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A COMMON ABSTRACTION OF BOOLEAN RINGS AND LATTICE ORDERED GROUPS

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Abstract

Lattice ordered partial semigroups are introduced as a common abstraction of Boolean rings and lattice ordered groups. Boolean rings and lattice ordered groups are characterized as lattice ordered partial semigroups with additional properties.

1. Introduction

An interesting problem posed in Birkhoff's book is the following: *Develop a common abstraction which includes Boolean algebras (rings) and lattice ordered groups as special cases* [1; p. 318]. In view of possible applications, such a common abstraction should not differ too much from Boolean rings and lattice ordered groups, and it also should be defined by a set of familiar axioms.

In the present note, we propose a solution which is motivated by a problem in the theory of measure and integration [4] and which is inspired by the work of Dinges [2], who suggested that the analogy between the disjoint union of sets and the addition of functions is more appropriate than the analogy between the union of sets and the supremum of functions.

The key to our solution of Birkhoff's problem is that we consider both of these analogies to be equally important. This leads us to the consideration of lattices on which a partial addition is defined such that order and lattice operations are compatible with addition. This concept is formalized in the notion of a *lattice ordered partial semigroup*, a structure which turns out to be only slightly more general than Boolean rings and lattice ordered groups.

Lattice ordered partial semigroups are defined and studied in Section 2. In Sections 3 and 4, respectively, Boolean rings and lattice ordered groups are characterized as lattice ordered partial semigroups with additional properties. We conclude with some remarks in Section 5, where

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lattice ordered partial semigroups are compared with earlier solutions of Birkhoff's problem due to Swamy [5], Wyler [6], Rama Rao [3], and Schmidt [4].

2. Lattice ordered partial semigroups

A *partial semigroup* is a set $\mathbb{E}$ with a distinguished element $0 \in \mathbb{E}$, a set $S \subseteq \mathbb{E} \times \mathbb{E}$, and a map (called *addition*) $+ : S \to \mathbb{E}$ such that the following axioms hold for all $x, y, z \in \mathbb{E}$:

(i) $(x, 0) \in S$ and $x + 0 = x$;
(ii) $(x, y) \in S$ implies $(y, x) \in S$ and $y + x = x + y$;
(iii) $(x, y) \in S$ and $(x + y, z) \in S$ implies $(y, z) \in S$, $(x, y + z) \in S$, and $x + (y + z) = (x + y) + z$.

A partial semigroup $\mathbb{E}$ has the *cancellation property* if

(iv) $(x, z) \in S$, $(y, z) \in S$ and $x + z = y + z$ implies $x = y$.

An *ordered partial semigroup* is a partial semigroup $\mathbb{E}$ with a partial ordering $\leqslant$ such that order and addition are compatible:

(v) $(x, z) \in S$, $(y, z) \in S$ and $x \leqslant y$ implies $x + z \leqslant y + z$.

The *positive cone* of an ordered partial semigroup $\mathbb{E}$ is defined to be the set $\mathbb{E}_+ := \{x \in \mathbb{E} \mid 0 \leqslant x\}$.

A *lattice ordered partial semigroup* is an ordered partial semigroup $\mathbb{E}$ which is a lattice:

(vi) $x \vee y$ and $x \wedge y$ exist for all $x, y \in \mathbb{E}$.

A lattice ordered partial semigroup $\mathbb{E}$ has the *difference property* if

(vii) for all $x, y \in \mathbb{E}$ there exists $z \in \mathbb{E}_+$ such that

$$(x, z) \in S,\ (x \wedge y, z) \in S,\ x + z = x \vee y, \text{ and } x \wedge y + z = y.$$

For the remainder of this section we suppose that $\mathbb{E}$ is a lattice ordered partial semigroup which has the cancellation property and the difference property.

CONVENTION: We shall simplify the notation by writing

$$x + y \quad \textit{has property} \quad (P)$$

instead of the full statement

$$(x, y) \in S \quad \textit{and} \quad (x + y) \quad \textit{has property} \quad (P).$$

There will be no source of confusion when this statement is used as a hypothesis; if it appears as an assertion, then the validity of

$$(x, y) \in S$$

is a consequence of the axioms (ii), (iii), and (vii).

2.1. THEOREM: *For all x, $y \in \mathbb{E}$ there exists a unique $u \in \mathbb{E}_+$ such that $x + u = x \vee y$ and $x \wedge y + u = y$.*

PROOF: The existence follows from (vii) and the uniqueness follows from (iv). □

2.2. COROLLARY: *For all x, $y \in \mathbb{E}$ such that $x \leqslant y$ there exists a unique $u \in \mathbb{E}_+$ such that $x + u = y$.*

PROOF: Note that $x \vee y = y$. □

2.3. THEOREM (*order cancellation property*): *If $x + z \leqslant y + z$, then $x \leqslant y$.*

PROOF: By Corollary 2.2, choose $u \in \mathbb{E}_+$ such that $x + z + u = y + z$. Then (iv) yields $x + u = y$, and (v) gives $x = x + 0 \leqslant x + u = y$. □

2.4. COROLLARY: *If $y \wedge z + v = y$ and $y \wedge z + w = z$, then $v \wedge w = 0$.*

PROOF: Note that $0 \leqslant v \wedge w$. From

$$y \wedge z + v \wedge w \leqslant y \quad \text{and} \quad y \wedge z + v \wedge w \leqslant z$$

we obtain

$$y \wedge z + v \wedge w \leqslant y \wedge z,$$

hence $v \wedge w \leqslant 0$, by Theorem 2.3. This proves $v \wedge w = 0$. □

2.5. THEOREM (*decomposition property*):
If $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n \in \mathbb{E}_+$ are such that

$$x_1 + x_2 + \ldots + x_m = y_1 + y_2 + \ldots + y_n,$$

then there exist $z_{ij} \in \mathbb{E}_+$, with $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, such that

$$x_i = z_{i1} + z_{i2} + \ldots + z_{in}$$

and

$$y_j = z_{1j} + z_{2j} + \ldots + z_{mj}$$

holds for all $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$.

PROOF: The assertion is trivial for $m = 1$ or $n = 1$. The general case can be proven by induction, and proving the induction step is equivalent to proving the assertion for $m = n = 2$. Choose $z_{21}, z_{12} \in \mathbb{E}_+$ such that

$$x_1 + z_{21} = x_1 \vee y_1, \quad x_1 \wedge y_1 + z_{21} = y_1$$

and

$$y_1 + z_{12} = x_1 \vee y_1, \quad x_1 \wedge y_1 + z_{12} = x_1.$$

Define $z_{11} := x_1 \wedge y_1$ and $z := x_1 + x_2 = y_1 + y_2$. Then we have

$$z_{11} + z_{12} + z_{21} = x_1 + z_{21} = x_1 \vee y_1 \leqslant (x_1 + x_2) \vee (y_1 + y_2) = z.$$

By Corollary 2.2, choose $z_{22} \in \mathbb{E}_+$ such that

$$z_{11} + z_{12} + z_{21} + z_{22} = z.$$

Now we have

$$x_1 = z_{11} + z_{12}, \qquad x_2 = z_{21} + z_{22}$$

and

$$y_1 = z_{11} + z_{21}, \qquad y_2 = z_{21} + z_{22},$$

as was to be shown. □

2.6. LEMMA: *If* $(u, v) \in S$, $(u, w) \in S$ *and* $v \wedge w = 0$, *then*

$$u + v + w = (u + v) \vee (u + w) \quad \textit{and} \quad u = (u + v) \wedge (u + w).$$

PROOF: Choose $x, y \in \mathbb{E}_+$ such that

$$u + v + x = (u + v) \vee (u + w), \quad (u + v) \wedge (u + w) + x = u + w$$

and

$$u + w + y = (u + v) \vee (u + w), \quad (u + v) \wedge (u + w) + y = u + v.$$

Then we have

$$v + x = w + y.$$

By Theorem 2.5, we may choose $z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{E}_+$ such that

$$v = z_{11} + z_{12}, \quad x = z_{21} + z_{22}$$

and

$$w = z_{11} + z_{21}, \quad y = z_{12} + z_{22}.$$

By assumption, we have

$$0 \leqslant z_{11} \leqslant v \wedge w = 0,$$

and Corollary 2.4 yields

$$0 \leqslant z_{22} \leqslant x \wedge y = 0.$$

Therefore we have $x = w$ and $y = v$, and the assertion follows from the defining identities for x and y. □

2.7. THEOREM: *If* $(x, y) \in S$ *and* $(x, z) \in S$, *then*

$$x + (y \vee z) = (x + y) \vee (x + z) \text{ and } x + (y \wedge z) = (x + y) \wedge (x + z).$$

PROOF: Choose $v, w \in \mathbb{E}_+$ such that

$$z + v = y \vee z, \quad y \wedge z + v = y$$

and

$$y + w = y \vee z, \quad y \wedge z + w = z.$$

Define $u := x + y \wedge z$. Then we have $(u, v) \in S$ and $(u, w) \in S$, and Corollary 2.4 yields $v \wedge w = 0$. Now the assertion follows from Lemma 2.6 and the identities $u + v + w = x + y \vee z$, $u + v = x + y$, $u + w = x + z$, and $u = x + y \wedge z$. □

2.8. THEOREM: *If* $(x, y) \in S$ *or* $(x \vee y, x \wedge y) \in S$, *then*

$$x + y = x \vee y + x \wedge y.$$

PROOF: Choose $u \in \mathbb{E}_+$ such that

$$x + u = x \vee y \quad \text{and} \quad x \wedge y + u = y.$$

If $(x, y) \in S$ *or* $(x \vee y, x \wedge y) \in S$, then all sums in the identity

$$x + y = x + (u + x \wedge y) = (x + u) + x \wedge y = x \vee y + x \wedge y$$

are defined. □

2.9. COROLLARY: *If* $x \wedge y = 0$, *then* $(x, y) \in S$ *and* $x + y = x \vee y$.

PROOF: Note that $(x \vee y, x \wedge y) = (x \vee y, 0) \in S$. □

2.10. THEOREM (*distributive laws*): *The identities*

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

and

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

hold for all $x, y, z \in \mathbb{E}$.

PROOF: It is sufficient to prove the first of these identities. To this end, choose $u, v, w \in \mathbb{E}_+$ such that

$$\begin{aligned} y \wedge z + u &= x \vee (y \wedge z), & x \wedge y \wedge z + u &= x, \\ x + v &= x \vee y, & x \wedge y + v &= y, \end{aligned}$$

and

$$x + w = x \vee z, \qquad x \wedge z + w = z.$$

Then we have

$$x \wedge y \wedge z + u \wedge v \wedge w \leqslant x \wedge y \wedge z,$$

hence

$$u \wedge v \wedge w = 0,$$

by Theorem 2.3, and

$$u + v \wedge w = u \vee (v \wedge w),$$

by Corollary 2.9. This yields, with Theorem 2.7,

$$\begin{aligned} (x \vee y) \wedge (x \vee z) &= (x + v) \wedge (x + w) \\ &= x + v \wedge w \\ &= x \wedge y \wedge z + u + v \wedge w \\ &= x \wedge y \wedge z + u \vee (v \wedge w) \\ &= (x \wedge y \wedge z + u) \vee (x \wedge y \wedge z + v \wedge w) \\ &\leqslant x \vee (y \wedge z). \end{aligned}$$

The converse inequality is obvious. □

We conclude this section with some remarks on invertible elements. An element $x \in \mathbb{E}$ is *invertible* if there exists $x' \in \mathbb{E}$ such that $x + x' = 0$. Let $\mathbb{E}_*$ denote the class of all invertible elements in $\mathbb{E}$.

2.11. Lemma: *If* $x \leqslant 0$, *then* $x \in \mathbb{E}_*$.

Proof: Apply Corollary 2.2. □

2.12. Theorem: *For each* $x \in \mathbb{E}$ *there exists* $y \in \mathbb{E}_+$ *and* $z \in \mathbb{E}_*$ *such that* $x = y + z$.

Proof: By Theorem 2.8, $x = x \vee 0 + x \wedge 0$. Then $y := x \vee 0 \in \mathbb{E}_+$ and $z := x \wedge 0 \leqslant 0$, hence $z \in \mathbb{E}_*$, by Lemma 2.11. □

2.13. Corollary: *If* $x \in \mathbb{E}$ *and* $y \in \mathbb{E}_*$, *then* $(x, y) \in S$.

Proof: Note that $(y, y') \in S$ and $(x, y + y') = (x, 0) \in S$, hence $(x, y) \in S$. □

3. Boolean rings

A *Boolean ring* is a distributive lattice which has relative complements and a least element.

3.1. Theorem: *Suppose* $\mathbb{E}$ *is a Boolean ring with the partial ordering* $\leqslant$ *and the least element* 0. *Define* $S := \{(x, y) \in \mathbb{E} \times \mathbb{E} \mid x \wedge y = 0\}$ *and* $+ : S \to \mathbb{E}: (x, y) \mapsto x \vee y$. *Then* $\langle \mathbb{E}, 0, S, +, \leqslant \rangle$ *is a lattice ordered partial semigroup which has the cancellation property and the difference property.*

Proof: The verification of axioms (i), (ii), (v), and (vi) is immediate. Consider $x, y, z \in \mathbb{E}$.

First, if $(x, y) \in S$ and $(x + y, z) \in S$, then

$$0 \leqslant y \wedge z \leqslant (x + y) \wedge z = 0,$$

hence $(y, z) \in S$, and

$$\begin{aligned} 0 &\leqslant x \wedge (y + z) \\ &= x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = x \wedge z \leqslant (x + y) \wedge z = 0, \end{aligned}$$

hence $(x, y + z) \in S$. Obviously, $x + (y + z) = (x + y) + z$. This proves (iii).

Next, if $(x, z) \in S$, $(y, z) \in S$ and $x + z = y + z$, then

$$x = x \wedge (x \vee z) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = x \wedge y$$

and, similarly, $y = y \wedge x$, hence $x = y$. This proves (iv).

Finally, since $\mathbb{E}$ has relative complements, we may choose $z \in \mathbb{E}$ such that $x \vee z = x \vee y$ and $x \wedge z = 0$. Then we have $(x, z) \in S$ and

$(x \wedge y, z) \in S$, hence

$$x + z = x \vee z = x \vee y$$

and

$$x \wedge y + z = (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$
$$= (x \vee y) \wedge (z \vee y) = (x \wedge z) \vee y = y.$$

This proves (vii). □

3.2. THEOREM: *Suppose $\langle \mathbb{E}, 0, S, +, \leqslant \rangle$ is a lattice ordered partial semigroup which has the cancellation property and the difference property. Then the following are equivalent*:

(*a*) $\mathbb{E}$ *is a Boolean ring with the partial ordering* $\leqslant$ *and the least element* 0.
(*b*) $x + y = x \vee y$ *holds for all* $(x, y) \in S$.
(*c*) $S \subseteq \{(x, y) \in \mathbb{E} \times \mathbb{E} \mid x \wedge y = 0\}$.
(*d*) $S = \{(x, y) \in \mathbb{E} \times \mathbb{E} \mid x \wedge y = 0\}$.

PROOF: Suppose first that (a) holds. Consider $(x, y) \in S$. From $\mathbb{E} = \mathbb{E}_+$ and Theorem 2.8 we have $x \vee y \leqslant x + y$. Since $\mathbb{E}$ has relative complements, we may choose $z \in \mathbb{E}$ such that

$$(x \vee y) \vee z = x + y \quad \text{and} \quad (x \vee y) \wedge z = 0,$$

and from Theorem 2.8 we obtain

$$x \vee y + z = x + y.$$

By Theorem 2.5, we may choose $z_{11}, z_{12}, z_{21}, z_{22} \in \mathbb{E}$ such that

$$x \vee y = z_{11} + z_{12}, \quad z = z_{21} + z_{22}$$

and

$$x = z_{11} + z_{21}, \quad y = z_{12} + z_{22}.$$

From $x \leqslant x \vee y$ and Theorem 2.3 we obtain $z_{21} \leqslant z_{12}$; similarly, $z_{22} \leqslant z_{11}$. Therefore we have

$$0 \leqslant z = z_{21} + z_{22} \leqslant (x \vee y) \wedge z = 0,$$

hence $z = 0$ and $x \vee y = x + y$. This proves (b).

Obviously, (b) implies (c), by Theorem 2.8, and (c) implies (d), by Corollary 2.9.

Finally, suppose that (d) holds. By Theorem 2.10, $\mathbb{E}$ is a distributive lattice. If $x, y \in \mathbb{E}$ are such that $x \leqslant y$, then we may choose $u \in \mathbb{E}_+$ such that $x + u = y$, by Corollary 2.2. By assumption, $x \wedge u = 0$, and Theorem 2.8 yields $x \vee u = x + u = y$. Therefore $\mathbb{E}$ has relative complements. Moreover, for all $x \in \mathbb{E}$, we have $(x, 0) \in S$, by (i), hence $x \wedge 0 = 0$, by assumption, and $0 \leqslant x \vee 0 = x$, by Theorem 2.8. Therefore 0 is the least element of $\mathbb{E}$. This proves (a). □

3.3. Corollary: *Suppose* $\langle \mathbb{E}, 0, S, +, \leqslant \rangle$ *is a lattice ordered partial semigroup which has the cancellation property and the difference property. Define* $S_+ := \{(x, y) \in \mathbb{E} \times \mathbb{E} \mid x \wedge y = 0\}$. *Then* $\langle \mathbb{E}_+, 0, S_+, +, \leqslant \rangle$ *is a lattice ordered partial semigroup which has the cancellation property and the difference property, and* $\mathbb{E}_+$ *is a Boolean ring with the partial ordering* $\leqslant$ *and the least element* 0.

Proof: The first assertion follows from $S_+ \subseteq \mathbb{E}_+ \times \mathbb{E}_+$ and the fact that $(x, y) \in S_+$ implies $x + y \in \mathbb{E}_+$. The second assertion follows from

$S_+ = \{(x, y) \in \mathbb{E}_+ \times \mathbb{E}_+ \mid x \wedge y = 0\}$ and Theorem 3.2. □

Therefore, each lattice ordered partial semigroup which has the cancellation property and the difference property contains a Boolean ring.

4. Lattice ordered groups

A *lattice ordered group* is a commutative group with a partial ordering such that axioms (v) and (vi) hold.

4.1. Theorem: *Suppose* $\mathbb{E}$ *is a lattice ordered group with addition* $+$, *zero element* 0, *and the partial ordering* $\leqslant$. *Define* $S := \mathbb{E} \times \mathbb{E}$. *Then* $\langle \mathbb{E}, 0, S, +, \leqslant \rangle$ *is a lattice ordered partial semigroup which has the cancellation property and the difference property.*

Proof: The verification of axioms (i) through (vi) is immediate.

For all $x, y, z \in \mathbb{E}$ we have $(-x) \vee (-y) = -(x \wedge y)$ and

$z + x \vee y = (z + x) \vee (z + y)$, hence

$$x + y = (x + y) + (-x) \vee (-y) + x \wedge y = y \vee x + x \wedge y.$$

Define $z := x \vee y - x$. Then $z \in \mathbb{E}_+$ and $x \wedge y + z = y$. This proves (vii). □

4.2. Theorem: *Suppose* $\langle \mathbb{E}, 0, S, +, \leqslant \rangle$ *is a lattice ordered partial semigroup which has the cancellation property and the difference property. Then the following are equivalent*:

(*a*) $\mathbb{E}$ *is a lattice ordered group with addition* $+$, *zero element* 0, *and the partial ordering* $\leqslant$.

(*b*) $\mathbb{E}_+ \subseteq \mathbb{E}_*$.

(*c*) $\mathbb{E} = \mathbb{E}_*$.

PROOF: Obviously, (a) implies (b).

Suppose now that (b) holds. Consider $x \in \mathbb{E}$: By Theorem 2.12, there exist $y \in \mathbb{E}_+ \subseteq \mathbb{E}_*$ and $z \in \mathbb{E}_*$ such that $x = y + z$, hence x is invertible. This proves (c).

Finally, suppose that (c) holds. Then $S = \mathbb{E} \times \mathbb{E}$, by Corollary 2.13. This proves (a). □

4.3. COROLLARY: *Suppose $\langle \mathbb{E}, 0, S, +, \leqslant \rangle$ is a lattice ordered partial semigroup which has the cancellation property and the difference property. Define $S_* := \mathbb{E}_* \times \mathbb{E}_*$. Then $\langle \mathbb{E}_*, 0, S_*, +, \leqslant \rangle$ is a lattice ordered partial semigroup which has the cancellation property and the difference property, and $\mathbb{E}_*$ is a lattice ordered group with addition $+$, zero element 0, and the partial ordering $\leqslant$.*

PROOF: Obviously, $(x, y) \in S_*$ implies $x + y \in \mathbb{E}_*$. Moreover, the verification of axioms (i) through (v) is immediate. Consider $x, y \in \mathbb{E}_*$. Then we have $x \vee y + (x \wedge y + x' + y') = 0$ and $x \wedge y + (x \vee y + x' + y') = 0$, by Theorem 2.8 and Corollary 2.13, hence $x \vee y \in \mathbb{E}_*$ and $x \wedge y \in \mathbb{E}_*$. This proves (vi). Furthermore, we may choose $u \in \mathbb{E}_+$ such that $x + u = x \vee y$ and $x \wedge y + u = y$. Then $((x \vee y)' + x) + u = 0$, hence $u \in \mathbb{E}_*$. This proves (vii). Therefore the first assertion holds, and the second one follows from $\mathbb{E}_* = (\mathbb{E}_*)_*$ and Theorem 4.2. □

Therefore, each lattice ordered partial semigroup which has the cancellation property and the difference property contains a lattice ordered group.

5. Remarks

The purpose of this final section is to compare lattice ordered partial semigroups which have the cancellation property and the difference property with earlier solutions of Birkhoff's problem, and to indicate an application of this new concept.

Swamy [5] considered *dually residuated lattice ordered semigroups*. Each Boolean ring $\mathbb{E}$ can be looked at as a dually residuated lattice ordered semigroup if the sum $x + y$ of $x, y \in \mathbb{E}$ is defined to be the supremum of x and y and if the difference $x - y$ is defined to be the relative complement of $x \wedge y$ in x. If $x, y, z \in \mathbb{E}$ are such that $x = 0 \neq y = z$, then $x + z = y + z$ and $x \neq y$. Therefore, there exist dually residuated lattice ordered semigroups in which the cancellation law does not hold.

Rama Rao [3] considered *algebras of species* $(2, 2, 2, -1)$ *with axioms* 1, 2, 3, *and* 4. If $\mathbb{E}$ is such an algebra and if $x + x = 0$ holds for all $x \in \mathbb{E}$, then $\mathbb{E}$ is a Boolean ring and the sum $x + y$ of $x, y \in \mathbb{E}$ is the symmetric difference of x and y. If $x, y, z \in \mathbb{E}$ are such that $x = 0 \neq y = z$, then $x \leqslant y$ and $x + z \nleqslant y + z$. Therefore, there exist algebras of species (2, 2, 2, -1) with axioms 1, 2, 3, and 4 in which order and addition are not compatible.

As another common abstraction of Boolean rings and lattice ordered groups, which is actually close to the notion of a lattice ordered partial semigroup which has the cancellation property and the difference property, Wyler [6] introduced clans. A *clan* is a lattice $\mathbb{E}$ with a set $T \subseteq \mathbb{E} \times \mathbb{E}$ and a map (called *subtraction*) $-: \mathrm{T} \to \mathbb{E}$ such that axioms $C1$, $C2$, $C3$, $C4$, and $C7$ hold. These axioms guarantee the existence of a zero element, and the partial subtraction induces a partial addition which in turn leads to an extension of the originally defined partial subtraction. The induced partial addition is also needed in the formulation of the axioms of symmetry and commutativity.

If $\mathbb{E}$ is a lattice ordered partial semigroup which has the cancellation property and the difference property, then a partial subtraction may be defined on the set $T := \{(x, y) \in \mathbb{E} \times \mathbb{E} | x \leqslant y\}$ by assigning to each pair $(x, y) \in T$ the unique relative complement of x in y, as given by Corollary 2.2. This way, $\mathbb{E}$ becomes a symmetric commutative clan such that the range of its partial subtraction is contained in $\mathbb{E}_+$.

If $\mathbb{E}$ is a lattice ordered group, then $\mathbb{E}$ can be looked at as a symmetric commutative clan in at least two ways: The first of these consists in defining a partial subtraction on the set T as described above, while the second one consists in defining subtraction on $\mathbb{E} \times \mathbb{E}$ by assigning to each pair $(x, y) \in \mathbb{E} \times \mathbb{E}$ the sum of y and the inverse of x. In the latter case, the range of the subtraction is not contained in $\mathbb{E}_+$ unless $\mathbb{E}$ is trivial.

Therefore, the class of all lattice ordered partial semigroups which have the cancellation property and the difference property is strictly smaller than the class of all symmetric commutative clans. In particular, it is free from the ambiguity which exists in the assignment of a clan to a lattice ordered group, and it has the additional advantage that its axioms are given in terms of partial addition alone.

A first step towards the definition of lattice ordered partial semigroups was made in [4] where *ordered partial semigroups* were introduced and used for a unified approach to the Jordan decomposition of signed measures on a Boolean ring and the corresponding decomposition theorem for functionals on a vector lattice. It turned out that the class of all ordered partial semigroups which have the decomposition property and the difference property as given in [4] has to be restricted in order to obtain characterizations of Boolean rings and lattice ordered groups by weak additional properties, and in order to make sure that each additive map on the positive cone of an ordered partial semigroup $\mathbb{E}$ has a unique extension to the whole of $\mathbb{E}$. These observations led us to the definition of lattice ordered partial semigroups which have the cancellation property and the difference property. These are in fact sufficiently close to

Boolean rings and lattice ordered groups, and they admit the solution of the problems described above.

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