COMPOSITIO MATHEMATICA

MARK L. GREEN

# The period map for hypersurface sections of high degree of an arbitrary variety

*Compositio Mathematica*, tome 55, nº 2 (1985), p. 135-156

<http://www.numdam.org/item?id=CM_1985__55_2_135_0>

Compositio Mathematica 55 (1984) 135–156. 


# THE PERIOD MAP FOR HYPERSURFACE SECTIONS OF HIGH DEGREE OF AN ARBITRARY VARIETY

Mark L. Green *

## Introduction

In this paper, for a fixed projective variety $Y$, we will say that a property holds for *sufficiently ample* analytic line bundles $L \to Y$ if there exists an ample bundle $L_0 \to Y$ so that the property holds for all $L$ on $Y$ with $L > L_0$, i.e. $L \otimes L_0^{-1}$ ample. We will denote this by saying the property holds for $L \gg 0$.

We will prove two theorems:

THEOREM 0.1: *Let $Y$ be a smooth complete algebraic variety of dimension $\geqslant 2$. Then for $L \to Y$ a sufficient ample line bundle, the Local Torelli Theorem is true for any smooth $Z$ in the linear system $|L|$.*

REMARK: What we will actually show is that the map

$$H^1(Z, \Theta_Z) \overset{P_*}{\to} \operatorname{Hom}\big(H^0(Z, K_Z), H^1\big(Z, \Omega_Z^{n-1}\big)\big) \tag{0.2}$$

is injective, where $n = \dim Z$; this is one piece of the derivative of the period map. Note that we are considering all first order deformations of $Z$ and not just those arising by varying $Z$ to first order in the linear system $|L|$.

THEOREM 0.3: *Let $Y$ be a smooth complete algebraic variety of dimension $\geqslant 2$, $L \to Y$ a sufficiently ample line bundle. Assume $K_Y$ is very ample. Let*

$$G = \{f \in \operatorname{Aut}_{\text{hol}}(Y) | f^*(L) \simeq L\} \tag{0.4}$$

*Then the period map has degree* 1 *on its domain in*

$$\mathbb{P}(H^0(Y, L))/G.$$

* Research partially supported by NSF Grant MCS 82-00924.

What we will actually show is stronger than (0.3). To state exactly what is proved, we make the following definition:

DEFINITION 0.5: Let $V_1, \dots, V_k$ be vector spaces. We say that two elements

$$\alpha, \beta \in V_1 \otimes V_2 \otimes \dots \otimes V_k$$

are $GL$-equivalent if they lie in the same orbit of $GL(V_1) \times \dots \times GL(V_k)$.

Let

$$T \subseteq H^1(Z, \Theta_Z)$$

be the image of $H^0(Y, L)$. The highest piece of the derivative of the period map $P_{*,Z}$ may be regarded as an element

$$P_{*,Z} \in T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})$$

Let

$$\mathbb{P}(H^0(Y, L))_{ns} \leftrightarrow \{\text{smooth, reduced } Z \in |L|\}.$$

Then what will actually be shown is that

*If $|K_Y|$ is very ample, then for $L$ a sufficiently ample line bundle, the map* (0.6)

$$\mathbb{P}(H^0(Y, L))_{ns}/G$$

$$\to \frac{T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})}{GL(T^*) \times GL(H^0(Z, K_Z)^*) \times GL(H^1(Z, \Omega_Z^{n-1}))} \quad (0.7)$$

*is injective.*

These theorems settle a conjecture in [C-G-Gr-H]; the method of proof of (0.1) is essentially a resurgence of an idea that occurs there. The proof of (0.3) is inspired by Donagi's proof of a similar result for hypersurfaces in projective space [*D*].

The author is grateful to Ron Donagi and Phillip Griffiths for their help and encouragement.

## §1. Hodge theory on a hypersurface of high degree

Given a short exact sequence of vector spaces

$$0 \to A \to B \to C \to 0$$

there is a long exact sequence

$$0 \to S^k A \to B \otimes S^{k-1} A \to \dots \to \Lambda^{k-1} B \otimes A \to \Lambda^k B \to \Lambda^k C \to 0 \quad (1.1)$$

for any $k \geqslant 1$. If $Y$ is a compact complex manifold of dimension $m$ and $Z \subset Y$ is a smooth submanifold of dimension $n$ with normal bundle $N_Z$ in $Y$, we have a short exact sequence

$$0 \to N_Z^* \to \Omega_Y^1 \otimes \mathcal{O}_Z \to \Omega_Z^1 \to 0. \quad (1.2)$$

Thus we have, for any $p \geqslant 1$, long exact sequences

$$0 \to S^p N_Z^* \to \dots \to \Omega_Y^{p-1} \otimes N_Z^* \to \Omega_Y^p \otimes \mathcal{O}_Z \to \Omega_Z^p \to 0. \quad (1.3)$$

The exact sequences (1.3) turn out to be quite useful in computations whenever we have an explicit form for $N_Z$, most notably in the case of complete intersections. To use these sequences, we need:

LEMMA 1.4: *Let*

$$0 \to F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \dots \to F_{k-1} \xrightarrow{f_{k-1}} F_k \to 0$$

*be an exact sequence of vector bundles on a compact complex manifold Z. Then there is a spectral sequence abutting to zero with*

$$E_1^{p,q} = H^q(Z, F_p)$$

$$E_2^{p,q} = \frac{\ker\left(H^q(Z, F_p) \xrightarrow{f_{p*}} H^q(Z, F_{p+1})\right)}{\operatorname{im}\left(H^q(Z, F_{p-1}) \xrightarrow{f_{p-1*}} H^q(Z, F_p)\right)}.$$

PROOF: Consider the bigraded complex

$$B^{p,q} = \mathscr{A}^{0,q}(Z, F_p)$$

where $\mathscr{A}^{0,q}$ denotes $\mathscr{C}^\infty$ $(0, q)$-forms, with maps

$$B^{p,q} \xrightarrow{d} B^{p+1,q} \qquad d = f_{p*}$$

$$B^{p,q} \xrightarrow{\delta} B^{p,q+1} \qquad \delta = \bar{\partial}.$$

There is then (see [G-H]) a pair of spectral sequences $'E_r^{p,q}$, $''E_r^{p,q}$ having the same abutment, with

$$'E_1^{p,q} = H_d^p(B^{\cdot,q}), \quad 'E_2^{p,q} = H_\delta^p H_d^p(B^{\cdot\cdot})$$
$$''E_1^{p,q} = H_\delta^q(B^{p,\cdot}), \quad ''E_2^{p,q} = H_d^p H_\delta^q(B^{\cdot\cdot}).$$

The rows of $B^{\cdot\cdot}$ are exact, so

$$'E_1^{p,q} = 0 \qquad \text{for all} \quad p,\ q.$$

Thus the spectral sequence $'E_r^{p,q}$ abuts to zero, and hence so does $''E_r^{p,q}$. Furthermore,

$$''E_1^{p,q} = H^q(Z,\ F_p)$$

$$''E_2^{p,q} = \frac{\ker\left(H^q(Z,\ F_p) \xrightarrow{f_{p*}} H^q(Z,\ F_{p+1})\right)}{\operatorname{im}\left(H^q(Z,\ F_{p-1}) \xrightarrow{f_{p-1*}} H^q(Z,\ F_p)\right)}$$

This completes the proof of the lemma. □

LEMMA 1.5: *Let Y be a compact Kähler manifold and $Z \subset Y$ a complex submanifold of dimension n. If*

$$H^i(Z,\ \Omega_Y^j \otimes S^m N_Z^*) = 0 \qquad \textit{for all} \quad i < n,\ 0 \leqslant j \leqslant n,\ 1 \leqslant m \leqslant n \tag{1.6}$$

*then*

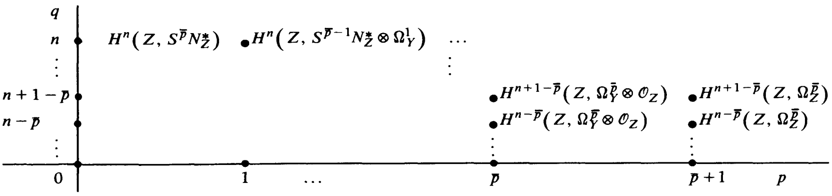
$$H^{p,q}(Z) \simeq H^q(Z,\ \Omega_Y^p \otimes \mathcal{O}_Z) \qquad \textit{if} \quad p+q < n \tag{1.7}$$

*and there is a short exact sequence*

$$0 \to \left(\frac{H^{n-p}(Z,\ \Omega_Z^p)}{\operatorname{im} H^{n-p}(Z,\ \Omega_Y^p \otimes \mathcal{O}_Z)}\right)$$
$$\to \left(\frac{H^0(Z,\ S^p N_Z \otimes K_Z)}{\operatorname{im} H^0(Z,\ S^{p-1} N_Z \otimes \Theta_Y \otimes K_Z)}\right)^*$$
$$\to \left(\ker H^{n+1-p}(Z,\ \Omega_Y^p \otimes \mathcal{O}_Z)\right) \to H^{n+1-p}(Z,\ \Omega_Z^p)) \to 0. \tag{1.8}$$

PROOF: We apply Lemma 1.4 to the exact sequence (1.3), taking $p = \bar{p}$.

We obtain a spectral sequence which abuts to zero whose $E_1$ term looks as follows:




The only non-zero differentials emerging from the position $(0, n)$ are $d_1$, $d_{\bar{p}-1}$, and $d_{\bar{p}}$. The only non-zero differentials whose target is the position $(\bar{p}+1, n-\bar{p})$ are $d_1$ and $d_{\bar{p}}$. We thus obtain maps

$$\begin{cases} \ker\left(H^n(Z, S^{\bar{p}}N_Z^*) \to H^n(Z, S^{\bar{p}-1}N_Z^* \otimes \Omega_Y^1)\right) \overset{d_{\bar{p}-1}}{\to} \\ \ker\left(H^{n+1-\bar{p}}(Z, \Omega_Y^{\bar{p}} \otimes \mathcal{O}_Z) \to H^{n+1-\bar{p}}(Z, \Omega_Z^{\bar{p}})\right) \\ \ker d_{\bar{p}-1} \overset{d_{\bar{p}}}{\to} \left( \dfrac{H^{n-\bar{p}}(Z, \Omega_Z^{\bar{p}})}{\operatorname{im} H^{n-\bar{p}}(Z, \Omega_Y^{\bar{p}} \otimes \mathcal{O}_Z)} \right) \end{cases} \tag{1.9}$$

where the second map is an isomorphism because the spectral sequence abuts to zero. Using Serre Duality, (1.9) gives (1.8).

There are no non-zero differentials other than $d_1$ coming into the positions $(\bar{p}, \bar{q})$ and $(\bar{p}+1, \bar{q})$ if $\bar{q} < n - \bar{p}$. This shows (1.7) and completes the proof of Lemma 1.5. □

A result similar to this is:

LEMMA 1.10: *Let $Y$ be a compact Kähler manifold and $Z \subset Y$ a complex submanifold of dimension $n$. If*

$$H^i(Z, \Omega_Y^j \otimes S^m N_Z^* \otimes K_Z^{-1}) = 0$$

$$\textit{for} \quad 0 < i < n,\ 1 \leqslant j \leqslant n,\ 1 \leqslant m \leqslant n-2$$

*then*

$$H^1(Z, \Theta_Z) \simeq \left( \frac{H^0(Z, S^{n-1}N_Z \otimes K_Z^2)}{\operatorname{im} H^0(Z, S^{n-2}N_Z \otimes \Theta_Y \otimes K_Z^2)} \right)^* \tag{1.12}$$

PROOF: We take the exact sequence (1.3) for $p = n-1$ and tensor with

$K_Z^{-1}$. This yields the exact sequence

$$0 \to S^{n-1}N_Z^* \otimes K_Z^{-1} \to S^{n-2}N_Z^* \otimes \Omega_Y^1 \otimes K_Z^{-1} \to \dots$$
$$\to \Omega_Y^{n-1} \otimes K_Z^{-1} \to \Theta_Z \to 0. \tag{1.13}$$

No apply Lemma 1.4 and observe that

$$\ker\big(H^n(Z, S^{n-1}N_Z^* \otimes K_Z^{-1})$$
$$\to H^n(Z, S^{n-2}N_Z^* \otimes \Omega_Y^1 \otimes K_Z^{-1})\big) \underset{\simeq}{\overset{d_{n-1}}{\to}} H^1(Z, \Theta_Z)$$

Now (1.12) follows by Serre Duality, completing the proof of Lemma 1.10.

LEMMA 1.14: *Let $Y$ be a smooth $(n+1)$-fold, and $L \to Y$ a sufficiently ample analytic line bundle. If $Z$ is a smooth reduced element of the linear system $|L|$, then for $n \geq 1$,*

$$H^1(Z, \Theta_Z) \simeq \left( \frac{H^0(Z, L^{(n-1)} \otimes K_Z^2)}{\operatorname{im} H^0(Z, L^{(n-2)} \otimes \Theta_Y \otimes K_Z^2)} \right)^* \tag{1.15}$$

*and there is a short exact sequence*

$$0 \to \left( \frac{H^1(Z, \Omega_Z^{n-1})}{\operatorname{im} H^1(Z, \Omega_Y^{n-1} \otimes \mathcal{O}_Z)} \right) \to \left( \frac{H^0(Z, L^{(n-1)} \otimes K_Z)}{\operatorname{im} H^0(Z, L^{(n-2)} \otimes \Theta_Y \otimes K_Z)} \right)^*$$
$$\to \left(\ker H^2(Z, \Omega_Y^{n-1} \otimes \mathcal{O}_Z) \to H^2(Z, \Omega_Z^{n-1})\right) \to 0 \tag{1.16}$$

*Furthermore, there is a commutative diagram*

$$\begin{array}{ccc} H^1(Z, \Omega_Z^{n-1})^* \otimes H^0(Z, K_Z) & \leftarrow & H^0(Z, L^{(n-1)} \otimes K_Z) \otimes H^0(Z, K_Z) \\ \downarrow & & \downarrow \\ H^1(Z, \Theta_Z)^* & \leftarrow & H^0(Z, L^{(n-1)} \otimes K_Z^2) \end{array}$$

*where the horizontal maps are induced by* (1.15) *and* (1.16), *the vertical map on the left is the dual of the highest piece*

$$H^1(Z, \Theta_Z) \overset{p_*}{\to} H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1}) \tag{1.18}$$

*of the derivative of the period map, and the vertical map on the right is multiplication.*

PROOF: From the restriction sequence

$$0 \to \mathcal{O}_Y(L^{-1}) \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0 \tag{1.19}$$

and the isomorphism of bundles

$$N_Z \simeq L|_Z \tag{1.20}$$

we have a long exact sequence

$$\begin{aligned} \dots &\to H^i(Y, \Omega_Y^j \otimes L^m) \to H^i(Z, \Omega_Y^j \otimes L^m \otimes \mathcal{O}_Z) \\ &\to H^{i+1}(Y, \Omega_Y^j \otimes L^{(m-1)}) \to \dots \end{aligned} \tag{1.21}$$

By the Kodaira Vanishing Theorem, we conclude that (1.6) holds when $L$ is sufficiently ample.

By the adjunction formula

$$K_Z \simeq K_Y \otimes L|_Z \tag{1.22}$$

and (1.19) we have a long exact sequence

$$\begin{aligned} \dots &\to H^i(Y, \Omega_Y^j \otimes K_Y^{-1} \otimes L^{-m-1}) \to H^i(Z, \Omega_Y^j \otimes S^m N_Z^* \otimes K_Z^{-1}) \\ &\to H^{i+1}(Y, \Omega_Y^j \otimes K_Y^{-1} \otimes L^{-m-2}) \to \dots \end{aligned} \tag{1.23}$$

so (1.11) holds when $L$ is sufficiently ample. Thus (1.15) and (1.16) follow from Lemmas (1.5) and (1.10). Since multiplication by $H^0(Z, K_Z)$ commutes with all the differentials of the spectral sequence, we conclude that (1.17) commutes, proving the lemma.

LEMMA 1.24: *Let $Y$ be a smooth $(n+1)$-fold with $n \geqslant 1$ and $L \to Y$ a sufficiently ample analytic line bundle. Then the Local Torelli Theorem holds for any smooth, reduced $Z \in |L|$ if the multiplication map*

$$H^0(Y, K_Y \otimes L^n) \otimes H^0(Y, K_Y \otimes L) \to H^0(Y, K_Y^2 \otimes L^{(n+1)}) \tag{1.25}$$

*is surjective.*

PROOF: To show that the map

$$H^1(Z, \Theta_Z) \overset{P_*}{\to} H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})$$

is injective, it is equivalent to show that the dual map

$$H^1(Z, \Omega_Z^{n-1})^* \otimes H^0(Z, K_Z) \to H^1(Z, \Theta_Z)^*$$

is surjective. By Lemma 1.14, it is enough to show that

$$H^0\big(Z,\ K_Z\otimes L^{(n-1)}\big)\otimes H^0(Z,\ K_Z)\to H^0\big(Z,\ K_Z^2\otimes L^{(n-1)}\big) \quad (1.26)$$

is surjective. From the restriction sequence (1.19), we have the long exact sequence

$$\begin{aligned}\dots\to H^0\big(Y,\ K_Y^2\otimes L^{(n+1)}\big)&\to H^0\big(Z,\ K_Z^2\otimes L^{(n-1)}\big)\\ &\to H^1\big(Y,\ K_Y^2\otimes L^n\big)\to\dots\end{aligned} \quad (1.27)$$

By the Kodaira Vanishing Theorem,

$$H^1\big(Y,\ K_Y^2\otimes L^n\big)=0$$

for $L$ sufficiently ample, and thus the map

$$H^0\big(Y,\ K_Y^2\otimes L^{(n+1)}\big)\to H^0\big(Z,\ K_Z^2\otimes L^{(n-1)}\big)$$

is surjective. The lemma follows.

LEMMA 1.28: *Let $Y$ be a smooth $(n+1)$-fold, $E_1$, $E_2$ analytic vector bundles over $Y$. For $L$ a sufficiently ample line bundle on $Y$, the multiplication map*

$$H^0(Y,\ E_1\otimes L^a)\otimes H^0\big(Y,\ E_2\otimes L^b\big)\to H^0\big(Y,\ E_1\otimes E_2\otimes L^{a+b}\big) \quad (1.29)$$

*is surjective when $a\geqslant 1$ and $b\geqslant 1$.*

PROOF *: Let $\Delta$ be the diagonal on $Y\times Y$, and $\pi_1$, $\pi_2$ the canonical projections. We then have a commutative diagram

$$\begin{array}{ccc} H^0\big(Y\times Y,\ \pi_1^*(E_1\otimes L^a)\otimes\pi_2^*(E_2\otimes L^b)\big) & \to & H^0\big(Y\times Y,\ \mathcal{O}_\Delta\big(\pi_1^*(E_1\otimes L^a)\otimes\pi_2^*(E_2\otimes L^b)\big)\big)\\ \simeq & & \simeq\\ H^0(Y,\ E_1\otimes L^a)\otimes H^0\big(Y,\ E_2\otimes L^b\big) & \to & H^0\big(Y,\ E_1\otimes E_2\otimes L^{a+b}\big)\end{array} \quad (1.30)$$

* This argument was suggested by Ron Donagi and replaces an earlier, more complicated proof.

From the restriction sequence

$$0 \to \mathscr{I}_\Delta \to \mathcal{O}_{Y\times Y} \to \mathcal{O}_\Delta \to 0$$

we see that to prove the surjectivity of (1.29), it suffices to prove

$$H^1\big(Y\times Y, \mathscr{I}_\Delta \otimes \pi_1^*(E_1\otimes L^a)\otimes \pi_2^*(E_2\otimes L^b)\big)=0 \tag{1.31}$$

Since $\pi_1^*L\otimes\pi_2^*L$ is sufficiently ample on $Y\times Y$ if $L$ is sufficiently ample on $Y$, (1.31) holds when $a\geqslant 1$ and $b\geqslant 1$. □

We conclude this section by noting that Theorem 0.1 is a direct consequence of Lemmas (1.24) and (1.28). The need to take $L\gg 0$ arose in satisfying (1.6), (1.11), and (1.25). In explicit situations, e.g. when $Y$ is a complete intersection in $\mathbb{P}_N$, these may be verified directly to yield many known results.

## §2. A global Torelli theorem

Let $Y$ be a smooth algebraic variety of dimension $n+1$ and $L\to Y$ a sufficiently ample line bundle. Let

$$\begin{cases} s\in H^0(Y, L) \\ Z=\operatorname{div} s, \quad Z \text{ smooth and reduced.}\end{cases} \tag{2.1}$$

We have

$$T_s\big(\mathbb{P}(H^0(Y, L))\big)\simeq H^0(Y, L)/(s) \tag{2.2}$$

and let $T$ be the image of the Kodaira-Spencer map

$$T_s\big(\mathbb{P}(H^0(Y, L))\big)\to H^1(Z, \Theta_Z). \tag{2.3}$$

The first derivative of the period map at $Z$ has as its leading piece

$$T \overset{P_{*,Z}}{\to} \operatorname{Hom}\big(H^0(Z, K_Z), H^1(Z, \Omega_Z^{n-1})\big) \tag{2.4}$$

which we may alternatively regard as an element

$$\hat{P}_{*,Z}\in T^*\otimes H^0(Z, K_Z)^*\otimes H^1(Z, \Omega_Z^{n-1}). \tag{2.5}$$

Using the notations introduced in the introduction, in order to show that $P$ has degree one on

$$\mathscr{M}=\mathbb{P}\big(H^0(Y, L)\big)_{ns}/G \tag{2.6}$$

it is sufficient to show that

*For $Z \in |L|$ generic, the GL-equivalence class of $\hat{P}_{*,Z}$ determines $Z$.* (2.7)

Let $\Sigma_Y$ denote the *first prolongation bundle* (see [A-C-G-H]) of $L$; it sits in the exact sequence

$$0 \to \mathcal{O}_Y \to \Sigma_Y \to \Theta_Y \to 0 \tag{2.8}$$

with extension class

$$-c_1(L) \in H^1(Y, \Omega_Y^1).$$

We can differentiate $s$ to obtain

$$\widetilde{ds} \in H^0(Y, \Sigma_Y^* \otimes L). \tag{2.9}$$

For any coherent analytic sheaf $\mathscr{F} \to Y$, we obtain a map

$$H^0(Y, \mathscr{F} \otimes \Sigma_Y \otimes L^{-1}) \overset{\rfloor \widetilde{ds}}{\to} H^0(Y, \mathscr{F}) \tag{2.10}$$

and we define the *pseudo-Jacobian system*

$$J_{\mathscr{F}} \subseteq H^0(Y, \mathscr{F}) \tag{2.11}$$

to be the image of the map (2.10). If $\dim Y \geqslant 2$, then for $L$ sufficiently ample, from the exact sequence

$$H^0(Y, \Theta_Y) \to H^0(Z, \Theta_Y) \to H^1(Y, \Theta_Y \otimes L^{-1})$$

and the Kodaira Vanishing Theorem we have that

$$H^0(Y, \Theta_Y) \twoheadrightarrow H^0(Z, \Theta_Y). \tag{2.12}$$

From the exact sequence

$$H^0(Z, \Theta_Y) \overset{\rfloor \widetilde{ds}|_Z}{\to} H^0(Z, L) \to H^1(Z, \Theta_Z) \tag{2.13}$$

we conclude from (2.3) that

$$T \simeq \frac{H^0(Y, L)}{J_L} \tag{2.14}$$

for $L$ sufficiently ample.

It is useful to have the following *Duality Theorem* or *Generalized Macaulay's Theorem*:

THEOREM 2.15: *For $Y$ a smooth $(n+1)$-fold, $E \to Y$ a fixed analytic vector bundle, and $L \to Y$ a sufficiently ample line bundle,*

$$\frac{H^0\left(Y, K_Y^2 \otimes L^{(n+2)}\right)}{J_{K_Y^2 \otimes L^{(n+2)}}} \simeq \mathbf{C} \tag{2.16}$$

*and the map*

$$\frac{H^0(Y, E \otimes L^a)}{J_E} \otimes \frac{H^0\left(Y, E^* \otimes K_Y^2 \otimes L^{(n+2-a)}\right)}{J_{E^* \otimes K_Y^2 \otimes L^{(n+2-a)}}}$$
$$\to \frac{H^0\left(Y, K_Y^2 \otimes L^{(n+2)}\right)}{J_{K_Y^2 \otimes L^{(n+2)}}} \simeq \mathbf{C} \tag{2.17}$$

*is a perfect pairing provided*

$$H^{a-1}(Y, E \otimes \Lambda^a \Sigma_Y) = H^a(Y, E \otimes \Lambda^a \Sigma_Y) = 0 \quad \textit{or} \quad a = 0. \tag{2.18}$$

*If only*

$$H^{a-1}(Y, E \otimes \Lambda^a \Sigma_Y) = 0 \tag{2.19}$$

*then the pairing* (2.17) *has no left kernel.*

PROOF OF 2.15: Using

$$\widetilde{ds} \in H^0(Y, \Sigma_Y^* \otimes L)$$

we may construct the Koszul complex

$$\begin{aligned} 0 \to \Lambda^{n+2}\Sigma_Y \otimes L^{-(n+2)} \quad & \overset{]\widetilde{ds}}{\to} \Lambda^{n+1}\Sigma_Y \otimes L^{-(n+1)} \overset{]\widetilde{ds}}{\to} \cdots \\ & \overset{]\widetilde{ds}}{\to} \Sigma_Y \otimes L^{-1} \overset{]\widetilde{ds}}{\to} \mathcal{O}_Y \to 0. \end{aligned} \tag{2.20}$$

Tensoring (2.20) with $E \otimes L^a$ and applying Lemma 1.4, we obtain a spectral sequence abutting to zero. For $L$ sufficiently ample, using the hypothesis (2.18), we get that

$$\ker\left(H^{n+1}\left(Y, \Lambda^{n+2}\Sigma_Y \otimes E \otimes L^{a-(n+2)}\right)\right.$$
$$\left.\to H^{n+1}\left(Y, \Lambda^{n+1}\Sigma_Y \otimes E \otimes L^{a-(n+1)}\right)\right)$$
$$\underset{\simeq}{\overset{d_{n+2}}{\to}} \frac{H^0(Y, E \otimes L^a)}{J_{E \otimes L^a}}$$

and thus

$$\left(\frac{H^0\left(Y, E^* \otimes K_Y^2 \otimes L^{((n+2)-a)}\right)}{J_{E^* \otimes K_Y^2 \otimes L^{(n+2-a)}}}\right)^* \simeq \frac{H^0(Y, E \otimes L^a)}{J_{E \otimes L^a}}.$$

When $E = 1$, $a = 0$, this gives (2.16). Moreover, because multiplication with $H^0(Y, E^* \otimes K_Y^2 \otimes L^{(n+2-a)})$ gives a map of the entire spectral sequence, we conclude that (2.17) gives the duality. The case where we have only the hypothesis (2.19) is similar. This proves (2.15).

We next generalize *Donagi's Symmetrizer Lemma* with the following two results:

THEOREM 2.21 (Generalized Symmetrizer Lemma): *Let $Y$ be a smooth $n+1$ fold, $L \to Y$ an analytic line bundle, $M \to Y$ an analytic line bundle with $|M|$ base-point free, and $E \to Y$ an analytic vector bundle. Then for $L$ sufficiently ample, the Koszul complex*

$$0 \to \frac{H^0(Y, E \otimes L)}{J_{E \otimes L}} \to H^0(Y, M)^* \otimes \frac{H^0(Y, E \otimes M \otimes L)}{J_{E \otimes M \otimes L}}$$

$$\to \Lambda^2 H^0(Y, M)^* \otimes \frac{H^0(Y, E \otimes M^2 \otimes L)}{J_{E \otimes M^2 \otimes L}} \tag{2.22}$$

*is exact as far as written above, provided that*

$$H^1(Y, E \otimes \Sigma) \to H^0(Y, M)^* \otimes H^1(Y, E \otimes M \otimes \Sigma) \tag{2.23}$$

*is injective.*

THEOREM 2.24: *For $L$ sufficiently ample, the Koszul complex*

$$0 \to H^0(Y, K_Y) \to \left(\frac{H^0(Y, L)}{J_L}\right)^* \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)}$$

$$\to \Lambda^2\left(\frac{H^0(Y, L)}{J_L}\right)^* \otimes H^1\left(Z, \Omega_Z^{n-1}\right) \tag{2.25}$$

*is exact as for as written, and*

$$0 \to \frac{H^0\left(Y, L \otimes K_Y^{-1}\right)}{J_{L \otimes K_Y^{-1}}} \to H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L)}{J_L}$$

$$\to \Lambda^2 H^0(Y, K_Y)^* \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \tag{2.26}$$

*is exact as far as written provided that* $|K_Y|$ *is base-point free and* $\dim \varphi_{K_Y}(Y) \geqslant 2$.

Proof of Theorems (2.21) and (2.24): Using the Generalized Macaulay's Theorem (2.15) and its proof, we have for $L$ sufficiently ample that the sequence (2.22) is dual to

$$\Lambda^2 H^0(Y, M) \otimes \frac{H^0\big(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}\big)}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}} + H^1(Y, E \otimes M^2 \otimes \Sigma)^*}$$

$$\to H^0(Y, M) \otimes \frac{H^0\big(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}\big)}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}} + H^1(Y, E \otimes M \otimes \Sigma)^*}$$

$$\to \frac{H^0\big(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}\big)}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}} + H^1(Y, E \otimes \Sigma)^*} \to 0 \tag{2.27}$$

The sequence

$$\Lambda^2 H^0(Y, M) \otimes H^0\big(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}\big)$$

$$\to H^0(Y, M) \otimes H^0\big(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}\big)$$

$$\to H^0\big(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}\big) \to 0 \tag{2.28}$$

is exact by considering the Koszul complex

$$\ldots \to \Lambda^2 H^0(Y, M) \otimes K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}$$

$$\to H^0(Y, M) \otimes K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}$$

$$\to K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \to 0$$

and applying Lemma 1.4 and the Kodaira Vanishing Theorem. Likewise,

$$H^0(Y, M) \otimes J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}} \twoheadrightarrow J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}}$$

by a similar argument, while

$$H^0(Y, M) \otimes H^1(Y, E \otimes M \otimes \Sigma)^* \twoheadrightarrow H^1(Y, E \otimes \Sigma)$$

by the dual of (2.23).

We are now done by

LEMMA 2.29: *Let $V$ be a vector space, $S(V)$ the symmetric algebra on $V$, and $A = \oplus_{q\in Z}A_q \subseteq B = \oplus_{q\in Z}B_q$ graded $S(V)$-modules. The Koszul complex*

$$\Lambda^2 V\otimes\left(B_q/A_q\right) \xrightarrow{F} V\otimes\left(B_{q+1}/A_{q+1}\right) \xrightarrow{G} B_{q+2}/A_{q+2}\to 0 \tag{2.30}$$

*is exact as far as written provided that*

$$\Lambda^2 V\otimes B_q \xrightarrow{\tilde F} V\otimes B_{q+1} \xrightarrow{\tilde G} B_{q+2}\to 0 \tag{2.31}$$

*is exact as far as written and*

$$V\otimes A_{q+1} \twoheadrightarrow A_{q+2}. \tag{2.32}$$

PROOF OF (2.29): As $\tilde G$ is surjective, so is $G$. If $\alpha\in\ker G$, then choosing $\tilde\alpha\in V\otimes B_{q+1}$ representing $\alpha$,

$$\tilde G(\tilde\alpha)\in A_{q+2}$$

By (2.32), we may modify $\tilde\alpha$ to $\tilde{\tilde\alpha}$ representing $\alpha$ so that

$$\tilde G(\tilde{\tilde\alpha})=0$$

So

$$\tilde{\tilde\alpha}=\tilde F(\tilde\beta)$$

for some $\tilde\beta\in\Lambda^2 V\otimes B_q$, and then

$$\alpha=F(\beta)$$

where $\beta$ is the projection of $\tilde\beta$ to $\Lambda^2 V\otimes(B_q/A_q)$. □

We have now proved (2.21). To prove (2.24), we will prove first the exactness of the sequence

$$0\to H^0(Y,K_Y)\to\left(\frac{H^0(Y,L)}{J_L}\right)^*\otimes\frac{H^0(Z,K_Z)}{H^0(Y,\Omega_Y^n)}$$

$$\to\Lambda^2\left(\frac{H^0(Y,L)}{J_L}\right)^*\otimes\frac{H^1\left(Z,\Omega_Z^{n-1}\right)}{H^1\left(Z,\Omega_Y^{n-1}\otimes\mathcal{O}_Z\right)}$$

which is stronger than showing exactness for (2.25). Using the dualities of §1, and the fact $H^0(Y,\Omega_Y^n)\twoheadrightarrow H^0(Z,\Omega_Y^n\otimes\mathcal{O}_Z)$ if $\dim Y\geqslant 2$ if $L\gg 0$

and the generalized Macauley's Theorem, we may dualize the above sequence to

$$\Lambda^2\left(\frac{H^0(Y, L)}{J}\right) \otimes \frac{H^0(Y, K_Y \otimes L^n)}{J_{K_Y \otimes L^n} + \text{more}}$$

$$\to \frac{H^0(Y, L)}{J_L} \otimes \frac{H^0\big(Y, K_Y \otimes L^{(n+1)}\big)}{J_{K_Y \otimes L^{(n+1)}} + \text{more}}$$

$$\to \frac{H^0\big(Y, K_Y \otimes L^{(n+2)}\big)}{J_{K_Y \otimes L^{(n+2)}}} \to 0$$

and we now proceed analogously to the preceding case, using Lemma 1.8, the Kodaira Vanishing Theorem, and Lemma 2.29. The one additional fact we require is that the sequence

$$\Lambda^2 H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^n)$$

$$\to H^0(Y, L) \otimes H^0\big(Y, K_Y \otimes L^{(n+1)}\big)$$

$$\to H^0\big(Y, K_Y \otimes L^{(n+2)}\big) \to 0$$

is exact as far as written for $L$ sufficiently ample. This follows from Lemma 2.47, which we have put at the end of this section.

The dual of (2.26) is

$$\Lambda^2 H^0(Y, K_Y) \otimes \frac{H^0\big(Y, K_Y \otimes L^{(n+1)}\big)}{J_{K_Y \otimes L^{(n+1)}} + \text{more}}$$

$$\to H^0(Y, K_Y) \otimes \frac{H^0\big(Y, K_Y^2 \otimes L^{(n+1)}\big)}{J_{K_Y^2 \otimes L^{(n+1)}} + H^1(Y, \Sigma)^*}$$

$$\to \frac{H^0\big(Y, K_Y^3 \otimes L^{(n+1)}\big)}{J_{K_Y^3 \otimes L^{(n+1)}} + H^1\big(Y, \Sigma \otimes K_Y^{-1}\big)^*} \to 0$$

and again we are done provided that

$$H^1\big(Y, \Sigma \otimes K_Y^{-1}\big) \to H^0(Y, K_Y)^* \otimes H^1(Y, \Sigma) \tag{2.33}$$

is injective. Applying Lemma 1.8 to the Koszul complex

$$0 \to \Sigma \otimes K_Y^{-1} \to H^0(Y, K_Y)^* \otimes \Sigma \to \Lambda^2 H^0(Y, K_Y)^* \otimes \Sigma \otimes K_Y \to \dots$$

we conclude that the injectivity of (2.33) is equivalent to proving

$$H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma) \to \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma \otimes K_Y)$$
$$\to \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma \otimes K_Y^2)$$

is exact at the middle term. From the exact sequence

$$0 \to \mathcal{O}_Y \to \Sigma_Y \to \Theta_Y \to 0$$

we have an exact sequence

$$0 \to H^0(Y, K_Y) \to H^0(Y, \Sigma \otimes K_Y) \to H^0(Y, \Omega_Y^n) \xrightarrow{\cup c_1(L)} H^1(Y, K_Y)$$

and the last map is an isomorphism by the Strong Lefschetz Theorem, so

$$H^0(Y, K_Y) \xrightarrow{\approx} H^0(Y, \Sigma \otimes K_Y)$$

and we are now reduced to showing

$$H^0(Y, K_Y)^* \otimes H^0(Y, \mathcal{O}_Y) \to \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y)$$
$$\to \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^2)$$

is exact at the middle term. However, for dim $\varphi_{K_Y} \geq 2$ this is true by the $\mathcal{K}_{p,1}$ Theorem of [G]. □

We have the following corollary of Theorem 2.21:

COROLLARY 2.34: *If $|K_Y|$ is base-point free and $Y$ is of general type, then for any $m \geq 1$ the Koszul complex*

$$0 \to \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \to H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L \otimes K_Y^{1-m})}{J_{L \otimes K_Y^{1-m}}}$$
$$\to \Lambda^2 H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L \otimes K_Y^{2-m})}{J_{L \otimes K_Y^{2-m}}}$$

*is exact as far as written for $k$ sufficiently large if* dim $\varphi_{K_Y}(Y) \geq 2$.

PROOF: Using (2.21), we need only show that hypothesis (2.23) holds in this case, i.e. that

$$H^1(Y, K_Y^{-m} \otimes \Sigma) \to H^0(Y, K_Y)^* \otimes H^1(Y, K_Y^{1-m} \otimes \Sigma)$$

is injective. As seen above, this is equivalent to the sequence

$$\begin{aligned} &H^0(Y, K_Y)^* \otimes H^0\big(Y, K_Y^{1-m} \otimes \Sigma\big) \\ &\quad \to \Lambda^2 H^0(Y, K_Y)^* \otimes H^0\big(Y, K_Y^{2-m} \otimes \Sigma\big) \\ &\quad \to \Lambda^3 H^0(Y, K_Y)^* \otimes H^0\big(Y, K_Y^{3-m} \otimes \Sigma\big) \end{aligned} \tag{2.36}$$

being exact at the middle term. For $m = 1$, we have already proved this.

As is well known,

$$H^0(Y, \Theta_Y) = 0 \quad \text{if } Y \text{ is of general type}$$

and thus

$$H^0\big(Y, K_Y^{2-m} \otimes \Sigma\big) = 0 \qquad \text{for} \quad m \geqslant 3$$

and

$$H^0\big(Y, K_Y^{2-m} \otimes \Sigma\big) \simeq H^0(Y, \mathcal{O}_Y) \qquad \text{for} \quad m = 2.$$

We are done in case $m \geqslant 3$, while for $m = 2$ we are reduced to the exactness of

$$0 \to \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, \mathcal{O}_Y) \to \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y)$$

at the middle term, and this follows from the fact that Koszul map $\Lambda^2 V \to \Lambda^3 V \otimes V^*$ is injective for any vector space. □

We are now ready to begin proving Theorem (0.3). The image of the derivative of the period map gives us the $GL$-equivalence class of the map

$$\frac{H^0(Y, L)}{J_L} \otimes H^0(Z, K_Z) \to H^1\big(Z, \Omega_Z^{n-1}\big). \tag{2.37}$$

We require

LEMMA 2.38: *The right kernel of* (2.37) *is the image of* $H^0(Y, \Omega_Y^n)$ *for* $L$ *sufficiently ample.*

PROOF: We know by Hodge theory that the image of $H^0(Y, \Omega_Y^n)$ in $H^0(Z, K_Z)$ is invariant as we deform $Z$ on $Y$. It remains to show that the map

$$\frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \to \left(\frac{H^0(Y, L)}{J_L}\right)^* \otimes H^1\big(Z, \Omega_Z^{n-1}\big)$$

is injective. A fortiori, it would be enough to show that

$$\frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \to \left(\frac{H^0(Y, L)}{J_L}\right)^* \otimes \frac{H^1(Z, \Omega_Z^{n-1})}{H^1(Z, \Omega_Y^{n-1} \otimes \mathcal{O}_Z)}$$

is injective. By the results of §1, for $L$ sufficiently ample this dualizes to the (quotiented) multiplication map

$$\frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Y, K_Y \otimes L^n)}{J_{K_Y \otimes L^n} + \text{more}} \to \frac{H^0(Y, K_Y \otimes L^{n+1})}{J_{K_Y \otimes L^{n+1}} + \text{more}}$$

which we must show is surjective. It suffices to show that

$$H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^n) \twoheadrightarrow H^0(Y, K_Y \otimes L^{n+1})$$

This follows from Lemma 1.28, for $L$ sufficiently ample. □

Thus, from (2.37), we can construct the map

$$\frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \to H^1(Z, \Omega_Z^{n-1}).$$

From this, we can construct the second map in the sequence (2.25), and thus can reconstruct the $GL$-equivalence class of the map

$$H^0(Y, K_Y) \otimes \frac{H^0(Y, L)}{J_L} \to \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)}. \tag{2.39}$$

From (2.39), we can construct the second map in the sequence (2.26), and thus can reconstruct the vector space

$$\frac{H^0(Y, L \otimes K_Y^{-1})}{J_{L \otimes K_Y^{-1}}}$$

and the $GL$-equivalence class of the map

$$\frac{H^0(Y, L \otimes K_Y^{-1})}{J_{L \otimes K_Y^{-1}}} \otimes H^0(Y, K_Y) \to \frac{H^0(Y, L)}{J_L}.$$

For any $m_0$ chosen in advance, we can choose $L$ sufficiently ample so that we can recover inductively using Corollary (2.24) the $GL$-equivalence classes of the maps

$$\frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \otimes H^0(Y, K_Y) \to \frac{H^0(Y, L \otimes K_Y^{1-m})}{J_{L \otimes K_Y^{1-m}}} \tag{2.40}$$

for all $m \leqslant m_0$.

However,

$$J_{L \otimes K_Y^{-m}} \simeq H^0(Y, \Sigma \otimes K_Y^{-m}).$$

If $K_Y$ is ample, we conclude that for some $m_1$,

$$J_{L \otimes K_Y^{-m}} = 0 \text{ for } m \geqslant m_1 \tag{2.41}$$

and the $m_1$ may be chosen independent of $L$. We now have the maps

$$H^0(Y, L \otimes K_Y^{-m}) \otimes H^0(Y, K_Y) \to H^0\left(Y, L \otimes K_Y^{1-m}\right) \tag{2.42}$$

for $m_0 \geqslant m \geqslant m_1 + 1$, where we can make $m_0$ as large as we like at the expense of our choice of how ample $L$ must be. For

$$W \subseteq H^0(Y, K_Y)$$

a linear subspace with base locus $B$, we have by making $L$ sufficiently ample that

$$H^0(Y, L \otimes K_Y^{-m}) \otimes W \twoheadrightarrow H^0\left(Y, L \otimes K_Y^{1-m} \otimes \mathscr{I}_B\right). \tag{2.43}$$

Thus we can detect from the map (2.42) which $W$'s have a base locus, and thus can determine the Chow form of $\varphi_{K_Y}(Y)$. Since $\varphi_{K_Y}(Y) \simeq Y$ by hypothesis, for each $p \in Y$ we can determine by (2.42) the subspaces

$$H^0\left(Y, L \otimes K_Y^{1-m} \otimes \mathscr{I}_p\right) \subset H^0\left(Y, L \otimes K_Y^{1-m}\right)$$

for $m_0 \geqslant m \geqslant m_1 + 1$. From this, we can determine the Chow form of $\varphi_{L \otimes K_Y^{1-m}}(Y)$. We thus can reconstruct the map (2.42) not merely up to $GL$-equivalence, but up to the action of $G$. Now, operating our induction in reverse, we can construct the projections

$$H^0(Y, L \otimes K_Y^{-m}) \to \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \tag{2.44}$$

for all $m \leqslant m_0$, modulo the action of $G$. Thus in particular, we can construct

$$J_L \subset H^0(Y, L)$$

modulo the action of $G$.

We now wish to recover $Z$ from $J_L$. In the case at hand, this is easy as $H^0(Y, \Theta_Y) = 0$; however, it is interesting to give the general argument.

To do this, we consider the group $\tilde{G}$ consisting of pairs of analytic isomorphisms

$$\begin{array}{ccc} L & \xrightarrow{\tilde{u}} & L \\ \downarrow & & \downarrow \\ Y & \xrightarrow{u} & Y \end{array}$$

so that the diagram commutes and $\tilde{u}$ is linear on each fiber. As

$$H^0(Y, \mathrm{Hom}(L, L)) \simeq \mathbf{C}$$

there is a natural exact sequence of Lie groups

$$0 \to \mathbf{C}^* \to \tilde{G} \to G \to 0. \tag{2.45}$$

Furthermore, we may make the identifications of the tangent spaces at the identity

$$\begin{array}{ccccc} T_e(\mathbf{C}^*) & \to & T_e(\tilde{G}) & \to & T_e(G) \\ \wr\| & & \wr\| & & \wr\| \\ H^0(Y, \mathcal{O}_Y) & \to & H^0(Y, \Sigma_Y) & \to & \ker\left(H^0(Y, \Theta_Y) \underset{\cup c_1(L)}{\to} H^1(Y, \mathcal{O}_Y)\right) \end{array}$$

so that the above diagram commutes. Further, $\tilde{G}$ acts on $H^0(Y, L)$ by pullback, so that the tangent space to the orbit of $s$ is

$$T_s(\tilde{G}s) \simeq J_L. \tag{2.46}$$

Gives this, the argument given by Donagi [D] adapting techniques of Mather and Yau [M-Y] goes through verbatim. This completes the proof of Theorem (0.3), once we have shown the lemma needed to prove the exactness of (2.25).

LEMMA 2.47: *Let $Y$ be a smooth $(n+1)$-fold, $E \to Y$ an analytic vector bundle. If $L \to Y$ is a sufficiently ample vector bundle, then for any $a \geqslant 1$, the Koszul sequence*

$$\Lambda^2 H^0(Y, L) \otimes H^0(Y, E \otimes L^a) \xrightarrow{\alpha_a} H^0(Y, L) \otimes H^0(Y, E \otimes L^{a+1}) \xrightarrow{\beta_a} H^0(Y, E \otimes L^{a+2}) \to 0 \tag{2.48}$$

*is exact as far as written.*

PROOF: Surjectivity of $\beta_a$ follows from Lemma 1.28. Exactness at the middle term would follow from the surjectivity of the map

$$H^0(Y, L)\otimes \ker \beta_{a-1} \xrightarrow{\gamma_a} \ker \beta_a \tag{2.49}$$

defined by

$$\gamma_a\left(l\otimes\left(\sum_i s_i\otimes r_i\right)\right)=\sum_i s_i\otimes (lr_i)$$

where $l\in H^0(Y, L)$, $s_0,\dots,s_r$ a basis for $H^0(Y, L)$ and $r_i\in H^0(Y, E\otimes L^a)$. To see this implication, we note that there is a commutative diagram

$$\begin{array}{ccc} & \overset{\delta_a}{\swarrow} & H^0(Y, L)\otimes \ker\beta_{a-1} \\ & & \downarrow \gamma_a \\ \Lambda^2 H^0(Y, L)\otimes H^0(Y, E\otimes L^a) & \xrightarrow{\alpha_a} & \ker\beta_a \end{array}$$

where

$$\delta_a\left(l\otimes\left(\sum_i s_i\otimes r_i\right)\right)=\sum_i (l\wedge s_i)\otimes r_i .$$

Thus

$$\operatorname{im}\gamma_a\subseteq \operatorname{im}\alpha_a$$

so it will suffice for our purposes to show that $\gamma_a$ is surjective.

On $Y\times Y$, let $\Delta$ be the diagonal and $\pi_1$, $\pi_2$ the canonical projections. From the exact sequence

$$0\to \mathcal{O}_{Y\times Y}(-\Delta)\to \mathcal{O}_{Y\times Y}\to \mathcal{O}_\Delta\to 0 \tag{2.51}$$

tensored with $\pi_1^*(L)\otimes\pi_2^*(E\otimes L^{a+1})$, we conclude that

$$\ker\beta_a\simeq H^0\big(Y\times Y, \pi_1^*(L)\otimes\pi_2^*(E\otimes L^{a+1})\otimes\mathcal{O}_{Y\times Y}(-\Delta)\big). \tag{2.52}$$

Further, $\gamma_a$ in these terms is the map

$$\begin{aligned} &H^0\big(Y\times Y, \pi_2^*(L)\big)\otimes H^0\big(Y\times Y, \pi_1^*(L) \\ &\quad\otimes\pi_2^*(E\otimes L^a)\otimes\mathcal{O}_{Y\times Y}(-\Delta)\big) \\ &\quad\xrightarrow{\tilde{\gamma}_a} H^0\big(Y\times Y, \pi_1^*(L)\otimes\pi_2^*(E\otimes L^{a+1})\otimes\mathcal{O}_{Y\times Y}(-\Delta)\big). \end{aligned} \tag{2.53}$$

On $Y\times Y\times Y$, let

$$\Delta_{ij}=\{(y_1, y_2, y_3)\in Y\times Y\times Y \mid y_i=y_j\}$$

and let $\pi_1$, $\pi_2$, $\pi_3$ be the projections. We may rewrite $\tilde{\gamma}_a$ equivalently as the map

$$H^0(Y\times Y\times Y, \pi_3^*(L))\otimes H^0(Y\times Y\times Y, \pi_1^*(L)$$

$$\otimes \pi_2^*(E\otimes L^a)\otimes \mathcal{O}_{Y\times Y\times Y}(-\Delta_{12}))$$

$$\xrightarrow{\hat{\gamma}_a} H^0\big(Y\times Y\times Y, \pi_1^*(L)\otimes \pi_2^*(E\otimes L^a)\otimes \pi_3^*(L)$$

$$\otimes \mathcal{O}_{Y\times Y\times Y}(-\Delta_{12})\otimes \mathcal{O}_{\Delta_{23}}\big). \tag{2.54}$$

From tensoring the restriction sequence for $\Delta_{23}$ on $Y\times Y\times Y$ appropriately, we see that $\hat{\gamma}_a$ is surjective if

$$H^1\big(Y\times Y\times Y, \pi_1(L)\otimes \pi_2(E\otimes L^a)\otimes \pi_3(L)$$

$$\otimes \mathcal{O}_{Y\times Y\times Y}(-\Delta_{12}-\Delta_{23})\big)=0. \tag{2.55}$$

For $L$ sufficiently ample, (2.55) holds for all $a\geqslant 1$. This proves Lemma 2.47. □

## References


[A-C-G-H] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris: *Special Divisors on Algebraic Curves*, to appear.

[C-G-Gr-H] J. Carlson, M. Green, P. Griffiths and J. Harris: Infinitesimal variations of Hodge structures. *Comp. Math.* 50 (1983) 109–205.

[D] R. Donagi: Generic Torelli for projective hypersurfaces. *Comp. Math.* 50 (1983) 325–353.

[G] M. Green: Koszul cohomology of projective varieties, J. Diff. Geom 19 (1984) 125–171.

[G-H] P. Griffiths and J. Harris: *Principles of Algebraic Geometry*, John Wiley and Sons (1978).

[M-Y] J. Mather and S. Yau: Classification of isolated hypersurface singularities by their moduli algebras: *Invent. Math.* 69 (1982) 243–251.


(Oblatum 20-IV-1983 & 14-II-1984)

Department of Mathematics
University of California at Los Angeles
Los Angeles, California
USA