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A CLASSIFICATION OF THE NILPOTENT TRIANGULAR MATRICES

Wim H. Hesselink

1. Introduction

1.1. Questions of linear algebra

Let x be a square matrix of order n with coefficients in a field K . Assume that x is nilpotent, say with $x^r = 0$. Since all eigenvalues of x are zero, x has a Jordan canonical form y with n zeros on the main diagonal. The matrices x and y are similar, so $y = gxg^{-1}$ where g is invertible. On the other hand, the matrix y is strictly upper triangular. So, one might ask for a description of the set of all invertible matrices g such that gxg^{-1} is strictly upper triangular, or for a classification of the strictly upper triangular matrices up to conjugation by invertible upper triangular matrices.

We consider the matrices of order n as endomorphisms of the vector space K^n . Let $F_\star = (F_r)_r$ be the standard flag in K^n ; so $F_r = \sum_{i=1}^r Ke_i$, where $e_1 \dots e_n$ is the standard basis of K^n . A matrix x is strictly upper triangular if and only if $xF_r \subset F_{r-1}$ for every index $r > 0$. Slightly more general, let F_\star be an arbitrary flag in a vector space V with $\dim(V) = n$. So we have

$$F_\star: 0 = F_0 \subset F_1 \subset F_2 \subset F_3 \subset \dots \subset F_n = V$$

with $\dim(F_r) = r$. The corresponding nilalgebra is defined by

$$N(F_\star) = \{x \in \text{End}(V) : \forall r > 0 : xF_r \subset F_{r-1}\}.$$

The above questions are almost equivalent to the following ones.

1. Given a nilpotent endomorphism x of V , describe the set $Y(x)$ of all flags F_\star with $x \in N(F_\star)$.

2. Characterize all pairs (x, F_\star) with $x \in N(F_\star)$, up to automorphisms of V .

1.2. Geometrical motivation

Let N be the set of the nilpotent endomorphisms of the vector space V . Let Y be the set of all flags F_* in V . Let X be the set of the pairs (x, F_*) with $x \in N(F_*)$. If the base field K is algebraically closed, then N is an irreducible algebraic variety with singularities, Y is a smooth projective variety, and X is a vector bundle over Y , isomorphic to the cotangent bundle T^*Y . The projection $\pi: X \rightarrow N$ given by $\pi(x, F_*) = x$ is a desingularization of the nilpotent variety N . It is called the Springer resolution. It is extensively studied in the literature, cf. [2,3,8,9,10,11,12,14]. In particular, one is interested in the structure of the fibers

$$\pi^{-1}(x) \cong Y(x) = \{ F_* \in Y : x \in N(F_*) \},$$

cf. question 1 of 1.1. It is known that $Y(x)$ is a connected algebraic variety and that its irreducible components are parametrized by the standard tableaux in the diagram of x , cf. [9,10,14]. We would like to describe the intersections of the components of $Y(x)$. This requires more detailed information about the set X . In this paper, we propose a finite classification of the elements of X , more or less in the direction of the above question 2.

1.3. Systems of partitions. Occurrence

Every pair $(x, F_*) \in X$ induces a family of nilpotent endomorphisms of the subquotients F_q/F_p . Since a nilpotent endomorphism is characterized by a partition, the pair (x, F_*) induces a system of partitions $(\tau[p, q])_{p < q}$. These systems form the subject of this paper.

If $\tau = (\tau[p, q])_{p < q}$ is a system of partitions, then we write $X(\tau)$ to denote the set of the pairs (x, F_*) which have τ as induced system of partitions. If $X(\tau)$ is non-empty, then we say that τ occurs. The partition $\lambda = \tau[0, n]$ is called the *global partition* of the system of partitions τ . If x is an endomorphism of V with partition λ , then we write $Y(x, \tau)$ to denote the set of the flags $F_* \in Y(x)$ such that $(x, F_*) \in X(\tau)$. It is clear that $Y(x, \tau)$ is non-empty if and only if τ occurs.

The question which systems τ occur, is wild. In 5.7, 9.3, 9.4, 9.6, we give examples to show that occurrence may depend on the choice of the base field K . On the other hand, we have obtained certain conditions, which are independent of the base field, and which are either necessary, or sufficient for occurrence.

1.4. Acceptable systems of partitions. Typrices

The information contained in a system of partitions τ is reorganized in the form of a strict upper triangular matrix $A = (a_{pq})$, with integer coefficients. If the system of partitions occurs, then the matrix A consists

only of zeros and ones. Therefore, we define a typrix to be a strict upper triangular matrix consisting of zeros and ones. We say that the system of partitions τ is *acceptable*, if the corresponding matrix A is a typrix.

Conversely, a typrix A is called *acceptable*, if there is a (necessarily unique) system of partitions τ which induces the matrix A . In this way, we obtain a bijective correspondence between the acceptable systems of partitions and the acceptable typrices. A typrix A is said to *occur*, if it is acceptable and if the corresponding system of partitions occurs. Now we also have a bijective correspondence between the occurring systems of partitions and the occurring typrices.

Let τ be an occurring system of partitions, say with $(x, F_*) \in X(\tau)$ and with a typrix A . The typrix A should not be thought of as representing a linear transformation. On an intuitive level, however, the features of the endomorphism $x \in N(F_*)$ are reflected brightly in the typrix, cf. 3.4 up to 3.7. Moreover, the typrix A is more easily memorized than the system of partitions τ .

1.5. Sufficient conditions

A typrix A is called *elementary*, cf. 4.4, if it corresponds to an equivalence relation on the indices $1 \dots n$. The construction of 4.3 shows that every elementary typrix occurs. This is independent of the base field K .

In 1975, N. Spaltenstein determined the irreducible components of the variety $Y(x)$, in the case that K is algebraically closed, cf. [9,10]. Roughly speaking, the components are indexed by the set $St(\lambda)$ of the (standard) tableaux T in the partition λ of x . In section 5, we construct an injective mapping γ which associates to a tableau $T \in St(\lambda)$ a system of partitions $\tau = \gamma(T)$ such that the set $Y(x, \tau)$ is open and dense in the "Spaltenstein component" $Y(x)_T$ of $Y(x)$. Therefore, a system of partitions τ will be called *generic*, if $\tau = \gamma(T)$ for some tableau T . Clearly, if the field K is algebraically closed, all generic systems of partitions occur. In fact, it suffices that the field K is infinite.

1.6. Necessary conditions for occurrence

In theorem 6.6, we obtain combinatorial conditions which are necessary for the occurrence of a given system of partitions τ . The conditions are expressed in matrices R and S , closely related to the typrix A of τ . The theorem is based on the following result.

PROPOSITION (6.2): *Let Q and R be subspaces of V . Let $V_1 \subset \dots \subset V_m$ be a nested sequence of subspaces of V . Then we have*

$$\sum_{i=1}^m (-1)^i (|V_i + Q| - |V_i + R|) \leq |(R + Q)/Q|,$$

where $|W| = \dim(W)$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Diagram 1. A Jordan matrix, $n = 7$, $\lambda_* = (3, 2, 1, 1)$, see 2.3.

The conditions described in theorem 6.6 can be verified by a computer. In this way, we generated lists of possibly occurring typrices of order $n \leq 7$. The methods and the results are discussed briefly in section 7.

1.7. One special component

For the investigation of at least some high dimensional cases, we have to make extra assumptions. In section 8, we therefore consider one special component of the variety $Y(x)$. As Spaltenstein has observed, the number of moduli of $Y(x)$ may grow quadratically with n . In section 9, the investigation is further specialized. There we show that the question which typrices occur, is equivalent to the question which systems of polynomial equations with integer coefficients have solutions in a given field. This proves that the question of occurrence is wild.

Recollections

Inspired by Spaltenstein's paper [9], I obtained the genericity theorem (5.5) in June 1976. My interest in the matter was kept alive by questions of De Concini (in Hamburg, November 1980) and Bürgstein (in Oberwolfach, March 1982). It was not before September 1982, however, that I could make real progress. Then the representation by means of typrices was discovered and a computer program was conceived.

2. Systems of partitions

2.1. CONVENTIONS: The set of the positive integers is denoted by \mathbb{N} . We write $[p \dots q]$ to denote the set of the integers $p, p+1, \dots, q$. The number of elements of a set S is denoted by $\square S$. The symbol K stands for an arbitrary field and V is a vector space over K with $\dim(V) = n$.

2.2. Partitions, cf. [7], p. 1

A *partition* $\lambda_* = (\lambda_1, \dots, \lambda_r, \dots)$ is a sequence of non-negative integers, in decreasing order, containing only finitely many non-zero terms,

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \dots \geq 0.$$

We do not distinguish between two such sequences which differ only by a string of zeros at the end. The sum of the terms is called the *order* or *weight* of the partition, denoted by

$$|\lambda| = \sum_{i=1}^{\infty} \lambda_i.$$

We define the corresponding *diagram* λ to be the subset of \mathbb{N}^2 consisting of the pairs (i, j) with $j \leq \lambda_i$. So λ is a finite subset of order $|\lambda| = |\lambda|$. The *conjugate* sequence $\lambda^* = (\lambda^1, \dots, \lambda^r, \dots)$ is defined by

$$\lambda^j = \square \{ i \mid j \leq \lambda_i \}$$

($= \lambda'_j$, cf. [7], p. 2). So we have

$$j \leq \lambda_i \Leftrightarrow (i, j) \in \lambda \Leftrightarrow i \leq \lambda^j.$$

Usually we identify the partition λ_* and the diagram λ .

2.3. The partition of a nilpotent endomorphism

Let x be a nilpotent endomorphism of the vector space V . Since all eigenvalues of x are zero, x has a Jordan canonical form which is a matrix with zeros on the main diagonal. See for example diagram 1. Let $\lambda_1, \dots, \lambda_r$ be the sizes of the Jordan blocks of this matrix, in decreasing order. Then λ_* is a partition of order n . The corresponding diagram λ is a convenient index set for a basis of the vector space V . In fact, it is easy to construct a basis $e(i, j)$, $(i, j) \in \lambda$, such that

$$\begin{aligned} xe(i, 1) &= 0 && \text{if } (i, 1) \in \lambda, \\ xe(i, j) &= e(i, j-1) && \text{if } j > 1. \end{aligned}$$

It follows that the conjugate sequence λ^* satisfies

$$\lambda^j = \dim(x^{j-1}V/x^jV) \quad \text{and} \quad \dim(x^jV) = \sum_{r=j+1}^{\infty} \lambda^r.$$

So the partition λ is completely determined by the endomorphism x , and we may write $\lambda = \lambda(x)$. The partition $\lambda(x)$ characterizes the similarity class of x . In fact, two nilpotent endomorphisms x and y of V are similar if and only if $\lambda(x) = \lambda(y)$.

2.4. Invariant subspaces

Let x be a nilpotent endomorphism of V . Let W be an invariant subspace of V . The induced endomorphisms of W and V/W are denoted

by $x: W$ and $x: V/W$, respectively. These endomorphisms are nilpotent. So we have induced partitions, say $\mu = \lambda(x: W)$ and $\nu = \lambda(x: V/W)$. The following lemma is easy and well known, cf. [7], p. 91.

LEMMA: $\mu \subset \lambda$ and $\nu \subset \lambda$ and $|\mu| + |\nu| = |\lambda|$.

REMARK: The assertion $\mu \subset \lambda$ is equivalent with $\forall i: \mu_i \leq \lambda_i$, and also with $\forall i: \mu^i \leq \lambda^i$.

2.5. Systems of partitions

Let $(x, F_*) \in X$, cf. 1.2. So F_* is a flag in V and x is an endomorphism of V with $xF_r \subset F_{r-1}$ for every index r . We associate to the pair (x, F_*) the discrete invariant $\tau = \tau(x, F_*)$, which is the family of the partitions

$$\tau[p, q] = \lambda(x: F_q/F_p), 0 \leq p \leq q \leq n.$$

It follows with 2.4 that we have

LEMMA: (a) $\tau[0, n] = \lambda(x)$ and $|\tau[p, q]| = q - p$. (b) If $1 \leq p \leq q \leq n$, then $\tau[p, q] \subset \tau[p-1, q]$ and $\tau[p-1, q-1] \subset \tau[p-1, q]$.

DEFINITION: A family of partitions $\tau[p, q]_{p \leq q}$ with the properties of this lemma is called a *system of partitions* of order n with global partition $\lambda(x)$. If τ is a system of partitions, we write $Y(x, \tau)$ to denote the set of the flags $F_* \in Y(x)$ with $\tau(x, F_*) = \tau$. The system τ is said to *occur* if $Y(x, \tau)$ is non-empty. If the field K is algebraically closed, the system τ is said to be *generic* if $Y(x, \tau)$ is dense in an irreducible component of the variety $Y(x)$.

REMARK: Occurrence and genericity are independent of the choice of x in its similarity class. It turns out that genericity of τ is independent of the field K , cf. 5.6 below.

2.6. How to represent a system of partitions?

Let $\tau = \tau[p, q]_{p \leq q}$ be an arbitrary system of partitions of order n . If n is not too small, then τ is an insurveyable heap of information. So we have to reorganize the information without disturbing the main structural features. To this end we introduce four upper triangular matrices of order n , each containing the same information.

The matrices $R = (r_{pq})$ and $S = (s_{pq})$ are derived from the pair of inclusions of lemma 2.5(b). In both cases, the complement of the subset in the containing diagram consists of one element. So, if we compare the conjugate sequences $\tau[p, q]^*$ for the respective values of p and q , then

we get unique numbers r_{pq} and s_{pq} such that

$$r = r_{pq} \Leftrightarrow \tau[p-1, q-1]^r < \tau[p-1, q]^r,$$

$$s = s_{pq} \Leftrightarrow \tau[p, q]^s < \tau[p-1, q]^s.$$

It follows that $r_{pp} = s_{pp} = 1$. If $p > q$ then we define $r_{pq} = s_{pq} = 0$. Now $R = (r_{pq})$ and $S = (s_{pq})$ are upper triangular matrices of order n . Both R and S determine the system of partitions τ . In fact, if $0 \leq p \leq q \leq n$, then

$$\tau[p, q]^r = \square\{j \in [p+1 \dots q] | r_{p+1, j} = r\},$$

$$\tau[p, q]^r = \square\{i \in [p+1 \dots q] | s_{i, q} = r\}.$$

In order to clarify the relationship between R and S , we introduce the upper triangular matrix $T = (t_{pq})$ of order n , given by

$$t_{pq} = \sum_{r=1}^{\infty} r \tau[p-1, q]^r.$$

If $1 \leq p < q \leq n$, then

$$r_{pq} = \sum_{r=1}^{\infty} r (\tau[p-1, q]^r - \tau[p-1, q-1]^r) = t_{pq} - t_{p, q-1}.$$

$$s_{pq} = \sum_{s=1}^{\infty} s (\tau[p-1, q]^s - \tau[p, q]^s) = t_{pq} - t_{p+1, q}.$$

It follows that R and S are equal to the matrix products $R = TE$ and $S = ET$, where $E = (e_{pq})$ is the fixed upper triangular matrix with $e_{pp} = 1$ and $e_{p, p+1} = -1$ and $e_{pq} = 0$ if $q \neq p, p+1$. In order to recover the symmetry, we introduce a fourth matrix

$$A = (a_{pq}) = ETE - I = ER - I = SE - I,$$

where I is the identity matrix of order n . Clearly, A is a strictly upper triangular matrix. If $p < q$, then

$$a_{pq} = r_{pq} - r_{p+1, q} = s_{pq} - s_{p, q-1}.$$

The matrices R and S are recovered as follows. If $p \leq q$ then

$$r_{pq} = 1 + \sum_{i=p}^{q-1} a_{iq} \quad \text{and} \quad s_{pq} = 1 + \sum_{j=p+1}^q a_{pj}.$$

The matrices R, S, T, A are called the *characteristic matrices* of the system of partitions τ . We use the matrix A to represent the information

contained in τ , rather than the matrices R , S , or T . This preference is justified by the remarkable fact, that if the system τ occurs, then the matrix A consists only of zeros and ones, cf. 3.1(c) below.

3. Typrices

3.1. PROPOSITION: *Let a pair $(x, F_*) \in X$ have the system of partitions $\tau = \tau(x, F_*)$. Let R, S, T, A be the characteristic matrices of τ . Let $1 \leq p \leq q \leq n$. Then we have*

- (a) $r_{pq} = \min\{i \mid x'F_q \subset x'F_{q-1} + F_{p-1}\}$.
- (b) $s_{pq} = \min\{i \mid F_p \not\subset x'F_q + F_{p-1}\}$.
- (c) $a_{pq} = 0$ or $a_{pq} = 1$.

PROOF: (a) By 2.6 we have

$$r_{pq} = \min\left\{i \mid \sum_{r=i+1}^{\infty} \tau[p-1, q-1]^r = \sum_{r=i+1}^{\infty} \tau[p-1, q]^r\right\}.$$

Using 2.3, we obtain

$$\begin{aligned} r_{pq} &= \min\left\{i \mid \text{rank}(x' : F_{q-1}/F_{p-1}) = \text{rank}(x' : F_q/F_{p-1})\right\} = \\ &= \min\left\{i \mid x'F_{q-1} + F_{p-1} = x'F_q + F_{p-1}\right\} = \\ &= \min\left\{i \mid x'F_q \subset x'F_{q-1} + F_{p-1}\right\}. \end{aligned}$$

(b) Similarly, we have

$$\begin{aligned} s_{pq} &= \min\left\{i \mid \sum_{s=i+1}^{\infty} \tau[p, q]^s = \sum_{s=i+1}^{\infty} \tau[p-1, q]^s\right\} = \\ &= \min\left\{i \mid \text{rank}(x^i : F_q/F_p) = \text{rank}(x^i : F_q/F_{p-1})\right\} = \\ &= \min\left\{i \mid \dim(x'F_q + F_p/x'F_q + F_{p-1}) = 1\right\} = \\ &= \min\left\{i \mid F_p \not\subset x'F_q + F_{p-1}\right\}. \end{aligned}$$

(c) We may assume $p < q$. So $a_{pq} = r - t$ where $r = r_{pq}$ and $t = r_{p+1, q}$. By part (a) we have

$$x'F_q \subset x'F_{q-1} + F_{p-1} \subset x'F_{q-1} + F_p.$$

By (a) this implies that $t \leq r$. Since $xF_p \subset F_{p-1}$, we have

$$x^{t+1}F_q \subset x(x^tF_{q-1} + F_p) \subset x^{t+1}F_{q-1} + F_{p-1}.$$

By (a) this implies that $r \leq t + 1$. This proves that r equals t or $t + 1$, so that a equals 0 or 1.

3.2. DEFINITION: A strictly upper triangular matrix only consisting of zeros and ones is called a *typrix*. A typrix A is said to be *acceptable*, if it is the characteristic A -matrix of a (necessarily unique) system of partitions τ . The typrix is said to *occur* (to be *generic*), if it is acceptable and the corresponding system of partitions occurs (is generic). Conversely, a system of partitions τ is said to be *acceptable*, if its characteristic A -matrix is a typrix.

REMARKS: By 3.1(c), every occurring system of partitions is acceptable. The acceptable systems of partitions are in bijective correspondence with the acceptable typrices. Similar statements hold for the occurring ones, and for the generic ones. The word typrix is a neologism, it is a contraction of the words type and matrix.

EXAMPLE: There are eight typrices of order three. There are six systems of partitions of order three. Five of the systems are acceptable. The only unacceptable one is the system τ with $\tau[0, 3]^* = (2, 1)$ and $\tau[0, 2]^* = \tau[1, 3]^* = (1, 1)$. In fact, the characteristic A -matrix of τ has the coefficient $a_{13} = -1$.

3.3. Typrices need not be self representing

Let x be a strictly upper triangular matrix with entries in the field K . The matrix x represents a linear transformation x of the space K^n which leaves the standard flag F_* of K^n invariant, cf. 1.1. So we have an occurring system of partitions $\tau = \tau(x, F_*)$ with a characteristic typrix A . Now A is a strictly upper triangular matrix of zeros and ones, which looks more or less the same as the original matrix x . Usually, however, A is of a different type, in the sense that $\tau \neq \tau(A, F_*)$. A typrix A is said to be *self representing*, if A is the characteristic typrix of the system of partitions $\tau(A, F_*)$.

EXAMPLE: Consider the typrices of order four:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A' = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A'' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The typrix A is not acceptable. A' is the characteristic typrix of the system of partitions $\tau(A, F_\star)$ induced by the matrix A . The system of partitions $\tau(A', F_\star)$ has the characteristic typrix A'' . The typrix A'' is self representing.

3.4. Duality

Let τ be a system of partitions of order n with characteristic matrices R, S, T, A . We define the dual system of partitions τ' by

$$\tau'[p, q] = \tau[n - q, n - p].$$

Let R', S', T', A' be the characteristic matrices of τ' . One easily verifies that

$$R' = f(S), \quad S' = f(R), \quad T' = f(T), \quad A' = f(A),$$

where f is the reflection in the skew diagonal which transforms a square matrix $M = (m_{ij})$ of order n into the matrix

$$f(M) = (m'_{ij}) \quad \text{with} \quad m'_{ij} = m_{n+1-j, n+1-i}.$$

If the system of partitions τ occurs, then the dual system τ' also occurs. In fact, assume that $\tau = \tau(x, F_\star)$. Let x' be the induced endomorphism of the dual vector space V' of V . Let F'_\star be the dual flag, which is given by

$$F'_i = \{u \in V' \mid u(F_{n-i}) = 0\}.$$

Then F'_\star is an x' -invariant flag with system of partitions $\tau' = \tau(x', F'_\star)$. In particular, $f(A)$ is the characteristic typrix.

3.5. Ancestry relations

If τ is a system of partitions of order n , then we define a *lefthand subsystem* $le(\tau)$ and a *righthand subsystem* $ri(\tau)$. These are the systems of partitions of order $n - 1$ given by

$$\left. \begin{aligned} le(\tau)[p, q] &= \tau[p, q] \\ ri(\tau)[p, q] &= \tau[p + 1, q + 1] \end{aligned} \right\} \quad \text{where } 0 \leq p \leq q \leq n - 1.$$

If τ has the characteristic matrices R, S, T, A , then $le(\tau)$ and $ri(\tau)$ have the characteristic matrices

$$le(R), \dots, le(A), ri(R), \dots, ri(A),$$

where le and ri also denote the corresponding stripping operators on matrices: the operator le strips the last row and the last column, whereas ri strips the first row and the first column.

If the system of partitions τ occurs, then the subsystems $le(\tau)$ and $ri(\tau)$ also occur. In fact, assume that $\tau = \tau(x, F_\star)$. Then $le(\tau) = \tau(x', F'_\star)$ where F'_\star is the flag $(F_i)_{i \leq n-1}$ in F_{n-1} and x' is the restriction $x: F_{n-1}$. Similarly, $ri(\tau) = \tau(x'', F''_\star)$ where x'' is the induced endomorphism of V/F_1 and $F''_i = F_{i+1}/F_1$. The pairs (x', F'_\star) and (x'', F''_\star) are called the *ancestors* of (x, F_\star) .

3.6. The remaining bit

If we want to determine the characteristic typrix A of a pair (x, F_\star) , then the minors $le(A)$ and $ri(A)$ are determined by the ancestors of (x, F_\star) . So the only new information is contained in the top righthand entry a_{1n} . The following result may be useful.

COROLLARY: *In the situation of 3.1, we have $a_{1n} = 1$ if and only if there exists a number m with $x^m V = F_1 \oplus x^m F_{n-1}$.*

PROOF: We have $a_{1n} = 1$ if and only if $r_{1n} > r_{2n}$. Now 3.1(a) reads

$$r_{1n} = \min\{i \mid x^i V \subset x^i F_{n-1}\},$$

$$r_{2n} = \min\{i \mid x^i V \subset x^i F_{n-1} + F_1\}.$$

Since $x^i F_{n-1} \subset x^i V$, the assertion follows.

3.7. The extreme cases

The analysis of the following extreme cases may be left to the reader.

COROLLARY: *Assume $(x, F_\star) \in X$ has characteristic typrix A .*

(a) $A = 0 \Leftrightarrow x = 0$.

(b) *We have $a_{ij} = 1$ for all pairs $i < j$, if and only if $F_i = x^{n-i} V$ for all indices i .*

3.8. Acceptability of a given typrix

Let $A = (a_{pq})$ be a typrix of order n . Let $R = (r_{pq})$ and $S = (s_{pq})$ be defined as follows, cf. 2.6. If $p > q$ then $r_{pq} = s_{pq} = 0$. If $p \leq q$ then

$$r_{pq} = 1 + \sum_{i=p}^{q-1} a_{iq}, \quad s_{pq} = 1 + \sum_{j=p+1}^q a_{pj}.$$

LEMMA: *The typrix A is acceptable if and only if it satisfies the following three conditions:*

- (a) *The minors $le(A)$ and $ri(A)$ are acceptable.*
- (b) *If $r_{1n} \geq 2$ then $\square\{j | r_{1j} = r_{1n} - 1\} \geq \square\{j | r_{1j} = r\}$.*
- (c) *$a_{1n} = 0$ or $r_{1n} = s_{1n}$.*

PROOF: Assume that A is acceptable. Let τ be the corresponding system of partitions. By 3.5, the minors $le(A)$ and $ri(A)$ are the characteristic typrices of the subsystems $le(\tau)$ and $ri(\tau)$, so they are acceptable. Condition (b) follows from 2.6. In fact, if $r = r_{1n} \geq 2$, then

$$\square\{j | r_{1j} = r - 1\} = \tau[0, n]^{r-1} \geq \tau[0, n]^r = \square\{j | r_{1j} = r\}.$$

In order to prove (c), we assume that $r_{1n} \neq s_{1n}$. We put

$$\lambda = \tau[0, n], \mu = \tau[0, n - 1], \nu = \tau[1, n], \xi = \tau[1, n - 1].$$

By 2.6 we have

$$\begin{aligned} r = r_{1n} &\Leftrightarrow \mu^r < \lambda^r, & r = r_{2n} &\Leftrightarrow \xi^r < \nu^r, \\ s = s_{1n} &\Leftrightarrow \nu^s < \lambda^s, & s = s_{1,n-1} &\Leftrightarrow \xi^s < \mu^s. \end{aligned}$$

Since $r_{1n} \neq s_{1n}$, we have $\mu \neq \nu$. It follows that

$$\lambda \setminus \mu = \nu \setminus \xi.$$

This implies $r_{1n} = r_{2n}$, so that $a_{1n} = 0$, proving (c).

Conversely, assume that the conditions (a), (b), (c) are satisfied. We have to construct a system of partitions τ . We define a family of sequences $\tau[p, q]^*$ with $0 \leq p \leq q \leq n$, by

$$\tau[p, q]^r = \square\{j \in [p + 1 \dots q] | r_{p+1,j} = r\}, \quad r \geq 1.$$

Since the typrices $le(A)$ and $ri(A)$ are acceptable, there are systems of partitions τ' and τ'' of order $n - 1$ with characteristic typrices $le(A)$ and $ri(A)$. So by 2.6 we have

$$\left. \begin{aligned} \tau'[p, q]^r &= \tau[p, q]^r \\ \tau''[p, q]^r &= \tau[p + 1, q + 1]^r \end{aligned} \right\} \text{ if } 0 \leq p \leq q \leq n - 1$$

If $r \geq 2$ and $r \neq r_{1n}$, then

$$\tau[0, n]^{r-1} \geq \tau[0, n - 1]^{r-1} \geq \tau[0, n - 1]^r = \tau[0, n]^r.$$

If $r = r_{1n} \geq 2$, then condition (b) reads

$$\tau[0, n]^{r-1} \geq \tau[0, n]^r.$$

So the sequence $\tau[0, n]^*$ is non-increasing. Now it is clear that there is a corresponding partition $\tau[0, n]$ of order n with

$$\tau[0, n-1] \subset \tau[0, n].$$

In order to prove that τ is a system of partitions, it remains to show that

$$\tau[1, n] \subset \tau[0, n].$$

Assume that this inclusion is false. Then there is a number t with

$$\tau[1, n]^t > \tau[0, n]^t.$$

Since we have

$$\tau[1, n-1]^t \leq \tau[0, n-1]^t \leq \tau[0, n]^t,$$

it follows from the definition of τ that $t = r_{2n}$, and that

$$\tau[1, n-1]^t = \tau[0, n-1]^t = \tau[0, n]^t.$$

By application of 2.6 on the acceptable system $le(\tau)$, the left hand equality implies that $t \neq s_{1,n-1}$. The righthand equality implies that $t \neq r_{1n}$. So we have $s_{1,n-1} \neq r_{2,n}$ and $r_{2,n} \neq r_{1,n}$, or equivalently

$$s_{1,n} \neq r_{1,n} \text{ and } a_{1,n} = 1,$$

contradicting condition (c). So the family τ is a system of partitions. It is clear that $R = (r_{pq})$ is the characteristic R -matrix of τ . Therefore A is the characteristic tripryx of τ . This proves that A is acceptable.

Tableaux and semitableaux

4.1. DEFINITION: By 2.2, a *diagram* λ is a finite subset of \mathbb{N}^2 satisfying the condition

$$(i, j) \in \lambda \ \& \ p \leq i \ \& \ q \leq j \Rightarrow (p, q) \in \lambda.$$

We define a *semidiagram* σ to be a finite subset of \mathbb{N}^2 satisfying

$$(i, j) \in \sigma \ \& \ q \leq j \Rightarrow (i, q) \in \sigma.$$

We define a *semitableau* (*tableau*) of order n to be an injective mapping $S: [1 \dots n] \rightarrow \mathbb{N}^2$ such that for every number $m \leq n$ the partial image $S[1 \dots m]$ is a semidiagram (diagram).

REMARK: Here we abandon the conventions of [7]. We think of a semidiagram as a wall of boxes in the positive quadrant. A semitableau is a way of building the wall. If T is a tableau, then the family of diagrams $(T[1 \dots m])_m$ is a standard tableau in the sense of [7], p. 5.

4.2. Straightening of a semitableau

If σ is a semidiagram, there is a unique diagram $\lambda = \pi(\sigma)$ which can be obtained from σ by permuting the columns of \mathbb{N}^2 (i.e. by permutation of the first coordinates of \mathbb{N}^2). The diagram λ may be characterized by its conjugate sequence:

$$\lambda' = \square\{i \mid (i, j) \in \sigma\}.$$

If S is a semitableau of order n , there is a unique tableau $T = \pi(S)$ of order n which satisfies

$$T[1 \dots m] = \pi(S[1 \dots m]), \quad m \leq n.$$

The tableau T is called the *straightening* of S , see diagram 2.

4.3. The typrix of a semitableau

Let σ be a semidiagram of order n . We use σ as an index set of a basis $e(i, j)$, $(i, j) \in \sigma$, of the vector space V . Since σ is a semidiagram, we have an endomorphism x of V satisfying

$$\begin{aligned} xe(i, 1) &= 0 && \text{if } (i, 1) \in \sigma, \\ xe(i, j) &= e(i, j-1) && \text{if } j \geq 2. \end{aligned}$$

Clearly, x is nilpotent and its diagram is equal to $\pi(\sigma)$.

Now assume that σ is the image $\text{Im}(S)$ of a semitableau S of order n .

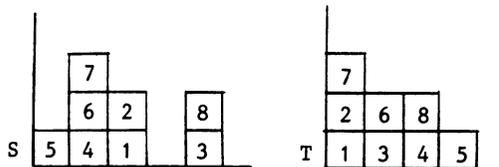


Diagram 2. A semitableau S and its straightening T .

Since the partial images $S[1 \dots m]$ are semidiagrams, the subspaces

$$F_m := \sum_{i=1}^m Ke(S(i))$$

form an x -invariant flag in the vector space V . We shall determine the corresponding system of partitions τ and the characteristic matrices $R = (r_{pq})$ and $A = (a_{pq})$. (We do not need the characteristic matrices S and T .) The conjugate sequences $\tau[p, q]^*$ are given by

$$\tau[p, q]^r = \dim(x^{r-1}F_q + F_p/x^rF_q + F_p).$$

It follows that $\tau[p, q]^r$ is the number of columns of the semitableau S which contain at least r boxes $S(t)$ with $p < t \leq q$. It follows that

$$r_{pq} = \square\{t \in [p \dots q] \mid S_1(t) = S_1(q)\},$$

where $S_1(i)$ denotes the first coordinate of the point $S(i)$ in \mathbb{N}^2 . Since $a_{pq} = r_{pq} - r_{p+1, q}$, we get

$$a_{pq} = 1 \Leftrightarrow S_1(p) = S_1(q).$$

Belonging to the same column of S is an equivalence relation on the set $[1 \dots n]$. So the typrix A is elementary in the sense of the following definition.

4.4. DEFINITION: A typrix $A = (a_{pq})$ is called *elementary* if there is an equivalence relation \sim on $[1 \dots n]$ such that

$$a_{pq} = 1 \Leftrightarrow p \sim q \& p < q.$$

Conversely, given an equivalence relation \sim on the set $[1 \dots n]$, it is easy to construct a semitableau S such that the columns of S are the equivalence classes of \sim . This proves:

LEMMA: *If A is an elementary typrix, then A occurs.*

4.5. Duality of tableaux

We define a partial order \leq on the set \mathbb{N}^2 by

$$(i, j) \leq (p, q) \Leftrightarrow i \leq p \& j \leq q.$$

So a diagram λ may be characterized as a finite subset of \mathbb{N}^2 such that if $x \in \lambda$ and $y \leq x$ then $y \in \lambda$.

PROPOSITION: Let T be a tableau of order n .

(a) There are unique tableaux $U = dT$ and $V = \delta T$ of order $n - 1$ with $U(i) = T(i)$ and $V(i) \leq T(i + 1)$ for all numbers $i \leq n - 1$.

(b) We have $\text{Im}(\delta T) \subset \text{Im}(T)$ and $\delta dT = d\delta T$.

(c) There is a unique tableau $W = DT$ of order n such that

$$\text{Im}(d^m W) = \text{Im}(\delta^m T) \text{ for all } m \in [0 \dots n].$$

(d) $dDT = D\delta T$ and $DdT = \delta DT$ and $D^2 T = T$.

REMARKS: If λ is a fixed diagram of order n , then D induces an involution of the set $St(\lambda)$ of the tableaux T with $\text{Im}(T) = \lambda$. It follows from theorem 5.5 below, that D is equal to the involution $\pi' \circ \pi^{-1}$ of [10], p. 92. In diagram 3, we give an example of a tableau T of order 7, with repeated application of δ , and with the resulting dual tableau DT .

PROOF: (a) It is trivial that U has to be the restriction $T[[1 \dots n - 1]]$. The unique existence of V is proved by induction. If $n = 1$, then it is trivial. So, by induction we may assume the unique existence of a tableau V of order $n - 1$ with $V(i) \leq T(i + 1)$ for all numbers $i \leq n - 2$. Since the image $T[1 \dots n - 1]$ is a diagram, we have

$$V[1 \dots n - 2] \subset T[1 \dots n - 1].$$

In order to extend V to a tableau of order $n - 1$, we have to prescribe the point $V(n - 1) \leq T(n)$. Since V must be injective and $\text{Im}(T)$ is a diagram, this point must be chosen in the difference set

$$\text{Im}(T) \setminus V[1 \dots n - 2].$$

This set contains two points: $T(n)$ and x , say. If $x \leq T(n)$, then since $V[1 \dots n - 1]$ must be a diagram, we have to prescribe $V(n - 1) = x$. If $x \not\leq T(n)$, then we have to choose $V(n - 1) = T(n)$, in view of the defining condition. In both cases the resulting map V is a tableau and it satisfies the requirements.

(b) Both assertions are easy.

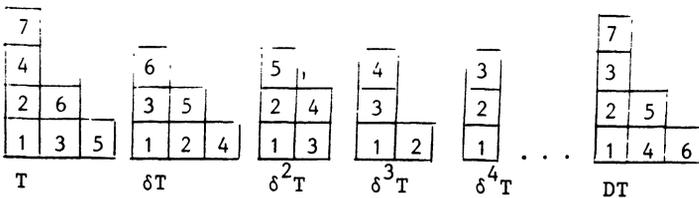


Diagram 3.

(c) It suffices to note that the sets $\text{Im}(\delta^m T)$ are diagrams of order $n - m$ which satisfy

$$\text{Im}(\delta^{m+1} T) \subset \text{Im}(\delta^m T).$$

(d) The equality $dDT = D\delta T$ is trivial. The equality $DdT = \delta DT$ is proved by induction. In fact, since δT is a tableau of smaller order, the induction yields that

$$Dd(\delta T) = \delta D(\delta T).$$

Using the equalities $dD = D\delta$ and $d\delta = \delta d$, we obtain

$$d(DdT) = d(\delta DT).$$

So the restrictions of DdT and δDT on the set $[1 \dots n - 2]$ are equal. By the definition of δ , it remains to prove that

$$DdT(n - 1) \leq DT(n).$$

Put $x = DT(n)$. We distinguish two cases:

(i) Assume $x \neq T(n)$. Since $x \in \text{Im}(T)$, it follows that $x \in \text{Im}(dT)$. Since $x \notin \text{Im}(d\delta T)$, it follows that $x = DdT(n - 1)$.

(ii) Assume $x = T(n)$. Put $y = \delta T(n - 1)$. Then $x \neq y$ and hence $y \in T[1 \dots n - 1] \setminus \delta T[1 \dots n - 2]$.

It follows that

$$DdT(n - 1) = y \leq T(n) = DT(n).$$

This concludes the proof of $DdT = \delta DT$.

The equality $D^2T = T$ is also proved by induction. In fact, induction yields $D^2dT = dT$. By the other two equalities, it follows that $dD^2T = dT$. So the restrictions of D^2T and T on the set $[1 \dots n - 1]$ are equal. Since the images of D^2T and T are also equal, it follows that $D^2T = T$.

4.6. COROLLARY: *Let T be a tableau of order n . Put $(r, s) = DdT(n - 1)$. Then $DT(n) \in \{(r, s), (r + 1, s), (r, s + 1)\}$.*

PROOF: By 4.5(d) we have

$$(r, s) = \delta DT(n - 1) \leq DT(n).$$

Assume $(r, s) \neq DT(n)$. Both points belong to the set difference

$$\text{Im}(T) \setminus \text{Im}(d\delta T).$$

It follows that

$$\text{Im}(d\delta T) = \text{Im}(T) \setminus \{(r, s), DT(n)\}.$$

So the righthand set is a diagram. Since $\text{Im}(T)$ is also a diagram and since $(r, s) < DT(n)$, it follows that

$$DT(n) \in \{(r+1, s), (r, s+1)\}.$$

4.7. Duality in terms of straightening

PROPOSITION: *Let T be a tableau of order n . Put $(p, q) = DT(n)$.*

(a) *Let S be a semitableau with $\pi(S) = T$ and $\text{Im}(S) = \text{Im}(T)$. Then $S(1) = (t, 1)$ with $t \leq p$.*

(b) *If $1 \leq t \leq p$, then there is a semitableau S with $\pi(S) = T$ and $\text{Im}(S) = \text{Im}(T)$ and $S(1) = (t, 1)$.*

PROOF: If $n = 1$, both assertions are trivial. In both cases we proceed by induction. We put

$$(r, s) = DdT(n-1).$$

So by 4.6 we have

$$(p, q) \in \{(r, s), (r+1, s), (r, s+1)\}.$$

(a) Now let the semitableau S be given. The relations $\pi(S) = T$ and $\text{Im}(S) = \text{Im}(T)$ remain undisturbed if we permute columns of S of equal length. So we may assume that the column of $S(1) = (t, 1)$ is as much to the right as possible. Putting $\lambda = \text{Im}(T)$, we have $\lambda_t > \lambda_{t+1}$. We fix the column of $S(1)$ and permute the other columns of S of equal length, in such a way that the point $S(n)$ comes as much to the right as possible.

First assume that $\lambda \setminus \{S(n)\}$ is a diagram. Then it is equal to the diagram $\text{Im}(dT)$. So by induction we have $t \leq r \leq p$, as required.

Now assume that $\lambda \setminus \{S(n)\}$ is not a diagram. Then the above permutations have established that $S(n) = (t-1, \lambda_t)$ and that $\lambda_{t-1} = \lambda_t$. Let S' be the semitableau of order $n-1$, obtained from S by interchanging the columns with the numbers $t-1$ and t , and then restricting to $[1 \dots n-1]$. Since $\text{Im}(S')$ is a diagram and hence equal to $\text{Im}(dT)$, and since $S'(1) = (t-1, 1)$, the induction hypothesis implies $t-1 \leq r \leq p$. So (p, q) is a point of the diagram λ with $p \geq t-1$. The set difference $\lambda \setminus \{(p, q)\}$ is equal to the diagram $\text{Im}(\delta T)$. Since $\lambda_{t-1} = \lambda_t$, it follows that $p \geq t$.

(b) Conversely, let $1 \leq t \leq p$ be given. We have to construct S . Put $t' = \min(r, t)$.

By induction we have got a semitableau S' of order $n - 1$ with $\pi(S') = dT$ and $\text{Im}(S') = \text{Im}(dT)$ and $S'(1) = (t', 1)$. Let S'' be the unique semitableau of order n such that S' is the restriction of S'' and that $\text{Im}(S'') = \lambda$. Then we have $\pi(S'') = T$. So if $t' = t$, then it suffices to put $S = S''$.

It remains to consider the case that $t' \neq t$. Then we have $t = p$ and $t' = t - 1$ and $(p, q) = (r + 1, s)$. Since the set

$$\lambda \setminus \{(p, q), (r, s)\} = \text{Im}(d\delta T)$$

is a diagram, it follows that the columns of S'' with the numbers t' and t have equal length $q = s$. Let S be the semitableau obtained from S'' by interchanging these two columns. Then S satisfies the requirements.

5. Generic systems of partitions

5.1. Until 5.6 we assume that the field K is algebraically closed. Recall that the set Y of the flags F_\star in the vector space V is a projective variety, cf. [1] 10.3. Let x be a fixed nilpotent endomorphism of V , say with partition $\lambda = \lambda(x)$. The set $Y(x)$ of the flags F_\star such that $x \in N(F_\star)$, is a closed subvariety of Y . Recall that if τ is a system of partitions with global partition λ , then $Y(x, \tau)$ is the set of the flags $F_\star \in Y(x)$ such that

$$\lambda(x : F_q/F_p) = \tau[p, q], \quad 0 \leq p \leq q \leq n.$$

Let T be a tableau of order n with $\text{Im}(T) = \lambda$. We define $Y(x)_T$ to be the set of the flags $F_\star \in Y(x)$ such that

$$\lambda(x : F_q) = T[1 \dots q], \quad 1 \leq q \leq n.$$

LEMMA: *The subsets $Y(x, \tau)$ and $Y(x)_T$ are localised closed.*

PROOF: We only consider the first case. Put

$$d(p, q, r) = p + \sum_{i=r+1}^{\infty} \tau[p, q]^i.$$

If $F_\star \in Y(x)$, then the condition $F_\star \in Y(x, \tau)$ is equivalent to

$$\dim(x'F_q + F_p) = d(p, q, r), \quad 0 \leq p \leq p + r \leq q \leq n.$$

Conditions like

$$\dim(x'F_q + F_p) \leq d \quad (\text{or } \leq d - 1)$$

are closed conditions. Therefore $Y(x, \tau)$ is locally closed. The other case is similar.

5.2. THEOREM (Spaltenstein, cf. [10] II5.21): *Let T be a tableau with $\text{Im}(T) = \lambda$. Then $Y(x)_T$ is an irreducible variety of dimension $\sum \frac{1}{2} \lambda^i (\lambda^i - 1)$, dense in some irreducible component of $Y(x)$. Every irreducible component of $Y(x)$ is the closure of a unique set $Y(x)_T$.*

REMARK: This version is dual to the original construction, which used the set of the flags F_* such that

$$\lambda(x, V/F_p) = T[1 \dots n - p], \quad 1 \leq p \leq n, \text{ cf. loc. cit. II5.5.}$$

5.3. PROPOSITION: *Let T be a tableau with $\text{Im}(T) = \lambda$. Put $DT(n) = (p, q)$. Then there is a flag $F_* \in Y(x)_T$ with $F_1 \not\subset x^q V$.*

PROOF: We use the Jordan basis of V introduced in 2.3. It consists of vectors $e(i, j)$, $(i, j) \in \lambda$, such that $xe(i, j) = e(i, j - 1)$ if $j \geq 2$, and that $xe(i, 1) = 0$. By proposition 4.7 there is a semitableau S with $\pi(S) = T$ and $\text{Im}(S) = \lambda$ and $S(1) = (p, 1)$. Using this semitableau we define a flag F_* by

$$F_r = \sum_{i=1}^r Ke(S(i)).$$

Since S is a semitableau, we have $x \in N(F_*)$ and

$$\lambda(x: F_r) = \pi(S[1 \dots r]) = T[1 \dots r].$$

This proves that $F_* \in Y(x)_T$. Since $F_1 = Ke(p, 1)$ and $(p, q + 1) \notin \lambda$, it is clear that $F_1 \not\subset x^q V$.

5.4. PROPOSITION: *Let T be a tableau with $\text{Im}(T) = \lambda$. Let S be a tableau of order $n - 1$. Put*

$$U(S) = \{F_* \in Y(x)_T \mid \forall m: \lambda(x: F_{m+1}/F_1) = S[1 \dots m]\}.$$

The set $U(S)$ is open and dense in $Y(x)_T$ if and only if $S = \delta T$.

PROOF: Since $Y(x)_T$ is irreducible and the disjoint union of the locally closed subsets $U(S)$, there is a unique tableau S of order $n - 1$ such that $U(S)$ is open and dense in $Y(x)_T$. It remains to prove that $S = \delta T$. This is done by induction.

Let $\mathbb{P}^*(V)$ be the projective space of the $n - 1$ dimensional subspaces of V . Let $f: Y(x)_T \rightarrow \mathbb{P}^*(V)$ be the projection given by $f(F_*) = F_{n-1}$.

Fix a flag $F_* \in U(S)$. The fiber $f^{-1}f(F_*)$ may be identified with the irreducible variety $Y'(x')_{dT}$ of flags in the space F_{n-1} , invariant under the restriction $x' = (x: F_{n-1})$. So by induction $f^{-1}f(F_*)$ contains $U'(\delta dT)$ as an open and dense subset. Since $U(S)$ is open, it follows that $U(S)$ intersects $U'(\delta dT)$. It follows that

$$dS = \delta dT = d\delta T.$$

It remains to prove that $\text{Im}(S) = \text{Im}(\delta T)$. Put $\mu = \text{Im}(S)$ and $\nu = \text{Im}(\delta T)$. Assume $\mu \neq \nu$. Put $\xi = \text{Im}(\delta dT)$. Then $\xi = \mu \cap \nu$ and $\lambda = \mu \cup \nu$. We have two different points

$$(p, q) = DT(n) \in \lambda \setminus \nu = \mu \setminus \xi,$$

$$(u, v) = \delta T(n-1) \in \nu \setminus \xi = \lambda \setminus \mu.$$

Since $(u, v) \leq T(n)$, we have $(p, q) \neq T(n)$ and hence $\nu \neq \text{Im}(dT)$. Since $\xi \subset \text{Im}(dT) \subset \lambda$, it follows that $\mu = \text{Im}(dT)$.

The set of the flags F_* with $F_1 \not\subset x^q V$ is open in the flag variety Y . By proposition 5.3, this set intersects $Y(x)_T$. Since $U(S)$ is dense in $Y(x)_T$, there is a flag $F_* \in U(S)$ with $F_1 \not\subset x^q V$. This flag satisfies

$$\lambda(x: V/F_1) = \mu = \lambda(x: F_{n-1}),$$

$$\lambda(x: F_{n-1}/F_1) = \xi.$$

Since $F_1 \not\subset x^q V$, we have

$$\sum_{i=q+1}^{\infty} \mu^i = \dim(x^q V + F_1/F_1) = \dim(x^q V) = \sum_{i=q+1}^{\infty} \lambda^i.$$

Since $(u, v) \in \lambda \setminus \mu$, it follows that $v \leq q$. The set μ is a diagram which contains (p, q) . Therefore $v < q$. It follows that

$$\dim(x^{q-1}F_{n-1}) = \sum_{i=q}^{\infty} \mu^i = \sum_{i=q}^{\infty} \lambda^i = \dim(x^{q-1}V).$$

This implies $x^{q-1}F_{n-1} = x^{q-1}V$, and hence

$$\sum_{i=q}^{\infty} \xi^i = \dim(x^{q-1}F_{n-1} + F_1/F_1) = \dim(x^{q-1}V + F_1/F_1) = \sum_{i=q}^{\infty} \mu^i,$$

contradicting the fact that

$$\xi^q = p - 1 < p = \mu^q.$$

This proves that $\mu = \nu$.

5.5. THEOREM (genericity): *Let T be a tableau with $\text{Im}(T) = \lambda$. There is a unique system of partitions τ such that $Y(x, \tau)$ is open and dense in $Y(x)_T$. This system τ is determined by*

$$\tau[p, q] = \text{Im}(d^{n-q}\delta^p T).$$

PROOF: The variety $Y(x)_T$ is irreducible and a disjoint union of its locally closed subsets $Y(x, \tau)$. This implies the first assertion. It remains to identify τ . Since $Y(x, \tau)$ is contained in $Y(x)_T$, we have

$$\tau[0, q] = \text{Im}(d^{n-q}T).$$

Since $Y(x, \tau)$ intersects the set $U(\delta T)$ of 5.4, we have

$$\tau[1, q] = \text{Im}(d^{n-q}\delta T).$$

We proceed by induction. Let $\mathbb{P}(V)$ be the projective space consisting of the one dimensional subspaces of V . Let $f: U(\delta T) \rightarrow \mathbb{P}(V)$ be the projection given by $f(F_*) = F_1$. The fiber $f^{-1}f(F_*)$ may be identified with the irreducible variety $Y'(x')_{\delta T}$ of flags in the space V/F_1 which are invariant under the endomorphism $x' = (x: V/F_1)$ and have tableau δT . So, by induction $f^{-1}f(F_*)$ contains the open and dense subset $Y'(x', \tau')$ where τ' is the system of partitions of order $n-1$ determined by

$$\tau'[p, q] = \text{Im}(d^{n-1-q}\delta^{1+p}T).$$

We may assume that $F_* \in Y'(x', \tau')$. Let F'_* be the induced flag with $F'_i = F_{i+1}/F_1$. For every index $p \geq 1$, we have

$$\begin{aligned} \tau[p, q] &= \lambda(x: F_q/F_p) = \lambda(x': F'_{q-1}/F'_{p-1}) \\ &= \tau'[p-1, q-1] = \text{Im}(d^{n-q}\delta^p T). \end{aligned}$$

This concludes the proof.

5.6. From now, the field K is again arbitrary. In fact, the following definition is independent of the field.

DEFINITION: Let T be a tableau of order n . The associated system of partitions $\tau = \gamma(T)$ is defined by

$$\tau[p, q] = \text{Im}(d^{n-q}\delta^p T).$$

Independently of theorem 5.5, it is clear that $\gamma(T)$ is a system of partitions of order n . By 2.5, 5.2, 5.5, we have

6	
4	
2	5
1	3

Tableau T.
Diagram 4.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

COROLLARY: *Let K be algebraically closed. A system of partitions τ of order n is generic if and only if there is a (necessarily unique) tableau T with $\tau = \gamma(T)$.*

REMARK: The theory can be extended in such a way that the field K is only supposed to be infinite. This follows from the fact that $Y(x)_\tau$ contains a dense open subset which is K -isomorphic to an affine space, cf. [9].

5.7. The Bürgstein family

Let T be the tableau of order 6, given in diagram 4. Put $\lambda = \text{Im}(T)$, so that $\lambda_* = (4, 2)$. One verifies that the system of partitions $\tau = \gamma(T)$ has the typrix A which is displayed in diagram 4. Let V be a vector space with $\dim(V) = 6$. Let x be a nilpotent endomorphism of V with partition λ . It turns out that the set $Y(x, \tau)$ consists of the flags F_* such that $F_1 = \text{Im}(x^3)$ and $F_3 = x^{-1}F_1$ and $F_5 = x^{-2}F_1$ and that F_2 and xF_4 are two different subspaces of F_3 which are unequal both to $\text{Ker}(x)$ and $\text{Im}(x^2)$. These four two-dimensional subspaces of F_3 all contain F_1 . So they induce four different points on the projective line $\mathbb{P}(F_3/F_1)$, and hence a projective cross ratio $t \in K \setminus \{0, 1\}$.

If K is infinite, this implies that $Y(x, \tau)$ contains infinitely many orbits with respect to the action of the centralizer $Z(x)$, the group of the automorphisms g of V with $gxg^{-1} = x$. This infinite family of orbits was discovered by H. Bürgstein. He proved that τ is the only system of partitions of order $n \leq 6$ such that the corresponding set $Y(x, \tau)$ has infinitely many orbits. On the other hand, it follows that $Y(x, \tau)$ is empty if the field K has only two elements. A related example was given by R. Steinberg in [14] 5.7.

6. Necessary conditions

6.1. In this section we shall derive conditions which are necessary for the occurrence of a given typrix. The conditions are sharper than the acceptability introduced in 3.2. For some months, our investigations were stimulated by the belief that the conditions might be sufficient.

EXAMPLE: Consider the typrices

$$A_a = \begin{pmatrix} 0 & 1 & 0 & a \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_b = \begin{pmatrix} 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{with } a, b \in \{0, 1\}.$$

All four typrices A_0, A_1, B_0, B_1 are acceptable. However, if (x, F_*) is a pair in X with characteristic typrix A_a , then it turns out that $x^2 \neq 0$ and that therefore $a = 1$. If the pair has characteristic typrix B_b , then $x^2 = 0$ and $b = 0$. So, the typrices A_0 and B_1 do not occur. The proof of this non-occurrence will be given in a highly generalized form. It gives information concerning the ranks of the powers of the induced endomorphisms $(x: F_q/F_p)$. The main tool is a dimension formula.

6.2. The dimension formula

Since the formulas tend to be large, we introduce some special notations. The dimension of a vector space W is denoted by $|W| = \dim(W)$. The rank of an endomorphism x of W is denoted by $|x|$ or $|x: W|$. The addition of subspaces is given higher priority than the formation of residue classes. So we write $P + Q/R + S$ instead of $(P + Q)/(R + S)$.

PROPOSITION: *Let Q and R be subspaces of V . Let $V_1 \subset \dots \subset V_m$ be a nested sequence of subspaces of V . Then we have*

$$\sum_{i=1}^m (-1)^i (|V_i + Q| - |V_i + R|) \leq |R + Q/Q|.$$

PROOF: If $m = 0$, the assertion is trivial. If $m = 1$, then it reads

$$|V_1 + R| - |V_1 + Q| \leq |R + Q/Q|.$$

This formula follows from the surjectivity of the canonical map

$$\varphi: R + Q/Q \rightarrow V_1 + R + Q/V_1 + Q.$$

We proceed by induction. Put $U = V/V_1$ and $U_i = V_{i+1}/V_1$ and $Q' = R + V_1/V_1$ and $R' = Q + V_1/V_1$. By induction, we have

$$\sum_{i=1}^{m-1} (-1)^i (|U_i + Q'| - |U_i + R'|) \leq |R' + Q'/Q'|,$$

or equivalently

$$\sum_{i=2}^m (-1)^i (-|V_i + R| + |V_i + Q|) \leq |Q + R + V_1/R + V_1|,$$

or equivalently

$$\sum_{i=1}^m (-1)^i (|V_i + Q| - |V_i + R|) \leq |Q + R + V_1/V_1 + Q|.$$

So, it remains to prove that

$$|Q + R + V_1/V_1 + Q| \leq |R + Q/Q|.$$

Again, this follows from the surjectivity of the above map φ .

6.3. COROLLARY: *If moreover m is even, then*

$$-|Q + R/R| \leq \sum_{i=1}^m (-1)^i (|V_i + Q/Q| - |V_i + R/R|) \leq |R + Q/Q|.$$

PROOF: The righthand inequality follows from 6.2 and the fact that $\sum_{i=1}^m (-1)^i = 0$. We obtain the lefthand inequality by interchanging Q and R .

6.4. REMARK: Although it has nothing to do with the case, we like to note another version. The Grassmann set of all linear subspaces of V can be equipped with a discrete metric d defined by

$$d(Q, R) = \dim(Q + R/Q \cap R) = 2|Q + R| - |Q| - |R|.$$

If m is odd, the formula of 6.2 is equivalent with

(*)

$$\sum_{i=1}^m (-1)^i (d(V_i, Q) - d(V_i, R)) \leq d(Q, R).$$

The case $m = 1$ is the triangle inequality. The case $m = 3$, however, is not a consequence of the triangle inequality. In fact, if the edges in the graph

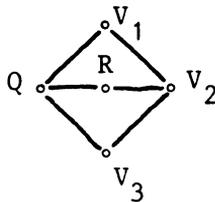


Diagram 5.

of diagram 5 all have equal length, then the corresponding discrete metric space with five points does not satisfy formula (*).

6.5. PROPOSITION: *Let x and y be endomorphisms of the space V . Let $V_1 \subset V_2 \subset \dots \subset V_{2m}$ be a nested sequence of subspace of V , which are invariant under both x and y . Then we have*

$$(a) \sum_{i=1}^{2m} (-1)^i (|x: V_i| - |y: V/V_i| - |V_i|) \leq |xy: V|.$$

(b) *If R is a subspace of V invariant under both x and y , then*
 $|x: V| - |x: R| - |V/R| \leq \sum_{i=1}^{2m} (-1)^i (|xy: V_i| - |y: V_i + R/R|) \leq |x: R|.$

PROOF: (a) Put $Q = \text{Ker}(x)$ and $R = \text{Im}(y)$. Then

$$|R + Q/Q| = |xy: V|,$$

$$|V_i + Q/Q| = |x: V_i|,$$

$$|V_i + R/R| = |y: V/V_i| + |V_i| - |R|.$$

Now the formula follows from corollary 6.3 together with $\sum_{i=1}^{2m} (-1)^i = 0$.

(b) Again we apply 6.3. We use $Q = \text{Ker}(x)$ and the nested sequence of subspaces $U_1 \subset \dots \subset U_{2m}$ given by $U_i = y(V_i)$. The formula follows from

$$|R + Q/Q| = |x: R|,$$

$$-|Q + R/R| = |x: V| - |x: R| - |V/R|,$$

$$|U_i + Q/Q| = |xy: V_i|,$$

$$|U_i + R/R| = |y: V_i + R/R|.$$

6.6. THEOREM: *Let A be an occurring typrix of order n with R -matrix (r_{ij}) and S -matrix (s_{ij}) , cf. 3.8. Let $a, b \in \mathbb{N}$. Put $c = a + b$.*

(a) *We have $\square\{j | r_{1j} > a \ \& \ s_{jn} > b\} \leq \square\{j | r_{1j} > c\}$.*

(b) *If $1 \leq p \leq n$, then*

$$(1) \square\{j \geq p | r_{pj} \leq b \ \& \ c < r_{1j}\} \leq \square\{j < p | a < r_{1j} \leq c\},$$

$$(2) \square\{i \leq p | s_{ip} \leq b \ \& \ c < s_{in}\} \leq \square\{i > p | a < s_{in} \leq c\},$$

$$(3) \square\{j \geq p | b < r_{pj} \ \& \ a < r_{1j} \leq c\} \leq \square\{j \geq p | r_{pj} \leq b \ \& \ r_{1j} \leq a\},$$

$$(4) \square\{i \leq p | b < s_{ip} \ \& \ a < s_{in} \leq c\} \leq \square\{i \leq p | s_{ip} \leq b \ \& \ s_{in} \leq a\}.$$

PROOF: Let A be the characteristic typrix of a pair (x, F_*) in X . Let τ be the corresponding system of partitions. In view of 2.3, we define, for every triple of numbers t, p, q with $t \geq 1$ and $0 \leq p \leq q \leq n$, the number

$$rk(t, p, q) = |x^t: F_q/F_p| = \sum_{r=t+1}^{\infty} \tau[p, q]^r.$$

By 2.6 we have

$$\begin{aligned} rk(t, p, q) &= \square\{j \in [p + 1 \dots q] \mid r_{p+1,j} \geq t\} \\ &= \square\{i \in [p + 1 \dots q] \mid s_{iq} > t\}. \end{aligned}$$

The formulas are proved by application of proposition 6.5 on the endomorphisms x^a and x^b of V and a sequence of subspaces $(V_i)_i$ given by $V_i = F_{q(i)}$ where $q(1) = q_1, \dots, q(2m) = q_{2m}$ is an arbitrary subsequence of $1, \dots, n$ of even length. We have $x^a x^b = x^c$. Now formula 6.5(a) yields

$$\sum_{i=1}^{2m} (-1)^i (rk(a, 0, q_i) - rk(b, q_i, n) - q_i) \leq rk(c, 0, n).$$

Let J be the alternating union of segments

$$J = \{j \mid \exists i: q_{2i-1} < j \leq q_{2i}\} = \bigcup_{i=1}^m [q_{2i-1} + 1 \dots q_{2i}].$$

Now the alternating sums can be worked out as follows.

$$\Sigma(-1)^i rk(a, 0, q_i) = \Sigma(-1)^i \square\{j \leq q_i \mid r_{1j} > a\} = \square\{j \in J \mid r_{1j} > a\}.$$

$$\begin{aligned} \Sigma(-1)^i rk(b, q_i, n) &= \Sigma(-1)^i \square\{j > q_i \mid s_{jn} > b\} \\ &= -\square\{j \in J \mid s_{jn} > b\}. \end{aligned}$$

$$\Sigma(-1)^i q_i = \square J.$$

In all five summations, the index i runs from 1 to $2m$. In a remarkable way, the above inequality reduces to

$$\square\{j \in J \mid r_{1j} > a\} + \square\{j \in J \mid s_{jn} > b\} - \square J \leq rk(c, 0, n).$$

By a suitable choice of the sequence (q_i) , the set J can be made equal to an arbitrary subset of the segment $[1 \dots n]$. In particular, we can accomplish

$$J = \{j \in [1 \dots n] \mid r_{1j} > a \ \& \ s_{jn} > b\}.$$

In this case the inequality reduces to

$$\square J \leq rk(c, 0, n).$$

This is formula (a).

In order to prove the inequalities (b1) and (b3), we apply proposition 6.5(b) on the subspace $R = F_{p-1}$. For convenience, we define $rk(t, p, q) = 0$ if $p \geq q$. Then we get

$$\begin{aligned} & rk(a, 0, n) - rk(a, 0, p-1) - (n-p+1) \\ & \leq \sum_{i=1}^{2m} (-1)^i (rk(c, 0, q_i) - rk(b, p-1, q_i)) \leq rk(a, 0, p-1). \end{aligned}$$

The alternating sums are worked out in the same way as above. Again, J is the alternating union of segments. If $p > q$ then $r_{pq} = 0$. Let us define

$$I_1 = \{j \geq p | r_{1j} \leq a\} \quad \text{and} \quad I_2 = \{j < p | r_{1j} > a\}.$$

The above inequality reduces to

$$-\square I_1 \leq \square \{j \in J | r_{1j} > c\} - \square \{j \in J | r_{pj} > b\} \leq \square I_2.$$

The set J may represent an arbitrary subset of $[1 \dots n]$. So we may substitute $J = J_1$ and also $J = J_2$, defined by

$$J_1 = \{j | r_{pj} > b \ \& \ r_{1j} \leq c\},$$

$$J_2 = \{j | r_{pj} \leq b \ \& \ r_{1j} > c\}.$$

These substitutions yield

$$-\square I_1 \leq -\square J_1 \leq \square J_2 \leq \square I_2.$$

So, the set differences satisfy

$$\square(J_1 \setminus I_1) \leq \square(I_1 \setminus J_1),$$

$$\square(J_2 \setminus I_2) \leq \square(I_2 \setminus J_2).$$

We have

$$J_1 \setminus I_1 = \{j \geq p | b < r_{pj} \ \& \ a < r_{1j} \leq c\},$$

$$I_1 \setminus J_1 = \{j \geq p | r_{pj} \leq b \ \& \ r_{1j}' \leq a\},$$

$$J_2 \setminus I_2 = \{j \geq p | r_{pj} \leq b \ \& \ c < r_{1j}\},$$

$$I_2 \setminus J_2 = \{j < p | a < r_{1j} \leq c\}.$$

This proves the inequalities (b3) and (b1). The inequalities (b4) and (b2) follow from (b3) and (b1) by duality, cf. 3.4.

6.7. EXAMPLES: The typrix A_0 of example 6.1 is rejected by condition 6.6(a) with $a = b = 1$. The typrix B_1 of 6.1 is rejected by 6.6(b1) with $a = b = 1$ and $p = 3$. The conditions (b3) and (b4) of 6.6 only reject typrices of order $n \geq 7$. For example, the typrix C is rejected by (b3) with $a = b = 1$ and $p = 5$.

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

7. Generating possibly occurring typrices

7.1. Motivation

When we had obtained a weak version of theorem 6.6(a), we felt the need for a list of the typrices which could occur according to the known conditions. Since the human mind is inadequate to verify conditions like 6.6, we used a computer. The first result was a list which contained the typrix B_1 of example 6.1. This typrix was easily rejected. We tried to generalize the argument and obtained the present form of theorem 6.6(a) and a weak version of 6.6(b1). The adapted program generated a list of 26 possibly occurring typrices of order 6. The typrix A was on this list, but it behaved suspiciously in the process of generating the list of typrices of order 7. An ad-hoc argument to reject A was found and generalized. In this way we obtained 6.5(b) and 6.6(b). In fact, the typrix A is rejected by 6.6(b1) with $a = b = 1$ and $p = 3$.

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So, previous lists of possibly occurring typrices have already proved fruitful. This seems to justify the inclusion in this paper of parts of our present list. It may be of equal importance, however, to give an indication of our methods to generate the list.

7.2. We start with a definition. A typrix A of order n is said to be *very acceptable*, if it is acceptable and if every subtyprix $A' = (a'_{ij})$ of order $k \leq n$ given by

$$a'_{ij} = a_{i+m, j+m}, \quad i, j \in [1 \dots k],$$

where $0 \leq m \leq n - k$, satisfies the inequalities of 6.6(a) and (b). If A is a very acceptable typrix, then clearly the minors $le(A)$ and $ri(A)$ are also very acceptable, see 3.8. By 3.2, 3.5, 6.6, every occurring typrix is very acceptable. In 9.3 below, we shall give very acceptable typrices A and B of order 21, such that A occurs over a field K if and only if $\text{char}(K) = 2$, and B occurs over K if and only if $\text{char}(K) \neq 2$.

7.3. Fusion of typrices

In order to get a list of the very acceptable typrices of order n , we may assume that the list for the order $n - 1$ is known already. So we assume that $x[1], \dots, x[N]$ are the very acceptable typrices of order $n - 1$. If A is a very acceptable typrix of order n , then its main minors $le(A)$ and $ri(A)$ are very acceptable typrices of order $n - 1$, so that $le(A) = x[i]$ and $ri(A) = x[j]$ for certain indices $i, j \in [1 \dots N]$. Moreover, the minors of $x[i]$ and $x[j]$ on the intersection in A are equal. That is

(*)

$$ri(x[i]) = le(x[j]).$$

Conversely, we obtain all very acceptable typrices of order n , if we fuse together all pairs $x[i]$ and $x[j]$ such that (*) holds, add an additional 0 or 1 in the top righthand corner, cf. 3.6, and verify whether the resulting typrix satisfies the conditions 3.8(b, c) and 6.6.

For the algorithm to be effective, we need bounds on the positive integers a and b of 6.6. Therefore we note the following facts.

(a) Condition 6.6(a) is only non-trivial if $a + b < n$. In fact, if the lefthand side of 6.6(a) is non-zero, then there is an index j such that $r_{1j} > a$ and $s_{jn} > b$. It follows that $j > a$ and $n - j \geq b$, and hence $n > a + b$.

(b) We may suppose that the main minors $le(A)$ and $ri(A)$ are already very acceptable. So condition (b1) need only be verified if $r_{pn} \leq b$ and $c < r_{1n}$, or equivalently if $r_{pn} \leq b < r_{1n} - a$.

(c) Similarly, condition (b3) need only be verified if $b < r_{pn}$ and $a < r_{1n} \leq c$, or equivalently if

$$1 \leq r_{1n} - a \leq b < r_{pn}.$$

(d) The conditions (b2) and (b4) are treated in the same way.

TABLE 1.
Counting the very acceptable typrices.

Order	Partition	N	N_g	N_e	N_{ge}	
2	(2)	1	1	1	1	+
	(1, 1)	1	1	1	1	
		2	2	2	2	
3	(3)	1	1	1	1	+
	(2, 1)	3	2	3	2	
	(1 ³)	1	1	1	1	
		5	4	5	4	
4	(4)	1	1	1	1	+
	(3, 1)	5	3	4	2	
	(2, 2)	3	2	3	2	
	(2, 1, 1)	6	3	6	3	
	(1 ⁴)	1	1	1	1	
		16	10	15	9	
5	(5)	1	1	1	1	+
	(4, 1)	7	4	5	2	
	(3, 2)	12	5	10	3	
	(3, 1, 1)	15	6	10	3	
	(2, 2, 1)	15	5	15	5	
	(2, 1 ³)	10	4	10	4	
	(1 ⁵)	1	1	1	1	
		61	26	52	19	
6	(6)	1	1	1	1	+
	(5, 1)	9	5	6	2	
	(4, 2)	26	9	15	3	
	(4, 1, 1)	28	10	15	3	
	(3, 3)	12	5	10	3	
	(3, 2, 1)	87	16	60	8	
	(3, 1 ³)	35	10	20	4	
	(2, 2, 2)	15	5	15	5	
	(2, 2, 1, 1)	45	9	45	9	
	(2, 1 ⁴)	15	5	15	5	
	(1 ⁶)	1	1	1	1	
		274	76	203	44	
7	(7)	1	1	1	1	+
	(6, 1)	11	6	7	2	
	(5, 2)	44	14	21	3	
	(5, 1, 1)	45	15	21	3	
	(4, 3)	57	14	35	4	
	(4, 2, 1)	240	35	105	8	
	(4, 1 ³)	84	20	35	4	
	(3, 3, 1)	130 *	21	70	7	
	(3, 2, 2)	148	21	105	10	
	(3, 2, 1, 1)	357	35	210	15	
	(3, 1 ⁴)	70	15	35	5	
	(2 ³ , 1)	105	14	105	14	
	(2, 2, 1 ³)	105	14	105	14	
	(2, 1 ⁵)	21	6	21	6	
	(1 ⁷)	1	1	1	1	
	1419 *	232	877	97		

* Two typrices do not occur.

7.4. Discussion of results

The results of the computer program are summarized in two tables. Table 1 gives the number N of the very acceptable typrices with a given global partition of order n where $2 \leq n \leq 7$. The number N_g is the corresponding number of generic typrices, N_e is the number of elementary ones, and N_{ge} is the number of typrices which are both generic and elementary. All very acceptable typrices of order $n \leq 6$ occur over the field \mathcal{Q} (H. Bürgstein). Recent results, cf. [4], show that there are two very acceptable typrices of order 7 which do not occur over any field, and that the other 1417 typrices of order 7 occur over \mathcal{Q} .

In Table 2, we give the occurring typrices of order $n \leq 5$. We only show the upper triangular part of the typrix. Consecutive typrices are separated by semicolons, commas or spaces. If they belong to different partitions, they are separated by a semicolon and the new partition is written under the typrix.

Typrices belonging to different components $Y(x)_T$ are separated by commas. Whether a typrix is generic or elementary is indicated by a letter “g” or “e” under its lowest entry.

8. A component with many moduli

8.1. In this section we investigate high dimensional cases of a special type. So, the number $n = \dim(V)$ may be large. We assume that x is a nilpotent endomorphism of V such that the diagram $\lambda = \lambda(x)$ is a rectangle, say

$$\lambda = [1 \dots u] \times [1 \dots v] \subset \mathbb{N}^2.$$

It follows that $n = u \cdot v$. If $0 \leq i \leq v$, then we put $Q_i = \text{Ker}(x^i)$. One verifies that $Q_i = \text{Im}(x^{v-i})$ and that $\dim(Q_i) = u \cdot i$. In the flag variety Y we consider the subset C of the flags F_\star with

$$F_{u \cdot i} = Q_i, \quad 0 \leq i \leq v.$$

If $F_\star \in C$ and $u \cdot i < p \leq u \cdot (i+1)$, then $F_p \subset Q_{i+1}$, so that

$$xF_p \subset xQ_{i+1} = Q_i \subset F_{p-1}.$$

This proves that C is contained in the subvariety $Y(x)$.

LEMMA: *If the base field K is algebraically closed, then the set C is an irreducible component of $Y(x)$.*

PROOF: It is clear that C is a closed subset of $Y(x)$. Since C is isomorphic to a product of flag varieties, the set C is irreducible. Let T

be the tableau shown in diagram 6. We claim that $Y(x)_T \subset C$. In fact, let $F_* \in Y(x)_T$. Then we have

$$\lambda(x: F_{u,i}) = T[1 \dots u \cdot i] = [1 \dots u] \times [1 \dots i].$$

It follows that $x^i F_{u,i} = 0$, so that $F_{u,i} \subset Q_i$. Since both spaces have the same dimension, it follows that $F_{u,i} = Q_i$. This proves that $F_* \in C$. Since $Y(x)_T$ is dense in an irreducible component of $Y(x)$, cf. 5.2, this proves that C is an irreducible component of $Y(x)$.

8.2. Analysis of the set C

If F_* is an arbitrary flag in C , then we define

$$E_{i,d} = x^i F_{u,i+d} \quad (0 \leq i < v; 0 \leq d \leq u)$$

It is clear that for every index i the sequence $E_{i,*} = (E_{i,d})_d$ is a flag in the vector space

$$Q := Q_1 = \text{Ker}(x).$$

Conversely, if $E_{0,*}, \dots, E_{v-1,*}$ is a v -tuple of flags in Q , then there is a unique corresponding flag $F_* \in C$, determined by

$$F_{u,i+d} = x^{-i} E_{i,d}.$$

This defines a bijective correspondence between the set C and the set $Y(Q)^v$ of the v -tuples of flags in the space Q .

8.3. The relative position of flags in Q

Recall that $Q = Q_1$ is a vector space with $\dim(Q) = u$. The relative position of two flags F_* and G_* in the space Q is characterized by the so-called *position matrix* $\Gamma = (\gamma_{pq})$ given by

$$\gamma_{pq} = \dim((F_p \cap G_q) / (F_{p-1} \cap G_q) + (F_p \cap G_{q-1})), \quad p, q \in [1 \dots u].$$

				n-1	n
				2u-1	2u
1	2	3		u-1	u

Diagram 6. The tableau T .

The following lemma is a direct consequence of the Bruhat decomposition, cf. [13], p. 225, ex. 7.

LEMMA (a) *There is a permutation σ of $[1 \dots u]$ and a basis e_1, \dots, e_u of Q such that F_p is spanned by $e_1 \dots e_p$ and that G_p is spanned by $e_{\sigma_1} \dots e_{\sigma_p}$.*

(b) *The matrix Γ is the permutation matrix of σ , that is $\gamma_{pq} = \delta_{p, \sigma q}$. In particular, σ is unique.*

(c) $\gamma_{pq} = 1 \Leftrightarrow (F_{p-1} \cap G_{q-1} = F_{p-1} \cap G_q) \& (F_p \cap G_{q-1} \neq F_p \cap G_q)$.

8.4. The typrix of a flag $F_* \in C$.

Let $F_* \in C$. We determine the system of partitions $\tau = \tau(x, F_*)$ in terms of the relative positions of the corresponding flags $E_{i,*}$ in the space $Q = \text{Ker}(x)$. The result is illustrated by diagram 7. Let $d, e \in [0 \dots u]$ and let $0 \leq i < j \leq v$. Putting

$$D = \dim(E_{i,d} \cap E_{j,e}),$$

we have

$$\begin{aligned} \tau[u \cdot i + d, u \cdot j + e]^r &= u, & \text{if } r < j - i, \\ &= u - d + D, & \text{if } r = j - i, \\ &= e - D, & \text{if } r = j - i + 1, \\ &= 0, & \text{if } r > j - i + 1. \end{aligned}$$

If $e > 0$, then we have

$$r_{u \cdot i + d + 1, u \cdot j + e} = j - i + h,$$

where h is 0 or 1, and $h = 1$ if and only if

$$E_{i,d} \cap E_{j,e-1} = E_{i,d} \cap E_{j,e}.$$

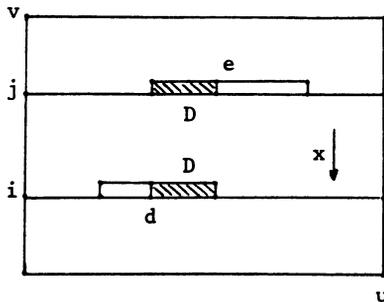


Diagram 7.

Therefore, if $d, e \in [1 \dots u]$, then

$$a_{u \cdot i + d, u \cdot j + e} = 1 \Leftrightarrow (E_{i, d-1} \cap E_{j, e-1} = E_{i, d-1} \cap E_{j, e})$$

$$\& (E_{i, d} \cap E_{j, e-1} \neq E_{i, d} \cap E_{j, e}).$$

By lemma 8.3(c), it follows that the $u \times u$ submatrix

$$\Gamma_{i, j} = (a_{u \cdot i + d, u \cdot j + e})_{d, e \in [1 \dots u]}$$

is equal to the position matrix of the pair of flags $E_{i, \star}$ and $E_{j, \star}$ in the space Q . This holds for $i < j$. Since the endomorphism x restricts to zero on the subquotient $F_{u \cdot i} / F_{u \cdot (i-1)}$, the $u \times u$ submatrix

$$\Gamma_{i, i} = (a_{u \cdot i + d, u \cdot i + e})_{d, e \in [1 \dots u]}$$

of the typrix A is the zero matrix. So, the typrix A is a block matrix with blocks $\Gamma_{i, j}$, where $i, j \in [0 \dots v - 1]$. Each block $\Gamma_{i, j}$ is a $u \times u$ matrix. If $i < j$, then $\Gamma_{i, j}$ is the position matrix of the pair of flags $E_{i, \star}$ and $E_{j, \star}$ in the space Q . If $i \geq j$, then $\Gamma_{i, j}$ is the zero block.

8.5. Permutation typrices

The above analysis suggests the following definition.

DEFINITION: A *permutation typrix of type (u, v)* is a typrix of order $n = u \cdot v$, which is a block matrix $(\Gamma_{i, j})_{i, j \in [0 \dots v - 1]}$ such that each block $\Gamma_{i, j}$ is a $u \times u$ matrix which is zero if $i \geq j$, and which is a permutation matrix if $i < j$.

PROPOSITION: (a) *If $F_{\star} \in C$, then the characteristic typrix of F_{\star} is a permutation typrix of type (u, v) .*

(b) *Let A be a permutation typrix of type (u, v) . Then A is acceptable, say with system of partitions τ . The flags $F_{\star} \in Y(x, \tau)$ are the flags $F_{\star} \in C$ such that the corresponding flags $E_{0, \star}, \dots, E_{v-1, \star}$ in the space Q have pairwise the position matrices $\Gamma_{i, j}$ which form the non-zero blocks of A .*

PROOF: (a) This is proved in 8.4.

(b) Let A have the blocks $\Gamma_{i, j}$. If $i < j$, assume that $\Gamma_{i, j}$ corresponds to the permutation $\sigma_{i, j}$ of the segment $[1 \dots u]$. If $i = j$, let $\sigma_{i, j}$ be the identity permutation. One verifies that A is the characteristic typrix of the system of partitions τ which is given as follows. If $0 \leq i \leq j < v$ and

$d, e \in [0 \dots u]$, then

$$\begin{aligned} \tau[u \cdot i + d, u \cdot j + e]^r & \\ &= u, & \text{if } < j - i; \\ &= u - d + D, & \text{if } r = j - i; \\ &= e - D, & \text{if } r = j - i + 1; \\ &= 0, & \text{if } r > j - i + 1 \end{aligned}$$

where

$$D = \square([1 \dots d] \cap \sigma_{i,j}[1 \dots e]).$$

This proves that A is acceptable. It remains to show that $Y(x, \tau)$ is contained in the set C . Let $F_* \in Y(x, \tau)$. We have

$$\lambda(x : F_{u,j}) = \tau[0, u \cdot j] = [1 \dots u] \times [1 \dots j].$$

It follows that $x'F_{u,j} = 0$, so that $F_{u,j} \subset Q_j$. Since both spaces have the same dimension, we get $F_{u,j} = Q_j$. This proves that $F_* \in C$.

REMARK: As the referee has remarked, this proposition allows to construct in a simple way acceptable typrices A which do not occur, e.g. $\Gamma_{1,2} = \Gamma_{2,3} = I$ and $\Gamma_{1,3} \neq I$. Here the non-occurrence does not require explicit calculations (as in 6.1) or the somewhat mysterious theorem 6.6.

8.6. The generic case

Assume that the field K is algebraically closed. The generic relative position of a pair of flags in the space Q is the opposite position. The corresponding permutation σ of the segment $[1 \dots u]$ is given by $\sigma(d) = u + 1 - d$. The position matrix B has ones on the skew diagonal. So, the generic typrix A in the component C of $Y(x)$ is the block matrix with blocks B above the main diagonal and all other blocks zero. Alternatively, this could have been derived from 5.5 and the tableau T of the proof of lemma 8.1.

$$B = \begin{pmatrix} 0 & 0 & 0 & & 0 & 0 & 1 \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & & 1 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 \\ 1 & 0 & 0 & & 0 & 0 & 0 \end{pmatrix}$$

Let τ be the corresponding generic system of partitions. The variety $Y(x, \tau)$ is isomorphic to the open subvariety U of $Y(Q)^v$ of the v -tuples of flags $E_{i,*}$, $i \in [0 \dots v-1]$, in Q which are pairwise opposite. The action of the centralizer

$$Z(x) = \{g \in \text{GL}(V) \mid gxg^{-1} = x\}$$

on the component C corresponds to the action of the group $\text{GL}(Q)$ on the product variety $Y(Q)^v$. Now assume that $v \geq 3$. Let U' be the open subvariety of U of the v -tuples $(E_{i,*})_i$ such that the one-dimensional subspace $E_{2,1}$ is in general position with respect to the opposite flags $E_{0,*}$ and $E_{1,*}$. If $(E_{i,*})_i$ is element of U' , then up to a common scalar, there is a unique basis e_1, \dots, e_u of Q such that the spaces $E_{0,d}$ are spanned by the vectors e_1, \dots, e_d , that the spaces $E_{1,d}$ are spanned by e_{u+1-d}, \dots, e_u , and that $E_{2,1}$ is spanned by the vector $e_1 + \dots + e_u$. The flag variety $Y(Q)$ has dimension $\frac{1}{2}u(u-1)$. The variety of the flags with a given one-dimensional part has dimension $\frac{1}{2}(u-1)(u-2)$. Therefore, the orbit space $U'/\text{GL}(Q)$ has dimension

$$\frac{1}{2}u(u-1)(v-3) + \frac{1}{2}(u-1)(u-2) = \frac{1}{2}(u-1)(n-2u-2).$$

Let n be fixed and divisible by 4. Then this dimension becomes maximal if we choose $u = n/4$ and hence $v = 4$. Then we have

$$\dim(U'/\text{GL}(Q)) = (u-1)^2 = (\frac{1}{4}n-1)^2.$$

In particular, this shows, as already observed by Spaltenstein, that the dimension of the orbit space may grow with a rate proportional to n^2 . In our example, the characteristic typrix is the block matrix

$$A = \begin{pmatrix} 0 & B & B & B \\ 0 & 0 & B & B \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{with } B \text{ as above.}$$

9. A component equivalent to plane projective geometry

9.1. Since plane projective geometry is better understood than the geometry of flags in an arbitrary space Q , we shall extend the analysis of section 8 in the special case $u = 3$. So we have $\dim(Q) = 3$. Every flag $E_{i,*}$ in Q can be written

$$E_{i,*} = (0, P_i, L_i, Q)$$

where $\dim(P_i) = 1$ and $\dim(L_i) = 2$ and $P_i \subset L_i$. We may consider P_i as a point in the projective plane $\mathbb{P}(Q)$ and L_i as a line in $\mathbb{P}(Q)$. So, the flag $E_{i,*}$ is identified with a pair $q_i = (P_i, L_i)$ consisting of a point incident with a line. In Table 3, we give the six incidence relations between such pairs which correspond to the six permutations σ of the set $[1 \dots 3]$. We represent the permutation σ by the sequence of images $[\sigma 1 \ \sigma 2 \ \sigma 3]$. The six position matrices get the names a, \dots, f . The incidence relations between the pairs q_0, \dots, q_{v-1} in the projective plane $\mathbb{P}(Q)$ can be represented by a graph Λ with the nodes $0, 1, \dots, v-1$, and with the edges as described in the last column of Table 3 below.

Conversely, let Λ be a graph with nodes $0, 1, \dots, v-1$, and edges of the types given in the table. We define a *drawing* of Λ to be a sequence of pairs $q_i = (P_i, L_i)$ with $i = 0, \dots, v-1$, in the projective plane $\mathbb{P}(Q)$ such that for every pair of indices $i < j$ the pairs q_i and q_j have the incidence relation described in the corresponding row of the table. We identify $\mathbb{P}(Q)$ with the projective plane $\mathbb{P}_2(K)$. We write $\text{Dr}(\Lambda/K)$ to denote the set of the drawings of Λ in $\mathbb{P}_2(K)$. We write $A(\Lambda)$ to denote the permutation typrix with the block minors $\Gamma_{i,j}$ given by the table. By proposition 8.5, the typrix $A(\Lambda)$ is acceptable. Let $\tau(\Lambda)$ be the corresponding system of partitions. By 8.5, we have a bijection

$$Y(x, \tau(\Lambda)) \cong \text{Dr}(\Lambda/K).$$

TABLE 3.
Dictionary between permutation, typrix, geometry and graph.

σ	Block $\Gamma_{i,j}$	Incidence relation $i < j$	Edge
[123]	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = a$	identity: $P_i = P_j, L_i = L_j$	$i \text{ --- } j$
[132]	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = b$	common point: $P_i = P_j, L_i \neq L_j$	
[213]	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = c$	common line: $P_i \neq P_j, L_i = L_j$	
[312]	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = d$	point on line: $P_i \subset L_j, P_j \not\subset L_i$	
[231]	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = e$	line through point: $P_i \not\subset L_j, P_j \subset L_i$	
[321]	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = f$	disjoint: $P_i \not\subset L_j, P_j \not\subset L_i$	

9.2. Triangles

For an arbitrary field K , the projective plane $\mathbb{P}_2(K)$ contains some triangle, see diagram 8. This triangle gives rise to a cyclic graph and a permutation typrix as shown in diagram 8. So, in particular, this typrix occurs. This is not surprising. In fact, the typrix is elementary, cf. 4.4.

However, if we start with the triangular graph of diagram 9, we find no corresponding drawing with points and lines in $\mathbb{P}_2(K)$. So, the corresponding typrix A does not occur. This fact is also not new. In fact, the minor $le(ri(A))$ of A of order 7 was already rejected in example 6.7.

9.3. The Fano configuration

The Fano configuration 7_3 is the configuration of the seven points and the seven lines of the projective plane $\mathbb{P}_2(\mathbb{F}_2)$ over the field with two elements, cf. [5] 14.1 ex. 2. This configuration can be realized in the plane $\mathbb{P}_2(K)$ if and only if the base field K has characteristic two. So the corresponding typrix A occurs if and only if $\text{char}(K) = 2$. Since this typrix is of order 21, we only give its block structure. The given typrix corresponds to the cyclic numbering of the vertices of the graph, see

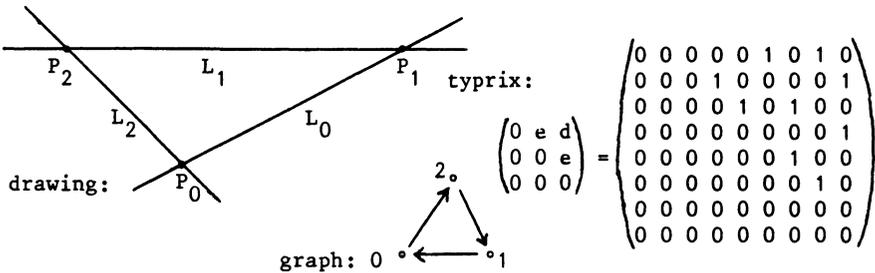


Diagram 8. A triangle with graph and permutation typrix.

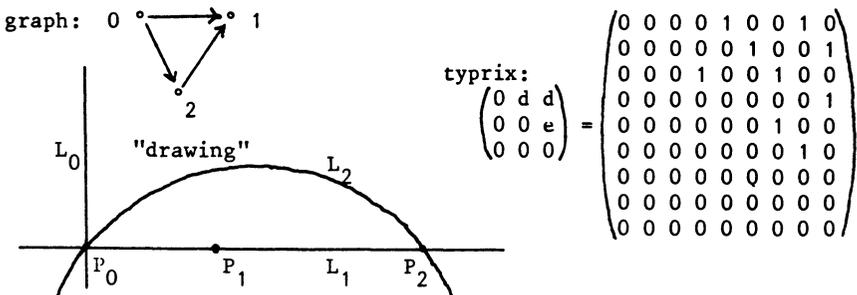


Diagram 9. Non-occurrence: a "false triangle".

diagram 10.

$$A = \begin{pmatrix} 0 & d & e & f & f & d & e \\ 0 & 0 & d & e & f & f & d \\ 0 & 0 & 0 & d & e & f & f \\ 0 & 0 & 0 & 0 & d & e & f \\ 0 & 0 & 0 & 0 & 0 & d & e \\ 0 & 0 & 0 & 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, let us omit the edge in the graph between the vertices 6 and 0. Then the typrix A is changed into the typrix B where the top righthand block $\Gamma_{0,6} = e$ is replaced by the block $\Gamma_{0,6} = f$. The drawing is modified such that the point P_6 is no longer on the line L_0 . It is clear that the typrix B occurs if and only if $\text{char}(K) \neq 2$. The typrices A and B are both very acceptable, since each occurs over some field.

9.4. Complex occurrence

The configuration 8_3 consists of 8 lines and 8 points, each point on three lines, each line through three points, see the graph of diagram 11. It occurs in the projective plane $\mathbb{P}_2(K)$ if and only if the equation $t^3 = 1$ has a solution $t \neq 1$ in K . In diagram 11, we also give the block structure of the corresponding typrix of order 24. So this typrix occurs over the complex numbers and it does not occur over the reals. Let τ be the corresponding system of partitions. If the field K is algebraically closed and of characteristic $\neq 3$, then the variety $Y(x, \tau)$ has two connected

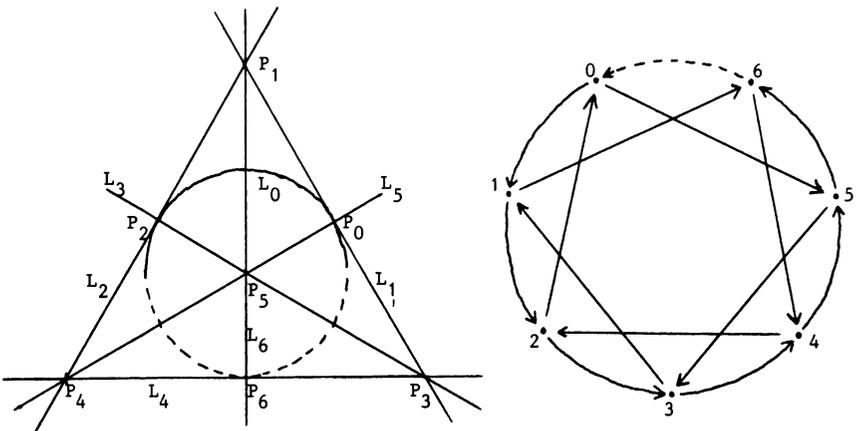


Diagram 10. The Fano configuration 7_3 .

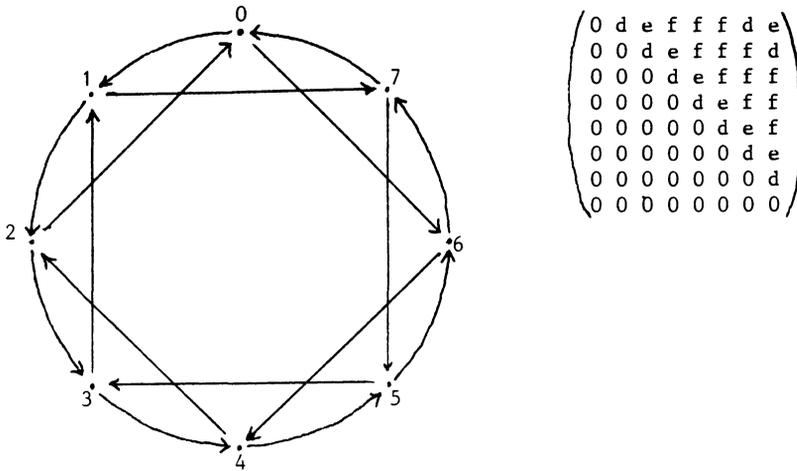


Diagram 11. The configuration 8_3 .

components. These components are irreducible. They correspond to the two different solutions of the equation $t^2 + t + 1 = 0$.

9.5. Plane configurations

In [6], we defined a *configuration* C to be a triple (C_1, C_2, C_0) such that C_1 and C_2 are disjoint finite sets and that C_0 is a subset of the Cartesian product $C_1 \times C_2$. Let $\mathbb{P}_2^*(K)$ denote the set of the lines in the plane $\mathbb{P}_2(K)$. A *drawing* S of a configuration C was defined to be a pair of injective maps

$$S_1: C_1 \rightarrow \mathbb{P}_2(K), \quad S_2: C_2 \rightarrow \mathbb{P}_2^*(K)$$

such that, for every pair $(a, b) \in C_1 \times C_2$, we have $(a, b) \in C_0$ if and only if the point $S_1(a)$ is incident with the line $S_2(b)$. The set of drawings S of a configuration C in $\mathbb{P}_2(K)$ is denoted by $\text{Dr}(C, K)$.

If C is a configuration, an element $a \in C_1$ is called *lonely* if $a \notin \text{pr}_1(C_0)$ where $\text{pr}_1: C_1 \times C_2 \rightarrow C_1$ is the projection mapping. Similarly, $b \in C_2$ is

TABLE 4.

$i < j$	i edge j
$a_i = a_j$ & $b_i \neq b_j$	$\circ \text{---} \text{---} \text{---} \circ$
$a_i \neq a_j$ & $b_i = b_j$	$\circ \text{---} \text{---} \text{---} \circ$
$(a_i, b_j) \in C_0$ & $(a_j, b_i) \notin C_0$	$\circ \text{---} \text{---} \text{---} \circ$
$(a_i, b_j) \notin C_0$ & $(a_j, b_i) \in C_0$	$\circ \text{---} \text{---} \text{---} \circ$
$(a_i, b_j) \notin C_0$ & $(a_j, b_i) \notin C_0$	$\circ \quad \circ$

called lonely if $b \notin pr_2(C_0)$. For our purposes, lonely elements are harmless. We just omit them. Then we have:

LEMMA: *Let C be a configuration without lonely elements. There is a graph Λ in the sense of 9.1 such that*

$$\text{Dr}(C, K) \cong \text{Dr}(\Lambda/K)$$

and that the number of vertices of Λ does not exceed $\square(C_1) + \square(C_2)$.

PROOF: Since C has no lonely elements, we may choose a subset J of C_0 such that $C_1 = pr_1(J)$ and $C_2 = pr_2(J)$. We may assume that J consists of v flags with $v \leq \square(C_1) + \square(C_2)$. Write $J = \{f_0, \dots, f_{v-1}\}$ with $f_i = (a_i, b_i)$. Let Λ be the graph with the nodes $0, \dots, v-1$. If $i < j$, let the node i be connected with the node j through an edge of the type determined by table 4. The bijective correspondence between the drawings S of C and the drawings (q_i) of Λ is determined by the equations

$$q_i = (S_1(a_i), S_2(b_i)).$$

REMARK: Conversely, it is clear that for every graph Λ , there is a configuration C with $\text{Dr}(C, K) \cong \text{Dr}(\Lambda/K)$.

9.6. Essentially, all varieties occur

By 9.5, the situation of 9.1 is reduced to our previous paper [6]. To avoid complicating details as well as unnecessary restrictions, we assume the base field K to be infinite and we define the concept of variety to mean "scheme of finite type over K ". Now, our sets $Y(x, \tau)$ may be considered as varieties. Recall that a morphism of varieties $f: Y \rightarrow X$ is called *free*, if it is surjective and if every pair of points y_1 and y_2 in Y has an open neighbourhood V in Y with an open immersion $j: V \rightarrow X \times \mathbf{A}^n$ such that the restriction of f to V is equal to the composition of j with the projection of $X \times \mathbf{A}^n$ onto the first factor, cf. [6] 4.1. It follows from [6] 6.1 and 8.3 that we have

THEOREM: *Let X be a variety isomorphic to a closed subset of \mathbf{A}^m or \mathbf{P}_m , which is defined by polynomial equations with integer coefficients. Then, for a sufficiently large number v , there exists a graph Λ with v vertices and with a free morphism of varieties $Y(x, \tau(\Lambda)) \rightarrow X$.*

COROLLARY (cf. [6] 8.2): *Let K be a number field. There is a graph Λ such that for every field extension L of \mathbf{Q} , we have that the typrix $A(\Lambda)$ occurs over L if and only if L contains a subfield isomorphic to K .*

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