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THE CARDINALITY OF THE SET OF INVARIANT MEANS ON A LOCALLY COMPACT TOPOLOGICAL SEMIGROUP

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Abstract

For a large class of locally compact topological semigroups, which include all non-compact, σ-compact and locally compact topological groups as a very special case, we show that the cardinality of the set of all invariant means on the space of weakly uniformly continuous functions is either 0 or $\geqslant 2^c$; where c denotes the cardinality of continuum.

1. Preliminaries

Throughout this paper, let S denote a (Hausdorff, jointly continuous and) locally compact topological semigroup, $C(S)$ the space of all bounded complex-valued continuous functions on S and $M(S)$ the Banach algebra of all complex-valued bounded Radon measures on S with convolution multiplication given by

$$\nu^*\mu(f) := \int\int f(xy)\mathrm{d}\nu(x)\mathrm{d}\mu(y) = \int\int f(xy)\mathrm{d}\mu(y)\mathrm{d}\nu(x)$$

for all ν, $\mu \in M(S)$ and $f \in C(S)$. Corresponding to each function f in $C(S)$, measure ν in $M(S)$ and point x in S we have the functions ${}_xf$, f_x, $\nu \circ f$ and $f \circ \nu$ in $C(S)$ given by

$${}_xf(y) := f(xy) \quad \text{and} \quad f_x(y) := f(yx) \quad (y \in S)$$

$$\nu \circ f(y) := \nu(f_y) \quad \text{and} \quad f \circ \nu(y) := \nu({}_yf) \quad (y \in S).$$

A function in $C(S)$ is said to be *weakly uniformly continuous* if it belongs to the set $\mathrm{WUC}(S) := \{f \in C(S)$: the maps $x \to {}_xf$ and $x \to f_x$ of S into $C(S)$ are weakly continuous$\}$.

For any subsets A and B of S and point x in S, we write $AB := \{ab$: $a \in A$ and $b \in B\}$, $A^{-1}B := \{y \in S$: $ay \in B$ for some $a \in A\}$, $x^{-1}B := \{x\}^{-1}B$ and $A^{-1}x := A^{-1}\{x\}$. By symmetry we similarly define BA^{-1}, Bx^{-1} and xA^{-1}. An object playing a pivotal role in this paper is the algebra of all *absolutely continuous measures* on S - namely $M_a(S) := \{\nu \in M(S)$: the maps $x \to |\nu|(x^{-1}K)$ and $x \to |\nu|(Kx^{-1})$ of S into $\mathbb{R}$ are continuous for all compact $K \subseteq S\}$. (Here $|\nu|$ denotes the measure arising from the total variation of ν.) For various studies on the algebra $M_a(S)$, see e.g. [1], [5] and the references cited there. Taking $\text{supp}(\nu) := \{x \in S$: $|\nu|(X) > 0$ for every open neighbourhood X of $x\}$ we define *the foundation* of $M_a(S)$, namely $F_a(S)$, to be the closure of the set $\cup\{\text{supp}(\nu)$: $\nu \in M_a(S)\}$. For a function f in $C(S)$ we also write $\text{supp}(f) :=$ closure of $\{x \in S: f(x) \neq 0\}$.

If $h \in M(S)^*$ and $\nu \in M(S)$ we define the functional $\nu \odot h$ on $M(S)$ and function $\nu \circ h$ on S by

$$\nu \odot h(\mu) := h(\nu * \mu) \quad \text{and} \quad \nu \circ h(x) := h(\nu * \bar{x})$$

$$(\mu \in M(S) \quad \text{and} \quad x \in S);$$

where $\bar{x}$ stands for the point mass at x. By symmetry one similarly defines $h \odot \nu$ and $h \circ \nu$. By a *mean* m on $M_a(S)^*$ (or WUC(S)) we mean a functional such that $m(1) = 1$ and $m(f) \geqslant 0$ if $f \geqslant 0$; for all f in $M_a(S)^*$ (or WUC(S), respectively). A mean m on WUC(S) is said to be *invariant* (or *topologically invariant*) if $m({}_xf) = m(f_x) = m(f)$ (or $m(\nu \circ f) = m(f \circ \nu) = m(f)\nu(S)$, respectively), for all $f \in$ WUC(S), $x \in S$ and $\nu \in M(S)$. Similarly one defines a (topologically) invariant mean on $M_a(S)^*$. We denote the set of all invariant means on WUC(S) by IM(WUC(S)).

For ease of reference we mention the following special case of a result proved in [6] and which can also be deduced from the two main theorems of [14].

1.1. PROPOSITION: *If S has an identity element and coincides with the foundation of $M_a(S)$, the following items are equivalent*:

(*a*) *There exists a topological invariant mean on $M_a(S)^*$.*

(*b*) *There exists an invariant mean on* WUC(S).

The following result is also proved in [6].

1.2. PROPOSITION: *Every invariant mean on* WUC(S) *is topologically invariant.*

For any subsets A, B, C of S we write $A \otimes B := \{AB, A^{-1}B, AB^{-1}\}$ and $A \otimes B \otimes C := (\cup\{A \otimes D\colon D \in B \otimes C\}) \cup (\cup\{D \otimes C\colon D \in A \otimes B\})$. So inductively we can define $A_1 \otimes A_2 \otimes \ldots \otimes A_n$ for any subsets $A_1, \ldots, A_n$ of S. Following [7], a subset B of S is said to be *relatively neo-compact* if B is contained in a (finite)union of sets in $A_1 \otimes A_2 \otimes \ldots \otimes A_2$ for some compact subsets $A_1, A_2, \ldots, A_n$ of S. In particular, as noted in [7]:

For a topological semigroup S such that $C^{-1}D$ and DC^{-1} are compact whenever C and D are compact subsets of S, we have that $B \subseteq S$ is relatively neo-compact if and only if B is relatively compact. (So relatively neo-compact subsets of a topological group are precisely the relatively compact subsets.)

The *centre* of $E \subset S$ is the set $Z(E) := \{x \in S\colon xy = yx \text{ for all } y \in E\}$. Let $P(M_a(S)) := \{\nu \in M_a(S)\colon \nu \geqslant 0 \text{ and } \|\nu\| = 1\}$. A net or sequence $(\mu_\alpha) \subset P(M_a(S))$ is said to be *weakly* (or *strongly*) *convergent to topological invariance* if $\nu * \mu_\alpha - \mu_\alpha \to 0$ and $\mu_\alpha * \nu - \mu_\alpha \to 0$ weakly (or strongly, respectively) in $M_a(S)$, for all $\nu \in P(M_a(S))$.

Let $F := \{\phi \in (l^\infty)\colon \phi > 0,\ \|\phi\| = 1 \text{ and } \phi(g) = 0 \text{ for all } g \in l^\infty \text{ such that } g(k) \to 0 \text{ as } k \to \infty\}$ and c be the cardinality of continuum.

Our aim in this paper is to prove the following result.

THEOREM: *Let S be a σ-compact locally compact topological semigroup such that S is not relatively neo-compact, $M_a(S)$ is non-zero and there exists an invariant mean on* WUC(S). *Suppose either* (a) *S has an identity element and coincides with the foundation of $M_a(S)$, or* (b) *the centre of $F_a(S)$ is not $M_a(S)$-negligible. Then there exists a linear isometry $\tau\colon (l^\infty)^* \to$ WUC$(S)^*$ such that*

$$\tau(F) \subset \mathrm{IM}(\mathrm{WUC}(S)) \quad \textit{and so} \quad \mathrm{card}\,(\mathrm{IM}(\mathrm{WUC}(S))) \geqslant 2^c.$$

2. Proof of the theorem

We partition part of our proof into some lemmas. In the following lemma $C_0(S)$ denotes the space of functions in $C(S)$ which are arbitrarily small outside compact sets and $C_{00}(S) := \{f \in C(S)\colon f \text{ vanishes outside some compact subset of } S\}$.

2.1. LEMMA: *If S is not relatively neo-compact, then*

$$m(f) = 0 \quad \textit{for all} \quad f \in C_0(S) \quad \textit{and} \quad m \in \mathrm{IM}(\mathrm{WUC}(S))$$

PROOF: Let $f \in C_{00}(S)$ be positive and take K to be the support of f (i.e. supp(f)). Since S is not relatively neo-compact, we can choose a sequence $\{x_n\}$ in S such that

$$x_{n+1} \notin K\left(x_1^{-1}K \cup \ldots \cup x_n^{-1}K\right)^{-1} \quad \text{for all} \quad n \in \mathbb{N}.$$

So if $n \neq k$ we have $x_n^{-1}K \cap x_k^{-1}K = \emptyset$ or, equivalently,

$$\mathrm{supp}\left({}_{x_n}f\right) \cap \mathrm{supp}\left({}_{x_k}f\right) = \emptyset.$$

consequently

$$nm(f) = m\left({}_{x_1}f + \ldots + {}_{x_n}f\right) \leqslant \|f\|_S \qquad \text{for all} \quad n \in \mathbb{N},$$

and so $m(f) = 0$.

Since $C_{00}(S)$ is dense in $C_0(S)$, the remainder of our result follows trivially.

2.2. LEMMA: *There is a net in $P(M_a(S))$ weakly convergent to topological invariance if and only if there is a net in $P(M_a(S))$ strongly convergent to topological invariance.*

PROOF: Suppose $(\eta_\alpha) \subseteq P(M_a(S))$ converges weakly to topological invariance. Setting $M_{\nu,\mu} := M_a(S)$ we form the locally convex product space $M := \Pi\{M_{\nu,\mu}: (\nu, \mu) \in P(M_a(S)) \times P(M_a(S))\}$ with the product of norm topologies and define the linear map $L: M_a(S) \to M$ by

$$L(\phi)(\nu, \mu) := \nu * \phi * \mu - \phi$$

$$\text{for all} \quad \phi \in M_a(S) \qquad \text{and} \quad \nu, \mu \in P\left(M_a(S)\right).$$

As noted in [11, page 160], the weak topology on M coincides with the product of the weak topologies on the $M_{\nu,\mu}$'s. Since $\nu * \eta_\alpha * \mu - \eta_\alpha \to 0$ weakly, for all ν, $\mu \in P(M_a(S))$, $0 \in$ weak-closure $L(P(M_a(S)))$. Since $L(P(M_a(S)))$ is a convex subset of the locally convex space M, we have weak-closure $L(P(M_a(S)))$ = strong-closure $L(P(M_a(S)))$, by the Hahn-Banach Theorem. So there exists a net $(\rho_\beta) \subseteq P(M_a(S))$ such that $L(\rho_\beta) \to 0$ in M or, equivalently, such that

$$\left\|\nu * \rho_\beta * \mu - \rho_\beta\right\| \to 0 \qquad \text{for all} \quad \nu, \mu \in P\left(M_a(S)\right).$$

Hence

$$\begin{aligned} \left\|\nu * \rho_\beta - \rho_\beta\right\| &\leqslant \left\|\nu * \rho_\beta - \nu * \nu * \rho_\beta * \nu\right\| + \left\|(\nu * \nu) * \rho_\beta * \nu - \rho_\beta\right\| \\ &\leqslant \left\|\rho_\beta - \nu * \rho_\beta * \nu\right\| + \left\|(\nu * \nu) * \rho_\beta * \nu - \rho_\beta\right\| \to 0 \end{aligned}$$

for all $\nu \in P(M_a(S))$. Similarly $\left\|\rho_\beta * \nu = \rho_\beta\right\| \to 0$, an the remainder of our lemma follows trivally.

2.3. LEMMA: *Let S be not relatively neo-compact and let $\{\mu_n\}$ be a sequence in $P(M_a(S))$ strongly convergent to topological invariance and such that $T_n :=$ supp(μ_n) is compact, for all $n \in \mathbb{N}$. Then, given $n_0 \in \mathbb{N}$ and $\epsilon > 0$, we can find $n > n_0$ such that*

$$\mu_n\left((T_1 \cup \ldots \cup T_{n_0}) \cap T_n\right) < \epsilon.$$

PROOF: Suppose, on the contrary, there exists an n_0 and $\epsilon > 0$ such that

$$\mu_n\big((T_1 \cup \ldots \cup T_{n_0}) \cap T_n\big) > \epsilon \qquad \text{for all} \quad n > n_0.$$

Let $f \in C_0(S)$ be a positive function with $f = 1$ on $T_1 \cup \ldots \cup T_{n_0}$ and note that

$$\mu_n(f) > \epsilon \qquad \text{for all} \quad n > n_0.$$

Let m be any weak* - cluster point of $\{\mu_n\}$ in WUC$(S)^*$ and note that m is such that $m(f) > \epsilon$ and $m(\nu \circ g) = m(g)(\nu \in P(M_a(S))$ and $g \in$ WUC(S). Since $M_a(S)$ is an ideal of $M(S)$, see e.g. [1] or [5], we have $\bar{x}*\nu \in P(M_a(S))$ whenever $x \in S$ and $\nu \in P(M_a(S))$. Now $x*\nu \circ g = \nu \circ {}_x g$ and so $m(g) = m(x*\nu \circ g) = m(\nu \circ {}_x g) = m({}_x g)$; similarly $m(g_x) = m(g)(g \in$ WUC(S) and $x \in S)$. By Lemma 2.1, we must have $m(f) = 0$. This contradiction implies our result.

2.4. LEMMA: *Let S be σ-compact with $M_a(S)$ non-zero and let there be an invariant mean on* WUC(S). *Then there exists a sequence (ρ_n) in $P(M_a(S))$ converging strongly to topological invariance and such that $K_n :=$* supp(ρ_n) *is compact, for all $n \in \mathbb{N}$, if any one of the following conditions holds*:

(*a*) *S is the foundation of $M_a(S)$ and S has an identity element.*

(*b*) *The centre of $F_a(S)$ is not $M_a(S)$-negligible.*

PROOF: First we show that there exists a net in $P(M_a(S))$ weakly convergent to topological invariance, if (a) or (b) holds. To this end, suppose (a) holds. Then there exists a topological invariant mean m on $M_a(S)^*$, by Proposition 1.1. Now $m \in$ weak*-closure $(P(M_a(S)))$ in $M_a(S)^{**}$. Consequently, there exists a net (μ_α) in $P(M_a(S))$ such that $\mu_\alpha(h) \to m(h)$ for all $h \in M_a(S)^*$. In particular, for each $\nu \in P(M_a(S))$ we have

$$|h(\nu*\mu_\alpha - \mu_\alpha)| = |\mu_\alpha(\nu \odot h) - \mu_\alpha(h)| \to |m(\nu \odot h) - m(h)| = 0,$$

for all $h \in M_a(S)^*$. Similarly $\mu_\alpha * \nu - \mu_\alpha \to 0$ weakly. Thus $((\mu_\alpha) \subseteq P(M_a(S))$ is weakly convergent to topological invariance.

Next suppose condition (b) holds. Then we can choose $\tau \in P(M_a(S))$ such that supp$(\tau) \subset Z(F_a(S))$ (, where $Z(F_a(S))$ denotes the centre of $F_a(S)$). We then have $\tau*\nu = \nu*\tau$ for all $\nu \in P(M_a(S))$. Now if m_0 is any invariant mean on WUC(S) we have that m_0 is topologically invariant, by proposition 1.2. Let (η_α) be a net in $P(M_a(S))$ such that $\eta_\alpha(f) \to m_0(f)$, for all $f \in$ WUC(S). Then for any $\nu \in P(M_a(S))$ and $h \in M_a(S)^*$, we have that $\tau*\nu \circ h \circ \tau$, $\tau \circ h \circ \tau \in$ WUC(S) by [7, Lemma 4.1].

Hence, if $\mu_\alpha := \tau^*\eta_\alpha^*\tau$,

$$\begin{aligned} h(\nu^*\mu_\alpha - \mu_\alpha) &= h(\nu^*\tau^*\eta_\alpha^*\tau - \tau^*\eta_\alpha^*\tau) \\ &= h(\tau^*\nu^*\eta_\alpha^*\tau - \tau^*\eta_\alpha{}^*\tau) \\ &= \eta_\alpha(\tau^*\nu \circ h \circ \tau) - \eta_\alpha(\tau \circ h \circ \tau) \\ &\to m_0(\tau^*\nu \circ h \circ \tau) - m_0(\tau \circ h \circ \tau) \\ &= m_0(\nu \circ (\tau \circ h \circ \tau)) - m_0(\tau \circ h \circ \tau) = 0. \end{aligned}$$

Similarly $h(\mu_\alpha^*\nu - \mu_\alpha) \to 0$. Thus (μ_α) is weakly convergent to topological invariance.

Now suppose either (a) or (b) holds. Then there exists a net $(\eta_\beta) \subset P(M_a(S))$ strongly convergent to topological invariance, by Lemma 2.2. Fix any $\lambda \in P(M_a(S))$ and set $\mu_\beta := \lambda^*\eta_\beta{}^*\lambda$. Note that (μ_β) is also strongly convergent to topological invariance. As S is σ-compact, we can choose an increasing sequence of compact neighbourhoods, $D_1 \subset D_2 \subset \ldots$, such that $S = \cup_{n=1}^{\infty} D_n$. Noting that the maps $(x, y) \to \bar{x}^*\mu_\beta{}^*\bar{y}$ of $S \times S$ into $M_a(S)$ are norm continuous (see e.g. [5, Corollary 4.10 (ii)]) we can choose a sequence $\{\mu_{\beta_1}, \mu_{\beta_2}, \ldots\}$ from the μ_β's such that

$$\|\bar{x}^*\mu_{\beta_n}{}^*\bar{y} - \mu_{\beta_n}\| < \frac{1}{5n} \qquad \text{for all} \quad x, y \in D_n.$$

Choose compact sets K_n such that

$$\mu_{\beta_n}(S \setminus K_n) < \frac{1}{5n} \qquad \text{for all} \quad n \in \mathbb{N}.$$

Setting $\rho := (\mu_{\beta_n}(K_n))^{-1}\mu_{\beta_n|K_n}$ we have $(\rho_n) \subset P(M_a(S))$, $\operatorname{supp}(\rho_n) \subset K_n$ and a standard technical argument shows that

$$\|\mu_{\beta_n} - \rho_n\| < 2\mu_{\beta_n}(S \setminus K_n) < \frac{2}{5n}.$$

Consequently

$$\begin{aligned} \|\bar{x}^*\rho_n{}^*\bar{y} - \rho_n\| &< \|\bar{x}^*\rho_n{}^*\bar{y} - \bar{x}^*\mu_{\beta_n}{}^*\bar{y}\| + \|\bar{x}^*\mu_{\beta_n}{}^*\bar{y} - \mu_{\beta_n}\| + \|\mu_{\beta_n} - \rho_n\| \\ &< 2\|\rho_n - \mu_{\beta_n}\| + \|\bar{x}^*\mu_{\beta_n}{}^*\bar{y} - \mu_{\beta_n}\| \\ &< \frac{4}{5n} + \frac{1}{5n} = \frac{1}{n} \qquad \text{for all} \quad x, y \in D_n. \end{aligned}$$

Now for any $\nu, \eta \in P(M_a(S))$ with compact supports we have $\operatorname{supp}(\nu) \cup \operatorname{supp}(\eta) \subset D_n$ for n larger than some n_0 and hence

$$\begin{aligned}\|\nu^*\rho_n{}^*\eta - \rho_n\| &\coloneqq \sup\left\{|\nu^*\rho_n{}^*\eta(f) - \rho_n(f)| : f \in C_0(S),\ \|f\|_S \leqslant 1\right\}\\ &\leqslant \sup\left\{\int |\bar{x}^*\rho_n{}^*\bar{y}(f) - \rho_n(f)| \mathrm{d}\nu(x)\mathrm{d}\eta(y) :\right.\\ &\qquad \left. f \in C_0(S),\ \|f\|_S \leqslant 1\right\}\\ &< \frac{1}{n}.\end{aligned}$$

It is now trivial to complete the proof of our lemma.

2.5. Proof of our theorem: Note that we have the hypothesis of Lemma 2.4 met and so let $\{\rho_n\}$ and $\{K_n\}$ be as in Lemma 2.4. Choose $\nu \in P(M_a(S))$ with $C \coloneqq \operatorname{supp}(\nu)$ compact and note that the sequence $\{\nu^*\rho_n{}^*\nu\}$ also converges strongly to topological invariance.

Observing that $T_n \coloneqq \operatorname{supp}(\nu^*\rho_n{}^*\nu) = CK_nC$ is compact $(n \in \mathbb{N})$ and recalling Lemma 2.3, there exist subsequences $\{T_{n_k}\}$ of $\{T_n\}$ and $\{\nu^*\rho_{n_k}{}^*\nu\}$ of $\{\nu^*\rho_n{}^*\nu\}$ such that

$$\nu^*\rho_{n_k}{}^*\nu\left(T_{n_k}\backslash\left(T_{n_0} \cup \ldots \cup T_{n_{k-1}}\right)\right) > \tfrac{1}{2}$$

$$\text{for} \quad k \in \mathbb{N}, \quad \text{where} \quad T_{n_0} \coloneqq \emptyset.$$

Let $F_k \coloneqq T_{n_k}\backslash(T_{n_1} \cup \ldots \cup T_{n_{k-1}})$ and $\mu_k \coloneqq \nu^*\rho_{n_k}{}^*\nu$, for all $k \in \mathbb{N}$.

Let Π: $\mathrm{WUC}(S) \to l^\infty$ be the linear mapping defined by

$$\Pi(f)(k) \coloneqq \rho_{n_k}(f)$$

$$\text{for all} \quad f \in \mathrm{WUC}(S) \quad \text{and} \quad k \in \mathbb{N}.$$

To see that Π is onto, let $g \in l^\infty$ be fixed. Since members of the sequence $\{F_k\}$ are clearly pairwise disjoint and $\mu_k(F_k) > \frac{1}{2}$, the function

$$h \coloneqq \sum_{k=1}^{\infty} \frac{g(k)}{\mu_k(F_k)} \chi_{F_k}$$

is a linear functional in $M(S)^*$. (Here χ_{F_k} is the characteristic function of

F_k.) Consequently $\nu \circ h \circ \nu \in \mathrm{WUC(S)}$, by [7, Lemma 4.1]. Now, for all $k \in \mathbb{N}$, we have

$$\begin{aligned}\Pi(\nu \circ h \circ \nu)(k) &:= \rho_{n_k}(\nu \circ h \circ \nu)\\ &= \nu^* \rho_{n_k}{}^* \nu(h)\\ &= \mu_k\left(\sum_{i=1}^{\infty} \frac{g(i)}{\mu_i(F_i)} \chi_{F_i}\right)\\ &= g(k).\end{aligned}$$

Thus Π maps the function $\nu \circ h \circ \nu$ onto g and Π is onto. Further, we clearly have

$$\|g\|_\infty = \|\Pi(\nu \circ h \circ \nu)\|_\infty = \|\nu \circ h \circ \nu\|_S.$$

It follows that the dual map Π^*: $(l^\infty)^* \to \mathrm{WUC(S)}^*$ is a linear isometry.

To see that $\Pi^*\mathrm{F} \subset \mathrm{IM(WUC(S))}$, let $\phi \in F$ be fixed. Then, clearly

$$\Pi^*\phi > 0 \quad \text{and} \quad \Pi^*\phi(1) = \phi(1) = 1.$$

Now, for any $\eta \in P(M_a(S))$ and $f \in WUC(S)$, we have

$$\begin{aligned}\Pi(\eta \circ f - f)(k) &:= \rho_{n_k}(\eta \circ f - f)\\ &= (\eta^* \rho_{n_k} - \rho_{n_k})(f) \to 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

Recalling the definition of F we have

$$\Pi^*\phi(\eta \circ f - f) = 0.$$

Similarly $\Pi^*\phi(f \circ \eta - f) = 0$ and so $\Pi^*\phi \in \mathrm{IM(WUC(}S\mathrm{))}$.

Taking $\beta\mathbb{N}$ to be the Stone-Chech compactification of $\mathbb{N}$, we have $\beta\mathbb{N} \setminus \mathbb{N} \subset F$. Since card $(\beta\mathbb{N} \setminus \mathbb{N}) = 2^c$ and Π^* is an isometry, we thus get card $(\mathrm{IM}(WUC(S))) \geqslant \mathrm{card}(F) \geqslant 2^c$. So $\tau := \Pi^*$ is the required map and our Theorem is proved.

3. Note on references

Many results on the sizes of sets of invariant means can be found in the literature: for discrete semigroups see e.g. Chou ([2] and [3]), Granirer [8] and Klawe [12]; and for locally compact topological groups see e.g. Chou ([3] and [4]) and Granirer [8]. Related results dealing with the difference between an invariant and a topologically invariant mean can be found in the papers of Rosenblatt [15] and Liu and Van Rooij [13]. Our main theorem generalizes some of these results and our techniques are inspired by the paper of Chou [4]. The proof of lemma 2.2 is closely related to that given in Greenleaf [9, Theorem 2.42].

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