O}_C(E + c) \rightarrow \mathcal{O}_C(E + c)|_c \rightarrow 0$  we have the long exact sequence

$$\begin{aligned}
 0 \rightarrow \Gamma(C, \mathcal{O}_C(E)) \rightarrow \Gamma(C, \mathcal{O}_C(E + c)) \rightarrow \Gamma(C, \mathcal{O}_C(E + c)|_c) \\
 \rightarrow H^1(C, \mathcal{O}_C(E)). \tag{1.2}
 \end{aligned}$$

Thus as  $\Gamma(C, \mathcal{O}_C(E + c)|_c)$  is one dimensional

$$E \text{ is in } Y \Leftrightarrow \dim \Gamma(C, \mathcal{O}_C(E + c)) = 2 \Leftrightarrow \delta_E = 0. \tag{1.3}$$

Next we will work out the variational for this calculation to get a global version of the last equation for  $Y$ .

Let  $D \subset C \times C^{(n+1)}$  be the universal effective divisor of degree  $n + 1$ . By 1.1, we have the natural isomorphism

$$\mathcal{O}_{C^{(n+1)}} \xrightarrow{\cong} \pi_{C^{(n+1)}} \star \mathcal{O}_{C \times C^{(n+1)}}(D). \quad (2.1)$$

Furthermore as  $\dim H^1(C, \mathcal{O}_C(E)) = 1 - (n + 1) + (2n + 1) - 1 = n$  for any choice of  $E$  in  $C^{(n+1)}$ , the sheaf

$$\begin{aligned} \mathcal{F} &\equiv R^1 \pi_{C^{(n+1)}} \star \mathcal{O}_{C \times C^{(n+1)}}(D) \quad \text{is locally free of rank } n \\ \text{and } \mathcal{F}_E &\xrightarrow{\cong} H^1(C, \mathcal{O}_C(E)) \quad \text{for all } E. \end{aligned} \quad (2.2)$$

Let  $K$  be the divisor  $D + c \times C^{(n+1)}$ . The short exact sequence

$$0 \rightarrow \mathcal{O}_{C \times C^{(n+1)}}(D) \rightarrow \mathcal{O}_{C \times C^{(n+1)}}(K) \rightarrow \mathcal{O}_{C \times C^{(n+1)}}(K) \Big|_{C \times C^{(n+1)}} \rightarrow 0$$

gives the long exact sequence

$$0 \rightarrow \mathcal{O}_{C^{(n+1)}} \rightarrow \pi_{C^{(n+1)}} \star \mathcal{O}_{C \times C^{(n+1)}}(K) \rightarrow \mathcal{L} \xrightarrow{\delta} \mathcal{F} \quad (2.3)$$

where  $\mathcal{L}$  is the invertible sheaf  $\pi_{C^{(n+1)}} \star \mathcal{O}_{C \times C^{(n+1)}}(K) \Big|_{C \times C^{(n+1)}}$  and  $\delta_E$  is isomorphic to  $\delta$  evaluated at  $E$ . The equations for  $Y$  as a closed subscheme of  $C^{(n+1)}$  is that  $Y$  is the scheme of zeroes of  $\delta$ .

By definition the scheme of zeroes of  $\delta$  is the closed subscheme of  $C^{(n+1)}$  whose ideal is the image of the homomorphism  $\delta': \mathcal{L} \otimes \mathcal{F} \rightarrow \mathcal{O}_{C^{(n+1)}}$  which is associated to  $\delta$ . Now  $Y$  is a smooth curve of codimension  $n$  in  $C^{(n+1)}$ . Thus we may compute the class  $[Y]$  of  $Y$  in the Chow ring of  $C^{(n+1)}$  by the rule

$$[Y] = c_n(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) \quad (3.1)$$

where  $c_i(\mathcal{W})$  denotes the  $i$ -th Chern class of a coherent sheaf  $\mathcal{W}$  [4]. Also we have an exact sequence,  $0 \rightarrow \mathcal{L} \otimes \mathcal{F} \Big|_Y \rightarrow \Omega_{C^{(n+1)}} \Big|_Y \rightarrow \Omega_Y \rightarrow 0$ . Consequently we have an isomorphism  $\Omega_Y = \Lambda^{n+1} \Omega_{C^{(n+1)}} \otimes \Lambda^n(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) \Big|_Y$ . Thus if  $K_Y$  is the canonical class  $c_1(\Omega_Y)$  of  $Y$  and we regard it as a cycle class on  $C^{(n+1)}$ , we have the relation

$$K_Y = [Y] \cdot [c_1(\Omega_{C^{(n+1)}}) + c_1(\mathcal{F} \otimes \mathcal{L}^{\otimes -1})]. \quad (3.2)$$

By routine methods one may verify the following two expressions for the Chern classes of  $\mathcal{F} \otimes \mathcal{L}^{\otimes -1}$ , which appear in the above formulas:

$$c_1(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) = c_1(\mathcal{F}) - n \cdot c_1(\mathcal{L}) \quad \text{and} \quad (3.3)$$

$$c_n(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) = \sum_{i=0}^n (-1)^i c_{n-i}(\mathcal{F}) c_1(\mathcal{L})^i. \quad (3.4)$$

We can determine the invertible sheaf  $\mathcal{L}$  more exactly. As  $c \times C^{(n+1)}$  has trivial self-intersection,  $K \cdot c \times C^{(n+1)} \sim D \cdot c \times C^{(n+1)} = c \times \{c + C^{(n)}\}$ . From its definition we have

$$\mathcal{L} \approx \mathcal{O}_{C^{(n+1)}}(c + C^{(n)}). \quad (4.1)$$

Furthermore we also have shown that

$$\mathcal{L} \approx \pi_{C^{(n+1)}} \star \left( \mathcal{O}_{C \times C^{(n+1)}}(D) \Big|_{c \times C^{(n)}} \right). \quad (4.2)$$

We will also denote by  $h$  the class of the cycle  $c \times C^{(n)}$  on  $C^{(n+1)}$ . Thus

$$h = c_1(\mathcal{L}). \quad (4.3)$$

We have the short exact sequence

$$0 \rightarrow \mathcal{O}_{C \times C^{(n+1)}} \rightarrow \mathcal{O}_{C \times C^{(n+1)}}(D) \rightarrow \mathcal{O}_{C \times C^{(n+1)}}(D)|_p \rightarrow 0.$$

By 2.1 the direct images under  $\pi_{C^{(n+1)}}$  of the first arrow is an isomorphism. Thus we have an exact sequence

$$0 \rightarrow \Omega_{C^{(n+1)}}^{\wedge} \rightarrow H^1(C, \mathcal{O}_C) \otimes \mathcal{O}_{C^{(n+1)}} \rightarrow \mathcal{F} \rightarrow 0 \quad (5.1)$$

as  $\Omega_{C^{(n+1)}}^{\wedge} = \pi_{C^{(n+1)}} \star \mathcal{O}_{C \times C^{(n+1)}}(D)|_D$  by [5]. Therefore we have a relation between Chern polynomials

$$c_t(\mathcal{F}) \cdot c_t(\Omega_{C^{(n+1)}}^{\wedge}) = 1. \quad (5.2)$$

In particular  $c_1(\mathcal{F}) + c_1(\Omega_{C^{(n+1)}}^{\wedge}) = 0$ . In other words

$$c_1(\mathcal{F}) = c_1(\Omega_{C^{(n+1)}}^{\wedge}). \quad (5.3)$$

This gives one relation satisfied by the Chern classes of  $\mathcal{F}$ .

We will use another such relation. First we will review some facts which are true for any smooth complete curve  $C$  of genus  $g$  with a marked point  $c$ . For any integer  $d$  the  $d$ -th Picard variety  $P_d$  of  $C$  is identified with the Jacobian  $J \equiv P_0$  of  $C$  by translating by an appropriate multiple of the class of  $c$ . For each integer  $i$  between zero and  $g$  we have the subvariety  $W_i$  of  $J$  which is identified with the variety of divisor classes of degree  $g - i$  which contain effective divisors. Thus  $W_i$  has codimension  $i$  in  $J$ . The theta divisor  $\theta$  is  $W_1$  and the  $W_i$  satisfy Poincare's relation.

$$W_i \text{ is numerical equivalent to } \frac{1}{i!} \theta^i. \quad (6.1)$$

A family  $\mathcal{N}$  of invertible sheaves on  $C$  parametrized by a variety  $X$  is an invertible sheaf  $\mathcal{N}$  on  $C \times X$ . The  $\text{deg}(\mathcal{N})$  of the family is equal  $\text{deg}(\mathcal{N}|_{C \times x})$  for each point  $x$  of  $X$ . Also we have the classifying morphism  $f_{\mathcal{N}}: X \rightarrow J$  which sends  $x$  to the isomorphism class of the invertible sheaf  $\mathcal{N}|_{C \times x}(-\text{deg}(\mathcal{N})c)$  of degree zero on  $C$ . The family  $\mathcal{N}$  is normalized if  $\mathcal{N}|_{c \times X}$  is a trivial sheaf on  $X$ . With these definitions we have

**THEOREM 6.2:** *If  $\mathcal{N}$  is a normal family parameterized by a smooth quasi-projective variety  $X$ , then*

$$\sum_{i=0}^g f_{\mathcal{N}}^{-1}(W_i)t^i = c_t(R^1\pi_{X*}\mathcal{N})/c_t(\pi_{X*}\mathcal{N}).$$

**PROOF:** If  $\text{deg}(\mathcal{N})=0$ , then  $\pi_{X*}\mathcal{N}=0$  and  $c_t(\pi_{X*}\mathcal{N})=1$  and the formation of  $\mathcal{N}$  commutes with base extension. Thus in this case we need only verify the relation for the universal normalized family parameterized by  $J$ . Indeed the relation is just Mattuck's calculation [6] of the Chern classes of Picard handles. The general case follows by degree shifting using the long exact sequence of  $\pi_X$  for the short exact sequence

$$0 \rightarrow \mathcal{N}(-c \times X) \rightarrow \mathcal{N} \rightarrow \mathcal{O}_{c \times X} \rightarrow 0.$$

Q.E.D.

We will apply the above theorem to the family  $\mathcal{O}_{C \times C^{(n+1)}}(D)$  of invertible sheaves of degree  $n+1$  on  $C$ . The classifying morphism  $f: C^{(n+1)} \rightarrow J$  sends  $(c_1 + \dots + c_{n+1})$  to the class of  $c_1 + \dots + c_{n+1} - (n+1)c$ . The normalized version of this family is  $\mathcal{N} \equiv \mathcal{O}_{C \times C^{(n+1)}}(D) \otimes \pi_{C^{(n+1)}}^* \mathcal{L}^{\otimes -1}$  by 4.2. By the projection formula, 2.1 and 2.2,  $\pi_{C^{(n+1)}}^* \mathcal{N} \approx \mathcal{L}^{\otimes -1}$  and  $R^1\pi_{C^{(n+1)}}^* \mathcal{N} = \mathcal{F} \otimes \mathcal{L}^{\otimes -1}$ . Thus from (6.2) we have the relation

$$\sum_{0 \leq i \leq 2n+1} W'_i t^i = c_t(\mathcal{F} \otimes \mathcal{L}^{\otimes -1})/c_t(\mathcal{L}^{\otimes -1}) \tag{7.1}$$

where  $W'_i$  denotes  $f^{-1}(W_i)$ . Thus from 4.3., we have

$$c_t(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) = (1 - ht) \left( \sum_{0 \leq i \leq 2n+2} W'_i t^i \right).$$

In particular we have

$$c_1(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) = W'_1 - h \quad \text{and} \tag{7.2}$$

$$c_n(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) = W'_n - h \cdot W'_{n-1} \quad \text{if } n \geq 1. \tag{7.3}$$

With these calculations we can easily finish our job. In fact by 3.2 and 7.3, we have

$$[Y] = W'_n - h \cdot W'_{n-1} \quad \text{if } n \geq 1. \quad (8.1)$$

By 3.2 to find  $K_Y$  we need only intersect this cycle with  $c_1(\Omega_{C^{(n+1)}}) + c_1(\mathcal{F} \otimes \mathcal{L}^{\otimes -1})$  which is  $c_1(\mathcal{F}) + c_1(\mathcal{F} \otimes \mathcal{L}^{\otimes -1})$  by 5.3 or, rather,  $2c_1(\mathcal{F} \otimes \mathcal{L}^{\otimes -1}) + nh$  by 3.3 and 4.3. Thus by 7.2 we get

$$\begin{aligned} K_Y &= (W'_n - h \cdot W'_{n-1})(2W'_1 + (n-2)h) \\ &= 2W'_1W'_n + [(n-2)W'_n - 2W'_1W'_{n-1}]h - (n-2)W'_{n-1}h^2 \\ &\text{if } n \geq 1. \end{aligned} \quad (8.2)$$

$$\begin{aligned} f_*K_Y &= 2W_1W_n f_*C^{(n+1)} + [(n-2)W_n - 2W_1W_{n-1}]f_*h \\ &\quad - (n-2)W_{n-1}f_*(h^2) \end{aligned} \quad (8.3)$$

where  $n \geq 1$ . Now  $f_*C^{(n+1)} = W_n$ ,  $f_*h = W_{n+1}$  and  $f_*(h^2) = W_{n+2}$ . Thus we have

$$\begin{aligned} f_*K_Y &= 2W_1W_n^2 + [(n-2)W_n - 2W_1W_{n-1}]W_{n+1} \\ &\quad - (n-2)W_{n-1}W_{n+2}. \end{aligned}$$

Using Poincaré's relation 6.1 several times we can count the number of points (i.e. multiples of  $W_{2n+1}$ ) in  $f_*K_Y = K_X$ . This gives

$$\begin{aligned} \deg K_X &= (2n+1)! \left\{ \frac{2}{(n!)^2} + \left[ \frac{(n-2)}{n!} - \frac{2}{(n-1)!} \right] \frac{1}{(n+1)!} \right. \\ &\quad \left. - \frac{(n-2)}{(n-1)!(n+2)!} \right\} \\ &= \frac{(2n+1)!}{n!(n+2)!} \{n^2 + n + 2\}. \end{aligned} \quad (8.4)$$

Using the relation  $2 \text{ genus } (X) - 2 = \deg K_X$ , we have  $\text{genus } (X) = 3$  if  $n = 1$  (in fact in this case  $X = -C + K_C$  is isomorphic to  $C$ ) and  $\text{genus } (X) = 11$  if  $n = 2$  (in this case  $X$  is double covering as it is invariant under the involution  $x \rightarrow K_C - x$ ).

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