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PICARD GROUPS OF ZARISKI SURFACES

Piotr Blass

To Kirsten

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Introduction

This paper originated in the following philosophical framework. Inspired by M. Noether's theorem [4] and facts about generic double planes over C, [10], as well as by Rudakov's and Šafarevič's computations in characteristic two [8] we were led to suspect that the ring $k[x,y,z]/(Z^P - f(x,y))$ is a UFD provided f is a sufficiently general polynomial of degree m. Here $k=\bar{k}$, characteristic $k=p \geqslant 5$, $m \geqslant 5$, say. The above conjecture still remains open but here we offer a result that in our opinion renders it extremely probable, to say the least, in the case when p divides m.

We consider in the paper a generic polynomial with indeterminate coefficients. $F=\Sigma T_{ij} X^i Y^j$, where $0 \leqslant i+j \leqslant p$, and the ring $R = L[X,Y,Z]/(Z^P - F)$, $L=\overline{k(T_{ij})}$, where T_{ij} are algebraically independent transcendentals over k, deg $F=p$. We show that R is a UFD. This is our main theorem (MTH) in its algebraic form. In a subsequent paper we hope to prove our original conjecture and also to generalize the result of this paper to degrees of $f \neq p$.

The proof given here follows very closely an outline shown to the author by Deligne. It is conceptually akin to SGA VII exp XIX. A new feature is, for example, DRL $\to$ MTH (see Chapter V).

Let us outline the main idea of the proof, which is simple, if somewhat lengthy in detail. Let $\bar{S} = \operatorname{Proj} L[X, Y, Z, Z_0]/(Z^P - F(X, Y, Z_0))$. $\bar{S}$ has $N = p^2 - 3p + 3$ rational double points of type A_{p-1}, (Sing $\bar{S}$). The galois group $G = \operatorname{Gal}(\overline{k(T_{ij})} : k(T_{ij}))$ acts on $\bar{S}$ and it permutes the N elements of Sing $\bar{S}$.

Step I is to show that G induces the full symmetric group on Sing $\bar{S}$.

Step II is to refine the above and to analyze more closely the action of G on Pic $\tilde{S}$ ($\tilde{S}$ is the minimal desingularization of $\bar{S}$). It turns out that there are elements of G which induce the identity on Sing $\bar{S}$ but nevertheless act non-trivially on certain exceptional curves contained in $\tilde{S}$. See Double Reversal Lemma (DRL) and Gal B below.

Step III is to deduce from DRL the fact that Pic $\tilde{S}$ is generated by the obvious curves, namely the exceptional curves for $\tilde{S} \to \bar{S}$ and a very ample curve on $\bar{S}$ pulled back to $\tilde{S}$.

Step III provides us already with the proof of our main theorem in its geometric form (MTHG). The equivalence of the geometric form with the algebraic from (R is a UFD) follows from a simple exact sequence which we had introduced in a previous paper [1], see (0.5.1) below.

Finally, let us remark that Jeff Lang has proven a number of related results in his Ph.D. thesis [6]. His method was to use differential equations in characteristic $p > 0$.

At a crucial point in our proof we use the fact that Pic$(\tilde{S})$ has no p-torsion. We had conjectured it, but the first proof is due to W.E. Lang whose result we quote from [7].

It is not hard to extend our main theorem to the case where F is replaced by the polynomial not of degree p but rather of degree pe, $e = 1, 2, \ldots$. We leave this extension as an exercise to the reader. In a subsequent paper we hope, among other things, to examine the transition from "generic" to "general" and to prove our original conjecture at least in the case when $p|m$. *

Chapter I. Notation; statement of the main theorem and of the principal theorems and lemmas used in its proof

0. Notation

(0.1) $k = \bar{k}$ is an algebraically closed field of characteristic $p \geqslant 5$. T_{ij} are

* See appendix to this paper, *Comp. Math.* 54 (1985) 37–40.

indeterminates algebraically independent over k, $0 \leqslant i+j \leqslant p$.

$$F(X, Y) = \sum_{0 \leqslant i+j \leqslant p} T_{ij} X^i Y^j$$

$\sum$ stands for $\sum_{0 \leqslant i+j \leqslant p}$ unless stated to the contrary.

F_x, F_y means $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}$, etc.

$$H(F) = \text{“hessian of } F\text{”} = \det \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$$

$L = \overline{k(T_{ij})}$, the algebraic closure of $k(T_{ij})$.

$G = \mathrm{Gal}(\overline{k(T_{ij})} : k(T_{ij}))$.

$A = \mathrm{Spec}\, k[T_{ij}]$

$E = \mathrm{Spec}(k[T_{ij}][X,Y]/(F_x, F_y))$

$D = \mathrm{Spec}(k[T_{ij}][X,Y,W]/(F_x, F_y, W^2 - H(F)))$.

We have the natural morphisms

$$D \xrightarrow{\phi} E \xrightarrow{\pi} A.$$

If $X \to A$ is a morphism we denote by E_X the scheme $E_A \times X$ and by π_X the projection π_X: $E_X \to X$.

If $U \subset A$ is open or closed π_U: $E_U \to U$ has the above meaning with respect to the inclusion map $U \to A$. We apply the same convention to maps $D \to E$, $D \to A$, $X \to E$, $X \to A$.

(0.2) We will identify closed points of A with polynomials of degree p in $k[X,Y]$. The following subset of A will be important to us, $V \leqslant A$ corresponding to polynomials $g \in k[X,Y]$ of degree p such that the surface $Z^p = g$ has no singularities at infinity. It is not hard to prove that V is open and dense in A. This follows from the following fact:

Let us write out $g(X, Y)$ in terms of homogeneous parts:

$$g = g_p + g_{p-1} + \dots + g_1 + g_0.$$

Then, $g \in V$ if and only if the system of equations $g_{p-1} = 0$, $\partial g_p/\partial x = 0$, $\partial g_p/\partial y = 0$ has only the solution $x = 0, y = 0$. Further, we define a subset

$U \subset V$ as follows $g \in U$ if and only if $g \in V$ and g has only non-degenerate singularities (i.e., $g_x = g_y = 0$ implies hessian of $g \neq 0$). We will show below that U is open and dense in V, see (3.1.3).

(0.3) In this paper we will constantly discuss an algebraic surface over L. Let us fix once and for all some notation relative to that surface. $R = L[X,Y,Z]/(Z^p - F(X, Y))$. S^{affine} or $S^{\text{aff}} = \operatorname{Spec}(L[X,Y,Z]/(Z^p - F(X, Y)))$ where F is as above. $S^{\text{aff}} = \operatorname{Spec} R$. $\bar{S} = \operatorname{Proj}(L[X,Y,Z,Z_0]/(Z^p - F(X, Y, Z_0)))$ where $F(X, Y, Z_0)$ is F homogenized (sometimes we will write $F(X, Y, X_0)$). η: $\tilde{S} \to \bar{S}$ a minimal desingularization of $\bar{S}$ over L. P_L^n denotes the projective n-space over L.

The following facts can be shown about S^{aff} and $\bar{S}$. There is a natural projection, a finite map:

$$\mu: \bar{S} \to P_L^2 \quad (\text{see [1] Section 1}).$$

We will denote by l the element of $\operatorname{Pic} \tilde{S}$ given by:

$$\eta^*\mu^*\left(O_{P_L^2}(1)\right).$$

$\bar{S}$ has a set of isolated singularities denoted $\operatorname{Sing}(\bar{S})$. One can show that:

$$\operatorname{Sing}(\bar{S}) \subset S^{\text{aff}} \quad (\text{see lemma (2.4.1) below}).$$

Thus

$$\operatorname{Sing}(S) = \operatorname{Sing}(\bar{S}).$$

Also, all the singularities are rational double points of type A_{p-1}, see (3.1.14). Further, the number of singularities is $p^2 - 3p + 3$, see (3.1.12).

The elements of Sing S will be denoted by Greek letters α, β, γ, etc. If we need to write out their coordinates in A_L^3 we will write $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, etc. In particular, we define

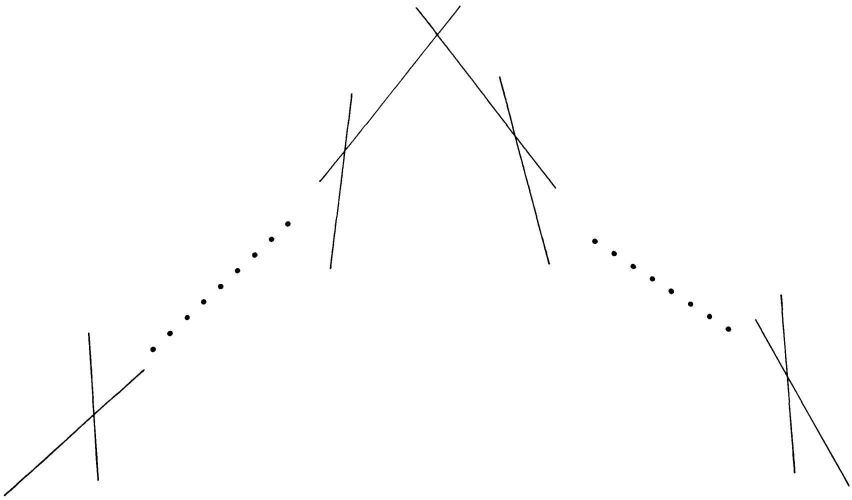
$$H_\alpha = \left(F_{xx}F_{yy} - F_{xy}^2\right)(\alpha_1, \alpha_2).$$

(0.4) The following facts about the minimal resolution map η: $\tilde{S} \to \bar{S}$ are well known, see [1] Section 1.

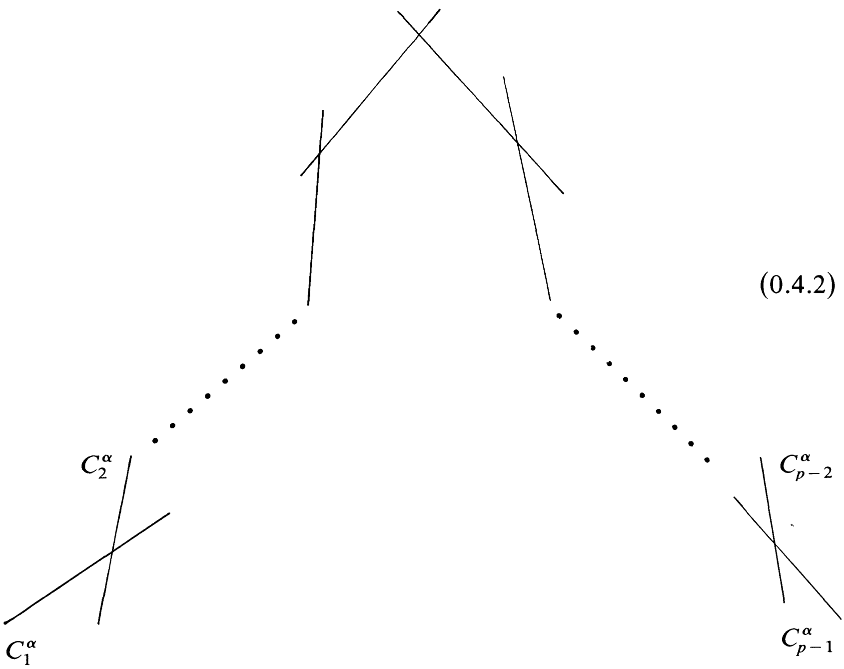
Let $\alpha \in \operatorname{Sing} \bar{S}$, then $\eta^{-1}(\alpha)$ consists of a tree of $(p-1)$ rational curves over L with intersection matrix (in $\operatorname{Pic} \tilde{S}$)

$$\begin{bmatrix} -2 & 1 & & & \mathbf{0} \\ 1 & -2 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots & 1 \\ \mathbf{0} & & & & 1 & -2 \end{bmatrix} \tag{0.4.1}$$

The tree can be pictured as follows:



We wish to emphasize the following subtle point which is of key importance in this paper. There are two possible natural choices of ordering the curves of the tree (from left to right or from right to left). We choose from the outset one orientation for each singularity $\alpha \in \operatorname{Sing} \bar{S}$ and we number the curves accordingly. Thus the tree will now be labelled



(0.4.2)

T_α stands for the ordered (tree) sequence $C_1^\alpha, C_2^\alpha, \ldots, C_{p-1}^\alpha$. T_α^{opp} stands for $C_{p-1}^\alpha, C_{p-2}^\alpha, \ldots, C_1^\alpha$.

For each $\alpha \in \operatorname{Sing} \bar{S}$ we now define a special element of Pic $\tilde{S}$

$$D_\alpha = C_1^\alpha + 2C_2^\alpha + \ldots + (p-1)C_{p-1}^\alpha \tag{0.4.3}$$

and

$$D_\alpha^{\text{opp}} = C_{p-1}^\alpha + 2C_{p-2}^\alpha + 3C_{p-3}^\alpha + \ldots + (p-1)C_1^\alpha \tag{0.4.4}$$

D_α has the following properties. First of all, $D_\alpha \cdot C_j^\alpha \equiv O(p)$ for all j, $1 \leqslant j \leqslant p-1$. Also

$$D_\alpha^2 = -p(p-1) \tag{0.4.5}$$

$$D_\alpha + D_\alpha^{\text{opp}} = pC_\alpha = p\left(C_1^\alpha + C_2^\alpha + \ldots + C_{p-1}^\alpha\right) \tag{0.4.6}$$

$$D_\alpha - D_\alpha^{\text{opp}} = 2D_\alpha - pC_\alpha.$$

(0.5) We will denote by

$$\operatorname{Pic}^{\text{obvious}}(\tilde{S}) \quad \text{or} \quad \operatorname{Pic}^{\text{ob}}(\tilde{S})$$

the subgroup generated by l and the C_j^α, $\alpha \in \operatorname{Sing}(\bar{S})$, $1 \leqslant j \leqslant p-1$. There is an intersection form on Pic $\tilde{S}$, we denote it by $A \cdot B$, $A, B \in \operatorname{Pic} \tilde{S}$. It can be shown that $l \cdot l = l^2 = p$. Also, we have the exact sequence

$$0 \to \operatorname{Pic}^{\text{ob}}\tilde{S} \to \operatorname{Pic} \tilde{S} \to \operatorname{Cl} R \to 0. \tag{0.5.1}$$

And Cl R is a finite elementary p group, see [6] or [9]. It was shown by W.E. Lang that Pic$(\tilde{S})$ has no torsion [7].

(0.6) Let X be a scheme (noetherian), Et(X) the category of finite étale coverings of X.

Let Ω be an algebraically closed field b: Spec $\Omega \to X$, a geometric point of X. Let $Y \in \operatorname{Et}(X)$.

$F_b^X(Y)$ is the set of liftings

$$\begin{tikzcd} & Y \arrow[d] \\ b:\operatorname{Spec}\Omega \arrow[ur] \arrow[r] & X \end{tikzcd}$$

If $W \to X$ is a morphism, we get the base change functor Et $X \to$ Et W denoted R_W or just R.

If X, Y are schemes we denote by $X \sqcup Y$ the disjoint union of X and Y.

(0.7) In the following definition the ground field is assumed algebraically closed of characteristics $\neq 2$.

A, B are smooth curves, B irreducible, π finite.

DEFINITION: $A \overset{\pi}{\dashrightarrow} B$ is called *very simple* if there exists a point $\bar{q} \in A$ with $\pi(\bar{q}) = q$ and such that $e(\bar{q}) = 2$ (ramification is two); further if $B^{\circ} = B - \{q\}$, $A^{\circ} = \pi^{-1}(B^{\circ})$, then $A^{\circ} \to B^{\circ}$ is finite and étale; also π is étale on $A - \{q\}$.

(0.8) DEFINITION: S, T two finite sets. m: $S \to T$ a two-to-one and onto mapping. We call the following commmutative diagram a *double flip*:

$$\begin{array}{ccc} S & \xrightarrow{m} & T \\ {\scriptstyle \sigma_S}\downarrow & & \downarrow{\scriptstyle id_T} \\ S & \xrightarrow{m} & T \end{array}$$

where σ_S is an automorphism of S such that there are two elements A, $B \in T$ with preimages $\{A_1, A_2\}$, $\{B_1, B_2\}$ in S and such that:

$$\sigma_S(A_1) = A_2, \quad \sigma_S(A_2) = A_1,$$

$$\sigma_S(B_1) = B_2, \quad \sigma_S(B_2) = B_1.$$

Also $\sigma_S | S - \{A_1, A_2, B_1, B_2\}$ is the identity. (Intuitively we think of T as the set of position of a certain number of coins; each coin has "heads" and "tails", hence the two-to-one map m: $S \to T$. A "double flip" corresponds to turning over exactly two coins without altering the order of the coins.)

1. Statement of principal results

Our Main Theorem (MTH) is this:

MAIN THEOREM (*MTH*) (1.1): *R is a unique factorization domain.*

We will restate the theorem and in fact prove it in its geometric form.

MAIN THEOREM (GEOMETRIC FORM) (MTHG) (1.2): $\mathrm{Pic}(\tilde{S}) = \mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$.

REMARK: Since there is no torsion we will, in fact, prove $p\ \mathrm{Pic}(\tilde{S}) = p\ \mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$.

The equivalence of MTH and MTHG is an immediate consequence of the exact sequence (0.5.1). We will deduce MTHG from the following theorems to which we give descriptive names $G1$, $G2$, $G2'$, DRL (double reversal lemma).

Here are the statements:

THEOREM G1 (1.3): *$G = \mathrm{Gal}(k(\overline{T_{ij}})\colon k(T_{ij}))$ acts on* $\mathrm{Sing}(\bar{S})$ *as the full symmetric group.*

THEOREM G2 (1.4): *There exists a $\sigma \in G$ such that σ induces the identity on* Sing S *but for some pair of singularities α, $\beta \in$* Sing S *$\sigma(\sqrt{H_\alpha}) = -\sqrt{H_\alpha}\,\sigma(\sqrt{H_b}) = -\sqrt{H_\beta}$ and for all $\beta \neq \gamma \neq \alpha$, $\gamma \in$* Sing S *$\sigma(\sqrt{H_\gamma}) = \sqrt{H_\gamma}$.*

THEOREM G2′ (1.5): *For every pair of singularities α, $\beta \in$* Sing S *there exists a $\sigma \in G$ such that σ induces the identity on* Sing S *but $\sigma(\sqrt{H_\alpha}) = -\sqrt{H_\alpha}$ $\sigma(\sqrt{H_b}) = -\sqrt{H_\beta}$ and $\sigma(\sqrt{H_\gamma}) = \sqrt{H_\gamma}$ for $\beta \neq \gamma \neq \alpha$ $\gamma \in$* Sing S.

DRL (DOUBLE REVERSAL LEMMA) (1.6): *For every pair of singularities α, β, $\alpha \neq \beta$ in* Sing $\bar{S}$ *there is an element $\sigma \in G$ such that the induced isometry $i(\sigma)$ of* $\mathrm{Pic}(\tilde{S})$ *preserves* $\mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$ *and satisfies: $i(\sigma)(D_\alpha) = D_\alpha^{op}$, $i(\sigma)(D_\beta) = D_\beta^{\mathrm{op}}$, $i(\sigma)(l) = l$ and for all $\gamma \in$* Sing $\bar{S}$ *with $\alpha \neq \gamma \neq \beta$ we have $i(\sigma)(D_\gamma) = D_\gamma$. The logical structure of the proof will be as follows:*

$$
\begin{array}{ccccccc}
\text{easy exercise} & & & & & & \\
\downarrow & & & & & & \\
G1, G2 \Rightarrow G2' & \Rightarrow & DRL & \Rightarrow & MTHG & \Rightarrow & MTH \\
 & \downarrow & & \downarrow & & & \\
 & \text{show} & & \text{show} & & &
\end{array}
$$

The implication $G1$, $G2 \Rightarrow G2'$ is a simple exercise and will not be described. The implication MTHG $\Rightarrow$ MTH has been indicated above. Thus, our paper consists in proving $G1$, $G2$, see Chapter IV. Further, in showing $G2' \Rightarrow$ DRL, this will be sketched at the end of Chapter IV below, and finally we will prove the implication DRL $\Rightarrow$ MTHG, in Chapter V.

Chapter II. Preliminaries

1. Fundamental group facts (see SGA I, Reference [3])

Let $i\colon Y \to X$ be a morphism of locally neotherian connected (regular) schemes. Let $b\colon \operatorname{Spec} \Omega \to Y$ be a geometric point of Y. Let us abuse notation and denote by b also the corresponding geometric point of X.

(2.1.0) We recall the definition of the induced homomorphism:

$$i^*: \pi_1(Y, b) \to \pi_1(X, b)$$

$$\begin{tikzcd} \mathrm{Et}(Y) \arrow[d, "F_b^Y"] & \mathrm{ET}(X) \arrow[l, "R"'] \arrow[d, "F_b^Y"] \\ \mathrm{ENS} & \mathrm{ENS} \end{tikzcd}$$

Now

$$\pi_1(Y, b) = \mathrm{Aut}(F_b^Y),$$

$$\pi_1(X, b) = \mathrm{Aut}(F_b^Y).$$

By SGA I, p. 142 we have the isomorphism of functors:

$$F_b^Y \circ R \underset{\tau}{\overset{\mu}{\approx}} F_b^X \quad \text{where } \mu\tau = \mathrm{id}\, F_b^X, \tau\mu = \mathrm{id}(F_b^Y \circ R)$$

If $\sigma \in \pi_1(Y, b) = \mathrm{Aut}\ (F_b^Y)$ we define $\bar{\sigma} = i^*(\sigma)$ by the diagram:

$$\begin{tikzcd} F_b^X(W) \arrow[r, dashed, "\tau"] \arrow[d, "\bar{\sigma}_W"] & F_b^Y(W_Y) \arrow[d, "\sigma_{W_Y}"] \\ F_b^X(W) & F_b^Y(W_Y). \arrow[l, dashed, "\mu"'] \end{tikzcd}$$

Proposition (2.1.1) *If $W \in \mathrm{Et}(X)$ is irreducible then $\pi_1(X, b)$ acts transitively on $F_b^X(W)$ for any base point b in X.*

Proof: W is connected, so Proposition 1 follows from SGA *I*, [3], p. 140.

Proposition (2.1.2): *Assume $W \in \mathrm{Et}(X)$ irreducible and that $R(W) = W_Y$ decomposes $W_Y = s(Y) \sqcup T$ where s: $Y \to W_Y$ is a section and T irreducible. Then for any base point $b \in X$ the action of $\pi_1(X, b)$ on $F_b(W)$ is transitive and twice transitive.*

Proof: Transitivity was shown above. Also the statement is independent of base point (by SGA *I*). So choose the base point to be in Y.

$$\begin{tikzcd} \mathrm{Et}(Y) \arrow[d, "F_b^Y"'] & \mathrm{Et}(X). \arrow[l, dashed, "R"'] \arrow[d, "F_b^X"] \\ \mathrm{ENS} & \mathrm{ENS} \end{tikzcd}$$

We have $F_b^Y \circ R \approx F_b^X$, see [3], p. 142.

$$\begin{aligned}(F_b^Y \circ R)(W) = F_b^Y(W_Y) = \\ &= F_b^Y(s(Y)) \sqcup F_b^Y(T) \\ &= \{A\} \sqcup F_b^Y(T)\end{aligned}$$

where $F_b^Y(s(Y))$ is the one element set $\{A\}$.

Now $\pi_1(Y, b)$ acts on $F_b^Y(W_Y)$, it fixes A and acts transitively on $F_b^Y(T)$ since T is irreducible, see [3], p. 140. Now the identification $F_b^Y(W) = F_b^Y(W_Y)$ shows that for some element A_1 of $F_b^Y(X)$, $\pi_1(X, b)$ acts on $F_b^Y(W)$ so that the stabilizer of A_1 acts transitively on the complement of A_1 in $F_b^Y(W)$. This, together with transitivity, establishes twice transitivity. Q.E.D.

PROPOSITION (2.1.3): *Let $W \in \mathrm{Et}(X)$, $R(W) = W_Y \in \mathrm{Et}(Y)$, let b be a base point in Y. Suppose that the action of $\pi_1(Y, b)$ on $F_b^Y(W_Y)$ includes a transposition, then so does the action of $\pi_1(X, b)$ on $F_b^Y(W)$. Also, if $\bar{b}$ is any other base point in X not necessarily in Y, the action of $\pi_1(X, \bar{b})$ on $F_{\bar{b}}^X(W)$ also includes a transposition.*

PROOF: Follows immediately from the definition of the induced homomorphism $\pi_1(Y, b) \to \pi_1(X, b)$ and the fact that the functors F_b^Y, $F_{\bar{b}}^X$ are isomorphic.

PROPOSITION (2.1.4): *Let V, $W \in \mathrm{Et}(X)$, m: $V \to W$ étale covering of degree two,*

Set $V_Y = R(V), W_Y = R(W)$,

$m_Y: V_Y \to W_Y$ the induced covering.

Suppose that there is a $\sigma \in \pi_1(Y, b)$ such that the diagram

$$\begin{tikzcd} F_b(V_Y) \arrow[r, dashed] \arrow[d, "\sigma V_Y"] & F_b(W_Y) \arrow[d, "\sigma W_Y"] \\ F_b(V_Y) \arrow[r, dashed] & F_b(W_Y) \end{tikzcd}$$

is a double flip. Then also the diagram

$$\begin{tikzcd} F_b(V) \arrow[r, dashed] \arrow[d, "\bar{\sigma}_V"] & F_b(W) \arrow[d, "\bar{\sigma}_W"] \\ F_b(V) \arrow[r, dashed] & F_b(W) \end{tikzcd}$$

is a double flip.

PROOF: Follows from the definition of $\bar{\sigma}$ and a diagram chase.

REMARK (2.1.4.1): Under the above assumptions for any base point $\bar{b}$ in X the the action of $\pi_1(X, \bar{b})$ on $F_{\bar{b}}(V) \to F_{\bar{b}}(W)$ includes a double flip.

PROOF: Diagram chase using the fact that F_b and $F_{\bar{b}}$ are isomorphic functors.

2. Preliminaries on curves

THEOREM C1 (2.2.1): *Let $A \to B$ be a very simple covering, then the action of $\pi_1(B^\circ, b)$ on $F_b(A^\circ)$ contains a transposition.*

THEOREM C2 (2.2.2): *Let $D \overset{\phi}{\dashrightarrow} A \overset{\pi}{\dashrightarrow} B$ be two very simple coverings. Assume that $\bar{\bar{q}} \in D$, $\bar{q} \in A$, $q \in B$ are such that $e_\phi(\bar{\bar{q}}) = 2$, $e_\pi(\bar{q}) = 2$, $e_\pi(\bar{q}) = 2$, $\phi(\bar{\bar{q}}) = \bar{q}$, $\pi(\bar{q}) = q$. Let $D^\circ \overset{\phi^\circ}{\dashrightarrow} A^\circ \overset{\pi^\circ}{\dashrightarrow} B^\circ$ be the induced étale coverings where $B^\circ = B - \{q\}$, $A^\circ = \pi^{-1}(B^\circ)$, $D^\circ = \phi^{-1}(B^\circ)$. Then for any base point b in B° the action of $\pi_1(B^\circ, b)$ on the diagram $F_b(D^\circ) \to F_b(A^\circ)$ includes a double flip.*

PROOF OF THEOREM C1: We denote $f\colon A \to B$, $\bar{q} \in A$, $f(\bar{q}) = q$, $e_f(\bar{q}) = 2$.

$$B^\circ = B - \{q\}$$

$$A^\circ = f^{-1}(B^\circ)$$

$f^\circ : A^\circ \to B^\circ$ the induced étale cover.

Let $\mathcal{O}_q$ be the local ring of q in B.

Let $\tilde{\mathcal{O}}_q$ be the henselization of $\mathcal{O}_q$. We may take $\tilde{\mathcal{O}}_q \subset \overline{k(B^\circ)}^{\mathrm{sep}}$. Consider the cartesian diagram

$$\begin{array}{ccc} \operatorname{Spec} \mathcal{O}_q \times_B A & \dashrightarrow & A \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_q & \dashrightarrow & B \end{array}$$

If $\tilde{K}$ is the field of fractions of $\tilde{\mathcal{O}}_q$ we get the induced diagram

$$\begin{array}{ccc} \operatorname{Spec} \tilde{K} \times_B A & \dashrightarrow & A^\circ \\ \downarrow & & \downarrow \\ \operatorname{Spec} \tilde{K} & \dashrightarrow & B^\circ \end{array}$$

Now Spec $\mathcal{O}_q XB_{\text{over } q} \approx S \sqcup \text{Spec } \mathcal{O}_q(\sqrt{t})$ where S is a union of sections over Spec $\mathcal{O}_q$, thus a trivial covering and t is a uniformizing parameter in $\tilde{\mathcal{O}}_q$. This follows from elementary theory of henselian rings. Correspondingly, we can write the second diagram as:

$$\begin{array}{ccc} \tilde{S} \sqcup \text{Spec } \tilde{K}(\sqrt{r}) & \dashrightarrow & A^\circ \\ \downarrow & & \downarrow \\ \text{Spec } \tilde{K} & \dashrightarrow & B^\circ \end{array}$$

where again $\tilde{S}$ is a union of several copies of Spec $\tilde{K}$.

Now we have Spec $\overline{k(B^\circ)} \to \text{Spec } \tilde{K} \to \text{Spec } k(B^\circ) \to B^\circ$). Thus we get a geometric point b of Spec $\tilde{K}$ and the corresponding geometric point b_1 of B°. $\pi_1(\text{Spec } \tilde{K}, b) \approx \text{Gal}(\overline{k(A^\circ)}\colon \tilde{K})$ acts on $F_b^{\text{Spec } \tilde{K}}$ ($\tilde{S} \sqcup$ Spec $\tilde{K}(\sqrt{t})$) and it is clear that it induced a transposition. Consequently, $\pi_1(B^\circ, b_1)$ induces a transposition in $F_{b_1}^{B^\circ}(A^\circ)$ by our discussion of the fundamental group (2.1.3).

Proof of Theorem C2: Choose a henselization

$$\mathcal{O}_q \subset \tilde{\mathcal{O}}_q \subset \overline{k(B^\circ)}$$

and consider the base change.

$$\begin{array}{ccccc} S_1 \sqcup S_2 & \sqcup & \text{Spec } \mathcal{O}_q(\sqrt[4]{t}) \approx \text{Spec}\tilde{\mathcal{O}}_q XD & \to & D \\ \downarrow & & \downarrow & & \downarrow \\ S & \sqcup & \text{Spec } \tilde{\mathcal{O}}_q(\sqrt{t}) \approx \text{Spec}\tilde{\mathcal{O}}_q XA & \to & A \\ \downarrow & & \downarrow B & & \downarrow \\ \text{Spec}\tilde{\mathcal{O}}_q & & \approx \text{Spec}\tilde{\mathcal{O}}_q & \dashrightarrow & B \end{array}$$

We get the leftmost column from elementary properties of henselian rings. S is a union of several copies of Spec $\tilde{\mathcal{O}}_q$, thus a trivial covering. Similarly, S_1 and S_2 are trivial coverings of S.

We get the induced diagram, where $\tilde{K}$ is the field of fractions of $\tilde{\mathcal{O}}_q$.

$$\begin{array}{ccc} \tilde{S}_1 \sqcup \tilde{S}_1 \sqcup \text{Spec } \tilde{K}(\sqrt[4]{t}) & \longrightarrow & D^\circ \\ \downarrow & & \downarrow \\ S \sqcup \text{Spec } \tilde{K}(\sqrt{t}) & \dashrightarrow & A^\circ \\ \downarrow & & \downarrow \\ \text{Spec}\tilde{K} & \dashrightarrow & B^\circ \end{array}$$

Choose an embedding $\tilde{K} \subset \overline{K(B^\circ)}$. This defines a geometric point b of Spec $\tilde{K}$ (and b_1 of B°). By [3], prop. 8.1, p. 143, we have π_1(Spec $\tilde{K}$,

$b) \approx \mathrm{Gal}(\overline{k(B^\circ)}: \tilde{K}\ )$ and this group acts on

$$\begin{array}{c} F_b\left(\tilde{S}_1 \sqcup \tilde{S}_2 \sqcup \operatorname{Spec} \tilde{K}\left(\sqrt[4]{t}\right)\right) \\ \downarrow \\ F_b\left(\tilde{S} \sqcup \operatorname{Spec} \tilde{K}(\sqrt{t})\right) \end{array}$$

and we get a double flip by simple Galois theory (just send $\sqrt[4]{t}$ to $-\sqrt[4]{t}$.). If b_1 is a the corresponding base point of B° we conclude that the $\pi_1(B^\circ, b_1)$ action on $F_{b_1}(D^\circ) \to F_{b_1}(A^\circ)$ includes a double flip (from our discussion of the fundamental group (2.1.4)).

3. Irreducibility of the hessian

We will need the following proposition in order to prove that D is irreducible and normal and ϕ: $D \to E$ is finite. A related question is discussed in Coolidge's book about curves [Treatise on Algebraic Plane Curves, Dover], p. 153.

PROPOSITION (2.3.1): *$F_{xx}F_{yy} - (F_{xy})^2 = H(F)$ is irreducible in $R_1 = k[T_{ij}, X, Y]$.*

LEMMA (2.3.2): *F_{xx} is irreducible and so is F_{yy}.*

PROOF: $F_{xx} = 2\ T_{20} + S_{xx}$ where S_{xx} does not involve T_{20}. $R_1/(F_{xx}) \approx R_2$; $R_2 = R_1/(T_{20})$, a domain. Q.E.D.

LEMMA (2.3.3.): *F_{yy} does not divide $H(F)$.*

PROOF: F_{yy} contains the term $2T_{02}$ and $(F_{xy})^2$ does not contain T_{02}.

COROLLARY (2.3.4): *Let V: $H(F) = 0$ so that $k[V] = R_1/H(F)$ then F_{yy} is not a zero divisor in $k[V]$.*

PROOF: Obvious.

COROLLARY (2.3.5): *$k[V]$ injects into the localization $k[V][1/F_{yy}]$*

LEMMA 2.3.6: *$k[V][1/F_{yy}]$ is a domain.*

PROOF: Define a homomorphism h of R_1 into R_1 by $h(X) = X$, $h(Y) = Y$, $h(T_{ij}) = T_{ij}$ for $(i, j) \neq (2,0)$ and $h(T_{20}) = H(F) = T_{20}\ F_{yy} + S_{xx}F_{yy} - (F_{xy})^2$. Note that F_{yy}, S_{xx}, F_{xy} do not involve T_{20}. Also $h(F_{yy}) = F_{yy}$. h induces an injective map $\bar{h}$: $R_1/(T_{20}) \to R_1/(H(F))$ which takes the class of F_{yy} to the class of F_{yy}. We therefore get the induced map:

$$h_1: R_1(T_{20})[1/F_{yy}] \to k[V][1/F_{yy}]$$

which is still injective. But now the image of h_1 includes not only T_{ij}, X, Y for $(i,j) \neq (2,0)$ and $S_{xx}F_{yy} - (F_{xy})^2$ hence also $T_{20}F_{yy}$ but also T_{20} since we have inverted F_{yy} in the image. Thus h_1 is an isomorphism. This proves Proposition (2.31). Q.E.D.

4. Non-singularity of $\bar{S}$ at infinity

For the sake of completeness we also include a simple lemma about the singularities of $\bar{S}$.

LEMMA (2.4.1): *There are no singularities of $\bar{S}$ in $\bar{S} - S$.*

PROOF: At such a singular point we would have:

$$\sum_{i+j=p-1} T_{ij}X^iY^j = 0$$

$$\sum_{\substack{i+j=p\\ i,j>0}} iT_{ij}X^{i-1}Y^{\gamma} = 0$$

$$\sum_{\substack{i+j=p\\ i,j>0}} jT_{ij}X^iY^{j-1} = 0$$

but it is clear that the only common solution in L is $X = Y = 0$ but then also $Z = 0$ and that does not define a point of the projective space over L. Q.E.D.

Chapter III. Geometry of the mappings

$$E \to A, E_V \to V,$$

$$E_U \to U, D \to E$$

(3.1) Generalities

The main results of this chapter are theorem (3.2.12) (existence of a special pencil) and twice transitivity result (3.3.1). In order to prove these results, we have to established certain basic facts about the maps π: $E \to A$ and ϕ: $D \to E$. These facts should also be of some independent interest in the theory of Zariski Surfaces. Theorem (3.2.12) will be combined with basic facts about the fundamental group to prove the crucial Theorem G2 (1.4). The twice transitivity result will be essential in proving the two-transitivity in Theorem G1 (1.3).

PROPOSITION (3.1.1): *E is smooth, irreducible; in fact E is isomorphic to an affine space over k of dimension equal to the dimension of A.*

PROOF: If we write out F_x and F_y we easily see that:

$$E = \operatorname{Spec} \frac{k[T_{00}, T_{10}, T_{01}, \ldots][X, Y]}{(F_X, F_Y)}$$

is isomorphic to

$$E = \operatorname{Spec} k[T_{00}, T_{20}, T_{11}, T_{02}, \ldots][X, Y]. \qquad \text{Q.E.D.}$$

REMARK (3.1.1.1.): the above observation is due to S. Mori. We will use his idea again in (3.2.10) below.

REMARK (3.1.1.2): π has at least one finite fiber.

PROOF: This follows from example (3.1.6) below.

COROLLARY (3.1.1.3): *π is dominating and generically finite. The field extension $k(A) \subset k(E)$ is a finite algebraic extension.*

PROOF: This follows from the above remark and [5].

PROPOSITION (3.1.2): *π_V: $E_V \to V$ is a finite map.*

PROOF: It is enough to show that π_V is quasi-finite and projective ([5], exercise 11.2, p. 280).

(i) π_V is quasi-finite. If $g \in V$ then the surface $Z^p = g$ can only have finitely many singularities. Otherwise, there would be a singularity at infinity which contradicts the definition of V.

(ii) π_V is projective. We have the commutative diagram:

$$\begin{tikzcd}
\operatorname{Sing}(\bar{S}_V)_{\mathrm{red}} \arrow[r, dashed] \arrow[d] & \bar{S}_V \arrow[d] & \subset & P^2_V \arrow[d] \\
E_V \arrow[r, dashed] & P^2_V \arrow[rr, dashed] & & V
\end{tikzcd}$$

where $\bar{S}_V$ is defined in P^3_V by $Z^p - F(X, Y, Z_0)$ and $\mathrm{Sing}(\bar{S}_V)$ is defined by the above homogeneous polynomial and its partials with respect to X, Y, Z, Z_0. The middle vertical arrow corresponds to projecting from the point at infinity on the Z axis in each fiber. The map $E_V \to P^2_V$ is projective because $\mathrm{Sing}(\bar{S}_V) \subset \bar{S}_V$ is closed and E_V is the reduced *image* of $\mathrm{Sing}(\bar{S}_V)$, thus closed in P^2_V. Therefore $E_V \to V$ is projective. Q.E.D.

COROLLARY (3.1.3): *$U \subset V$ is open and dense. π_U: $E_U \to U$ is étale.*

PROOF: E_V is open and dense in E. The polynomial $H(F)$ is not identically zero on E. Thus, it defines a proper closed subscheme $\sigma_V \subset E_V$. Since π_V is finite, $\pi_V(\sigma_V)$ is a proper and closed subset of V. Now $U = V - \pi_V(\sigma_V)$. This shows that U is open and dense in V. The fact that π_U is étale follows from the Jacobian criterion.

COROLLARY (3.1.4): *For any base point b in U the action of $\pi_1(U, b)$ on $F_b(E_U)$ is transitive.*

PROOF: E_U is open and dense in E and thus it is irreducible and therefore connected. The corollary follows now from SGA I, [3], p. 140.

PROPOSITION (3.1.5): *D is irreducible and normal. ϕ: $D \to E$ is finite of degree two.*

PROOF: This follows from the irreducibility of the polynomial $H(F)$ in the ring $k[E] \simeq k[T_{00}, T_{20}, T_{11}, T_{01}, \dots X, Y]$, see (2.3.1) and [5], p. 147, exercise 6.4.

Next we given an example of a point $\tau \in U$. By counting the points in $\pi^{-1}(\tau)$ we will find the degree of π to be $p^2 - 3p + 3$. This result was originally proven by J. Sturnfield, see [1].

EXAMPLE (3.1.6): For almost every choice of $A \in k$ the polynomial $h(x, y) = xy + Ax^{p-1} + y^{p-1} + xy^{p-1} + yx^{p-1}$ defines a point $\tau \in U$. Further we have that the cardinality of $\pi^{-1}(\tau)$ is $p^2 - 3p + 3$. The proof will follow from the following computational lemmas:

PROOF OF EXAMPLE (3.1.6)

LEMMA (3.1.7):

$$g = xy + Ax^{p-1} + y^{p-1} + xy^{p-1} + yx^{p-1}.$$

For all but finitely many choices of $A \in k$ the system of equations

$$g_x = 0$$

$$g_y = 0$$

$$\det\begin{vmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{vmatrix} = 0$$

has no solution.

LEMMA (3.1.8): *For all but finitely many $A \in K$ the intersection number of $g_x = 0$ and $g_y = 0$ at infinity is $p - 2$.*

PROOF OF LEMMA (3.1.7): Consider the system:

$$y - Ax^{p-2} + y^{p-1} - yx^{p-2} = 0$$

$$x - y^{p-2} - xy^{p-2} + x^{p-1} = 0$$

$$\left(2Ax^{p-3} + 2yx^{p-3}\right)\left(2y^{p-3} + 2xy^{p-3}\right) +$$
$$-\left(1 - y^{p-2} - x^{p-2}\right)^2 = 0$$

$x = 0$ implies $y = 0$ but $(x, y) = (0, 0)$ is not a solution. Assume $x \neq 0$, and rewrite the last equation,

$$4\left(Ax^{p-2} + yx^{p-2}\right)\left(y^{p-3} + xy^{p-3}\right) +$$
$$-\left(1 - y^{p-2} - x^{p-2}\right)^2 x = 0$$

eliminate Ax^{p-2}

$$4\left(y + y^{p-1} - yx^{p-2} + yx^{p-2}\right)\left(y^{p-3} + x^{p-3}\right) +$$
$$-\left(1 - y^{p-2} - x^{p-2}\right)^2 x = 0.$$

It is enough to show that

$$4\left(y + y^{p-1}\right)\left(y^{p-3} + x^{p-3}\right) +$$
$$-\left(1 - y^{p-2} - x^{p-2}\right)^2 = 0$$

and

$$x - y^{p-2} - xy^{p-2} + x^{p-1} = 0$$

have no common component. ($x = 0$ is excluded again). We get finitely many values from A from the first equation once this is established.

At common points at infinity of the two curves

$$-xy^{p-2} + x^{p-1} = 0$$

$$4y^{p-1}\left(y^{p-3} + x^{p-3}\right) - \left(y^{p-2} + x^{p-2}\right)^2 = 0$$

If $x = 0$ then

$$4y^{p-1}\left(y^{p-3}\right) - \left(y^{p-2}\right)^2 = 0$$

$$4y^{2p-4} - y^{2p-4} = 0$$

since $p \neq 3$. This gives

$$y^{2p-4} = 0.$$

Thus no point with $x = 0$. If $x \neq 0$ then

$$\frac{y^{p-2}}{x^{p-2}} = 1.$$

Set $T = y/x$ so that $T^{p-2} = 1$. The second equation gives

$$4T^{p-1}(T^{p-2} + 1) - (T^{p-2} + 1)^2 = 0 \quad \text{so that}$$

$$8T - 4 = 0 \quad \text{or} \quad T = \tfrac{1}{2} \quad \text{so that}$$

$$2^{p-2} = 1 \quad \text{which gives} \quad 2 = 1. \quad \text{Contradiction.}$$

Therefore the two curves have no common point at infinity and they cannot have a common component. Q.E.D. (for lemma 5.1.).

PROOF OF (3.1.8): This proof is based on an idea of Jim Sturnfield. At a common point at infinity we must have:

$$\begin{aligned} &y\left(y^{p-2} - x^{p-2}\right) = 0 \\ &x\left(-y^{p-2} + x^{p-2}\right) = 0 \end{aligned} \tag{3.1.8.1}$$

Thus there are $p - 2$ common points.

$$\frac{y}{x} = \lambda \quad \text{where} \quad \lambda^{p-2} = 1.$$

Homogenize

$$g_x, yW^{p-2}\text{-}Ax^{p-2}W + y^{p-1} - yx^{p-2}$$

Set

$$y = 1$$

Set

$$\tau = \frac{1}{\lambda}$$

Rewrite

$$W^{p-2} - Ax^{p-2}W + 1 - x^{p-2} = 0$$

as

$$W^{p-2} - A[(x-\tau)+\tau]^{p-2}W + 1 + -[(x-\tau)+\tau]^{p-2} = 0.$$

The tangent line at $W = 0$, $x = \tau$ is

$$-A\tau^{p-2}W - (p-2)\tau^{p-3}(x-\tau) \quad \text{or}$$

$$-A\tau^{p-2}W + 2\tau^{p-3}(x-\tau) \quad \text{or}$$

$$-A\tau W + 2(x-\tau) = 0.$$

Similarly for the *second curve*

$$xW^{p-2} - W - x + x^{p-1} = 0$$

and we consider $W = 0$. We obtain:

$$x^{p-1} - x = 0$$

$$x^{p-2} = 1$$

$$x = \tau$$

and we have:

$$\tau^{p-1} - \tau = 0.$$

We compute the tangent line at $W = 0$, $x = \tau$. From

$$-W - [(x-\tau)+\tau] + [(x-\tau)+\tau]^{p-1} + \dots$$

we obtain:

$$-W-(x-\tau)+(p-1)\tau^{p-2}(x-\tau)=0$$

or

$$-W+(x-\tau)(-2)=0.$$

Finally

$$-W+(x-\tau)(-2)=0$$

defines the tangent line. We compare the two obtained tangent ines: Suppose

$$\det\begin{vmatrix} -1, & (-2) \\ -A\tau, & 2 \end{vmatrix}=0.$$

So that $2+2\tau A=0$. But choose A so that $A \neq -\dfrac{1}{\tau}$ for every τ such that $\tau^{p-2}=1$. Then we get a contradiction. Thus for all but finitely many $A \in k$ we get no common tangent, Q.E.D.

LEMMA (3.1.9): *For all but finitely many choices of $A \in k$ we have $g \in V$.*

PROOF is by computation. By (0.2) we must consider the system,

$$\begin{aligned} &Ax^{p-1}+y^{p-1}=0 \\ &y^{p-1}-yx^{p-2}=0 \\ &x^{p-1}-xy^{p-2}=0. \end{aligned} \tag{3.1.9.1}$$

Suppose that there is a solution with $x \neq 0$, then

$$Ax^{p-1}+y^{p-1}=0$$

and

$$y^{p-2}-x^{p-2}=0.$$

Set $T=y/x$ so that

$$T^{p-2}=1.$$

But $AT^{p-1} = -1$ and we obtain

$$AT = -1$$

$$-\frac{1}{A} = T$$

$$\left(-\frac{1}{A}\right)^{p-2} = 1$$

Contradiction for almost every choice of A. Q.E.D.

COROLLARY (3.1.10): *There exists a point $g \in V$ such that $\pi^{-1}(g)$ consists of $p^2 - 3p + 3$ unramified points (at which π is étale)* $\deg \pi = p^2 - 3p + 3$.

COROLLARY (3.1.10.1): *π is generically étale.*

COROLLARY (3.1.10.2): *U is open and dense in A (and in V).*

COROLLARY (3.1.11): *The degree of the field extension $k(A) \subset k(E)$ is $p^2 - 3p + 3$.*

COROLLARY (3.1.12): *The surface S^{aff} has $p^2 - 3p + 3$ singularities at finite distance.*

PROOF: Sing $S \approx F_b^U(E_U)$ where b: spec $\overline{k(U)} \to \operatorname{Spec} k(U) \to \operatorname{Spec} U$. Now the cardinality of $F_b^U(E_U)$ is equal to the degree of π_U: $E_U \to U$ and that is $p^2 - 3p + 3$. Q.E.D.

COROLLARY (3.1.13): *The map $D_U \to E_U$ is finite and étale of degree two. The degree of the field extension*

$$k(A) \subset k(D) \quad \text{is} \quad 2p^2 - 6p + 6.$$

PROOF: Follows from definition of D, and (3.1.11) above.

COROLLARY (3.1.14): *All the singularities of S are non-degenerate.*

PROOF: The maps $D_U \to E_U \to U$ are étale coverings. Let b: Spec$\overline{k(U)} \to U$ be as above (3.1.12). Now we have m: $F_b(D_U) \to F_b(E_U)$ and this map is two-to-one and onto. But $F_b(D_U) \approx$ set of pairs

$$\langle \alpha, t \rangle, \quad \alpha \in \text{Sing } S,$$

$$t \in L, \quad t^2 = H_\alpha$$

Furthermore, $F_b(E_U) \approx$ Sing S, and m corresponds to projecting $\langle \alpha, t \rangle$ to α. By a counting argument we must have all $t \neq 0$ so all $H_\alpha \neq 0$. Q.E.D.

(3.2) Existence of a special pencil

In order to prove the main result of this section Theorem (3.2.12) below our next objective is to find a point $q \in V - U$ such that the cardinality of $\pi^{-1}(q) = (\deg \pi) - 1$. We will do this by by by producing an explicit example of such a q.

EXAMPLE (3.2.1): Construction of the Special Polynomial (point of V) q.
The point q will be constructed as follows:

$$q \leftrightarrow y^2 + x^3 + Ax^{p-1} + By^{p-1} + xy^{p-1} + yx^{p-1} = g(x, y)$$

where $B = +1$ if $p \neq 13$ and $B = -1$ if $p = 13$. $A \in k$ can be chosen so that the cardinality of $\pi^{-1}(q) = (\deg \pi) - 1$, $q \in V$. Also the surface $Z^p = g$ has $(\deg \pi) - 1$ non degenerate singularities and one degenerate singularity (in other words $H(F)$ is equal to zero at exactly one point of $\pi^{-1}(q)$).

LEMMA (3.2.2): *$q \in V$ for almost every choice of $A \in k$.*

PROOF: Identical to the proof of 5.3.

LEMMA (3.2.3): *For almost every $A \in k$ the system of equations* $(**)$ *below only has the solutions* $(0, 0, t)$, $t \in k$

$$\begin{aligned} (**)\, g_{xx}g_{yy} - (g_{xy})^2 &= 0 \\ g_x &= 0 \\ g_y &= 0. \end{aligned} \tag{3.2.3.1}$$

PROOF: Consists of the following computation:

$$g = y^2 + x^3 + Ax^{p-1} + By^{p-1} + xy^{p-1} + xy^{p-1}$$

$$g_x = 3x^2 - Ax^{p-2} + y^{p-1} - yx^{p-2}$$

$$g_y = 2y - By^{p-2} - xy^{p-2} + x^{p-1}$$

$$g_{xx} = 6x - 2Ax^{p-3} + 2yx^{p-3}$$

$$g_{yy} = 2 + 2By^{p-3} + 2xy^{p-3}$$

$$g_{xy} = -y^{p-2} - x^{p-2}$$

$$\begin{aligned} g_{xx}g_{yy} - (g_{xy})^2 &= 0 \\ (**)\, g_x &= 0 \\ g_y &= 0 \end{aligned}$$

$(0, 0, t)$ is always a root. Also $x = 0 \Rightarrow y = 0$.

Let us find all solutions with $x \neq 0$. We eliminate A from the first two equations as follows:

$$Ax^{p-2} = 3x^2 + y^{p-1} - yx^{p-1}$$

$$xg_{xx}g_{yy} - x(g_{xy})^2 = 0$$

or

$$(6x^2 + 2Ax^{p-2} + 2yx^{p-2})g_{yy} - x(g_{xy})^2 = 0$$

And we obtain:

$$2(3x^2 + 3x^2 + y^{p-1} - yx^{p-2} + yx^{p-2})g_{yy} - x(g_{xy}) = 0$$

$$g_y = 0$$

We wish to show that the curves

$$2(6x^2 + y^{p-1})(2 + 2By^{p-3} + 2xy^{p-3}) + \\ -x(y^{p-2} + x^{p-2})^2 = 0 \qquad (***) \tag{3.2.3.2}$$

and

$$g_y = 2y - By^{p-2} - xy^{p-2} + x^{p-1} = 0$$

have only finitely many points in common. Again $x = 0$ implies $y = 0$ so enough to prove this for $x \neq 0$.

LEMMA (3.2.4): *The curve $g_y = 0$ is smooth and irreducible.*

PROOF: Consider the projective curve (closure of $g_y = 0$)

$$2yW^{p-2} - By^{p-2}W - xy^{p-2} + x^{p-1} = 0.$$

At points at infinity, i.e., where $W = 0$, we have:

$$x(x^{p-2} - y^{p-2}) = 0$$

$$W = 0, \quad x = 0, \quad y = 1$$

or if

$$x \neq 0 \quad \left(\frac{y}{x}\right)^{p-2} = 1.$$

$$\frac{\partial}{\partial W} = 0 \quad \text{at infinity implies} \quad y = 0.$$

Thus, there are no singularities at infinity. We dehomogenize:

$$2y - By^{p-2} - xy^{p-2} + x^{p-1} = 0$$

and check for singularities at finite distance:

$$\begin{aligned} &2y - By^{p-2} - xy^{p-2} + x^{p-1} = 0 \\ &2 + 2By^{p-3} + 2xy^{p-3} = 0 \\ &\quad -y^{p-2} - x^{p-2} = 0 \end{aligned} \tag{3.2.4.1}$$

First of all $x = 0$ is impossible if $y = 0$ then $x = 0$, contradiction. Assume $y \neq 0$ and transform

$$\begin{aligned} &2y - By^{p-2} - xy^{p-2} + x^{p-1} = 0 \\ &2y + 2By^{p-2} + 2xy^{p-2} = 0 \\ &y^{p-2} = -x^{p-2} \end{aligned} \tag{3.2.4.2}$$

$$\begin{aligned} &2y + Bx^{p-2} + x^{p-1} + x^{p-1} = 0 \\ &2y - 2Bx^{p-2} - 2x^{p-1} = 0 \\ &y^{p-2} = -x^{p-2} \end{aligned} \tag{3.2.4.3}$$

$$\begin{aligned} &Bx^{p-2} + 2x^{p-1} = -2Bx^{p-2} - 2x^{p-1} \\ &y^{p-2} = -x^{p-2} \\ &2y = 2Bx^{p-2} + 2x^{p-1} \end{aligned} \tag{3.2.4.4}$$

$$\begin{aligned} &3Bx^{p-2} + 4x^{p-1} = 0 \\ &y^{p-2} = -x^{p-2} \\ &y = Bx^{p-2} + x^{p-1} \end{aligned} \tag{3.2.4.5}$$

$x = 0$ is excluded. We obtain

$$x = \frac{-3B}{4}, \quad x^{p-2} = \frac{-4}{3B}, \quad x^{p-1} = 1$$

$$y = -\tfrac{4}{3} + 1 = -\tfrac{1}{3}$$

$$y^{p-2} = -3.$$

Now we must have

$$-3 = +\frac{4}{3B} \quad \text{or}$$

$$9B = -4.$$

If $p \neq 13$ then $B = 1$ by our assumption and $13 = 0$ a contradiction. If $p = 13$ then $B = -1$, $-9 = -4$ or $5 = 0$, again a contradiction.

Q.E.D. (for lemma 3.2.4).

In order to complete the proof of (3.2.3) it is enough to find one point on the curve $g_y = 0$ which is not on the first one in the system $(***)$ (3.2.3.2).

Take a point with

$$x = 0, \quad 2y = By^{p-2}$$

and with

$$y \neq 0$$

so that

$$2 = By^{p-3}.$$

The first curve gives

$$(2y^{p-1})(2 + 2By^{p-3}) = 0$$

or

$$4y^{p-1} + 4By^{2p-4} = 0$$

since

$$y \neq 0, \quad 1 + By^{p-3} = 0$$

$$By^{p-3} = -1.$$

Thus

$$2 = -1 \quad \text{or} \quad 3 = 0 \quad \text{contradiction since } p \geqslant 5.$$

Q.E.D. (for 3.2.3).

COROLLARY (3.2.5): *For almost every* $A \in k$ $(**)$ (3.2.3.1) *has only the* (0, 0) *solution in* (x, y).

LEMMA (3.2.6): *For almost every choice of* $A \in k$ *the intersection number of* $g_x = 0$ *and* $g_y = 0$ *at the points at infinity is* $p - 2$.

PROOF: Analogous to (3.1.8). We omit it.

LEMMA (3.2.7): *Assumptions as in* (3.2.6). *The intersection number at the origin is two.*

PROOF: Simple computation using $p \neq 2$ and $p \neq 3$ (omitted).

COROLLARY (3.2.8): *The cardinality of* $\pi^{-1}(q)$ *is* $p^2 - 3p + 2 = \deg \pi - 1$. *The hessian is non-zero at all but one point of* $\pi^{-1}(q)$. *It is zero at one point which we call* $\bar{q}$, $\bar{q} \in \pi^{-1}(q)$.

PROOF: The total intersection number of the curves $g_x = 0$, $g_y = 0$ is $(p-1)^2$ by Bezout's theorem. The number of intersections at finite distance is $(p-1)^2 - (p-2) - 1$ by the above. Thus it is $p^2 - 3p + 2$. Also the local intersection number is equal to one at every point where the hessian is non-zero. Q.E.D.

PROPOSITION (3.2.9): *Let* A *be as in example* (3.2.1). *Let* L *be the line in* $A = \operatorname{Spec} k[T_{ij}]$ *corresponding to polynomials of the form*:

$$y^2 + x^3 + \lambda x + Ax^{p-1} + By^{p-1} +$$

$$yx^{p-1} + xy^{p-1}$$

then π_L: $E_L \to L$, E_L *is an irreducible and smooth curve. There is exactly one ramified point lying over* q, *let us call it* $\bar{q}$ *with ramification index* $e(\bar{q}) = 2$.

LEMMA (3.2.10): E_L *is smooth and irreducible.*

PROOF:

$$E_L \approx (k[x, y, \lambda])(\lambda + 3x^2 - Ax^{p-2} - yx^{p-2} + y^{p-1},$$

$$2y - {}^{B}y^{p-2} + x^{p-1} - y^{p-2}x)^{-1}$$

$$\approx (k[x, y])(2y - y^{p-2} + x^{p-1} - y^{p-2}x)^{-1}$$

Now we have shown above that the last curve is smooth and irreducible, see (3.2.4). $E_L \to L$ is isomorphic to the projection to the Spec $k[\lambda]$ axis of the space curve defined in Spec $k[x, y, \lambda]$ by the two equations:

$$\lambda + 3x^2 - Ax^{p-2} + y^{p-1} - yx^{p-2} = 0$$

$$2y - By^{p-2} - xy^{p-2} + x^{p-1} = 0$$

q corresponds to the point $\lambda = 0$. The matrix of partials with respect to x, y, λ is:

$$\begin{bmatrix} 6x + 2Ax^{p-3} + 2yx^{p-2} & -y^{p-2} - x^{p-2} & 1 \\ -y^{p-2} - x^{p-2} & 2 + 2By^{p-3} - 2xy^{p-3} & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} g_{xx} & g_{xy} & 1 \\ g_{yx} & g_{yy} & 0 \end{bmatrix}$$

We have shown above that if

$$\lambda = 0 \quad \text{then} \quad \det\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} \neq 0$$

for every point of the space curve except $\lambda = x = y = 0$. Thus, all of those points are smooth. Also, $\lambda = x = y = 0$, the point $\bar{q}$, is smooth and x is a uniformizing parameter at $\bar{q}$. Finally, we see that the hessian has a simple zero at $\bar{q}$ and $e(\bar{q}) = 2$. All this follows from the matrix.

Similarly, we consider the map

$$D_L \to E_L \to L.$$

D_L is isomorphic to the curve in 4-space Spec $k[x, y, \lambda, W]$

$$\lambda + 3x^2 - Ax^{p-1} + y^{p-1} + yx^{p-2} = 0$$

$$2y - By^{p-2} - xy^{p-2} + x^{p-1} = 0$$

$$W^2 = g_{xx}g_{yy} - (g_{xy})^2$$

The Jacobian matrix is:

$$\begin{bmatrix} g_{xx} & g_{xy} & 1 & 0 \\ g_{yx} & g_{yy} & 0 & 0 \\ \alpha & \beta & 0 & 2W \end{bmatrix} \tag{3.2.10.1}$$

$$\alpha = g_{xxx}g_{yy} + g_{xx}g_{yyx} - 2g_{xy}g_{xxy}$$

$$\beta = g_{xxy}g_{yy} + g_{xx}g_{yyy} - 2g_{xy}g_{xyy}$$

all points with $W \neq 0$ are clearly smooth.... If $\lambda = 0$ and $W = 0$ then we must be at the point $\bar{\bar{q}}$ given by $x = y = \lambda = W = 0$ by (3.2.9). We compute matrix there:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 12 & 0 & 0 & 0 \end{bmatrix}$$

it has rank 3, thus the point is non-singular.

W is a uniforming parameter at $\bar{\bar{q}}$ and $g_{xx}g_{yy} - (g_{xy})^2 = W^2$ has a zero of order two at $\bar{\bar{q}}$. From this we see that the projection $\pi_L \circ \phi_L$: $D_L \to L$ (L corresponds to the Spec $k[\lambda]$ axis) has ramification equal to four at $\bar{\bar{q}}$. Also $\phi_L(\bar{\bar{q}}) = \bar{q}$ and $e_{\pi_L}(\bar{q}) = 2$, therefore, $e_{\phi_L}(\bar{q}) = 2$.

Lemma (3.2.11): *D_L is Irreducible.*

Proof: ϕ_L: $D_L \to E_L$ is finite and generically two to one. Thus D_L may have at most two components. Let C_L be the component which contains $\bar{\bar{q}}$. Now $\bar{\bar{q}}$ is a smooth point of C_L and $e(\bar{\bar{q}}) = 2$. Also, $C_L \to E_L$ is a finite map (proper and quasi finite), thus $C_L \to E_L$ must have degree two. Therefore, $C_L = D_L$ (inclusion of a closed subscheme is proper $-$[5]. p. 102) 48 (a).

The following theorem summarizes what we have shown:

Theorem (3.2.12) (*Existence of a Special Pencil*): *There exists a point in $q \in V - U$ and a smooth rational curve L closed in V such that $q \in L$ and $L_U = L \cap U$ is open and dense in L and closed in U. Further, let L_1 be the open subset of L defined by $L_1 = L_U \cup \{q\}$. We have the induced coverings*:

$$D_L \to E_l \to L$$

$$U \quad U \quad U$$

$$D_{L_1} \to E_{L_1} \to L_1$$

$$U \quad U \quad U$$

$$D_{L_U} \to E_{L_U} \to L_U$$

$$D_{L_1} \to E_{L_1} \quad \text{and} \quad E_{L_1} \to L_1$$

are very simple coverings

$$\bar{\bar{q}} \in D_{L_1}, \quad \bar{\bar{q}} \to \bar{q}, \quad \bar{q} \to q$$

$$e(\bar{\bar{q}}) = 2 \quad e(\bar{q}) = 2$$

REMARK (3.2.12.1): $L_U \subset U$ is closed and the bottom line may be induced directly by the base change $L_U \to U$ from E_U. $D_{L_U} \to E_{L_U} \to L_U$ are étale maps.

REMARK (3.2.12.2): E_L, E_{L_1}, D_{L_1} are smooth irreducible curves.

PROOF:To obtain $L_U = L \cap U$ remove from L the finitely many points over which π: $E_L \to L$ is ramified (π is not everywhere ramified, for example, it is not ramified at N-2 points of $\pi^{-1}(q)$). Now q corresponds to $\lambda = 0$. The smoothness of D_{L_U} can be seen from the matrix (3.2.10.1) above.

All the other assertions have been shown before.

COROLLARY (3.2.13): *Let b: Spec $\Omega \to L_U$ be any geometric base point, then a) the action of $\pi_1(L_U, b)$ on $F_b^{D_U}(E_{L_U})$ includes a transposition; b) the action of $\pi_1(L_U, b)$ on $F_b^{L_U}(D_{l_U}) \to F_b^{L_U}(E_{L_U})$ includes a double flip.*

PROOF: a), see Theorem C1 (2.2.1) above; b), see Theorem C2 (2.2.2) above.

COROLLARY (3.2.14): *For any geometric base point b in U we have*
a) the action of $\pi_1(U, b)$ on $F_b(E_U)$ includes a transposition;
b) the action of $\pi_1(U, b)$ on $F_b(D_U) \to F_b(E_U)$ includes a double flip.

PROOF: See propositions (2.1.3), (2.1.4), and (2.1.4.1).

(3.3) Twice transitivity of the action of $\pi_l(U, b)$ and $F_b(E_U)$

THEOREM (3.3.1): *For any base point b in Z_U $\pi_1(Z_U, b)$ acts on $F_b^{Z_U}(E_{Z_U})$ in such a way that there is an element $A \in F_b^{Z_U}(E_{Z_U})$ whose stabilizer in $\pi_1(Z_U, b)$ acts transitively on $F_b(E_{Z_U}) - \{A\}$.*

COROLLARY (3.3.2): *$\pi_1(U, b)$ acts on $F_b(E_U)$ transitively and twice transitively for any base point b in U.*

PROOF OF THE COROLLARY: Transitivity has been shown before. If b is in Z_Y we have $F_b^{Z_U}(E_{Z_U}) \approx F_b^U(E_U)$ and we recall the induced homomorphism $\pi_1(Z_U, b) \to \pi_1(U, b)$ which is such that the actions of $\pi_1(U, b)$ on $F_b^U(E_U)$ and of $\pi_1(Z_U, b)$ on $F_b^{Z_U}(E_{\underline{Z_U}})$ are compatible so that by the above theorem there is an element $\overline{A} \in \underline{F}_b^U(E_u)$ whose stabilizer in in $\pi_1(U, b)$ acts transitively on $F_b^U(E_U) - \{\overline{A}\}$. This shows transitivity and twice transitivity. If b is not in Z_U then the conclusion still holds since all the functors F_b^U are isomorphic by [3].

PROOF OF THEOREM:

$$Z = \operatorname{Spec} k[T_{00}, T_{20}, T_{11}, T_{02}, \ldots][X, Y]$$

$$E_Z = \operatorname{Spec} \frac{k[T_{00}, T_{20}, T_{11}, T_{02}, \ldots][X, Y]}{(2T_{20}X + T_{11}Y + \ldots, 2T_{02}Y + T_{11}X + \ldots)}$$

We get a section from sending $X \to 0$, $Y \to 0$. Let S: $Z_U \to E_{Z_U}$ be the section. By SGA I, Cor. 5.3, p. 7, we have

$$E_{Z_U} = S(Z_U) \sqcup T$$

T is reduced, smooth by SGA I 9.2, p. 16. We only need to show that T is topologically connected. Now T is covered by two opens T_X where $X \neq 0$ and T_Y where $y \neq 0$. T_X is homeomorphic to Spec $k[T_{00}, T_{02}, \ldots][X, Y][\frac{1}{X}]$ thus connected. T_Y is homeomorphic to Spec $k[T_{00}, T_{20}, \ldots][X, Y][\frac{1}{Y}]$ again connected.

To see that $T_X \cap T_Y \neq 0$ note that for the point τ from example (3.1.6), at least one point in $\pi^{-1}(\tau)$ has $X \neq 0$, $Y \neq 0$. Thus, T is connected and therefore irreducible by SGA I, Prop. 10.1, p. 21.

Now:

$$F_b^{Z_U}(E_{Z_U}) = F_b^{Z_U}(S(Z_U)) \sqcup F_b^{Z_U}(T)$$

$$= \{A\} \sqcup F_b^{Z_U}(T)$$

Now $\pi_1(Z_U, b)$ stabilizes A and acts transitively on $F_b^{Z_U}(T)$ by SGA I, p. 140. Q.E.D.

REMARK (3.3.2): The above theorem was inspired by reading J. Harris' paper "Galois Groups of Enumerative Problems," *Duke Journal of Math*, 1980.

Chapter IV

Proof of Theorems G1, G2 and G2′ and of DRL

THEOREM (4.1): *For any geometric base point b*: Spec $\Omega \to U$ *the action of* $\pi_1(U, b)$ *on* $F_b(E_U)$ *is the full symmetric group.*

PROOF: It is transitive by (3.1.4). It is twice transitive by (3.3.2). It contains a transposition by (3.2.14). All the above properties do not depend on the choice of the base point. Q.E.D.

THEOREM (4.2): *In the notations of Theorem* 1, *the action of* $\pi_1(U, b)$ *on* $F_b^U(D_U) \to F_b^U(E_U)$ *contains a double flip.*

PROOF: This is a restatement of (3.2.14) b).

PROOF OF THEOREM G1 (1.3): Let b: Spec $\overline{k(T_{ij})} \to \mathrm{k}(T_{ij})$ be the base point of U. We have a surjective map $G \to \pi_1(U, b) \to (e)$ by SGA I, p. 143. Also Sing $S \leftrightarrow F_b(E_U)$ and this identification is G equivariant where G acts on $F_b(E_U)$ via $G \to \pi_1(U, b)$ this follows from the proof on page 143 of SGA I. This proves G1.

PROOF OF THEOREM G2 (1.4): Choose b as in the proof of G1. Now $F_b(D_U) \leftrightarrow$ set of pairs $\langle \alpha, t \rangle$ where $\alpha \in$ Sing S, $t^2 = H_\alpha$, $t \in L = \overline{k(T_{ij})}$ (see 0.3 for the definition of H_α). Let us call this set of pairs P. The map

$$\begin{array}{c} F_b(D_U) \\ \downarrow \\ F_u(E_U) \end{array}$$

corresponds to projection $\langle \alpha, t \rangle$ to α. The diagram

$$\begin{array}{ccc} P & \approx F_b(D_U) & \\ \downarrow & \downarrow & \text{is } G \text{ equivariant,} \\ \text{Sing } S & \approx F_b(E_U) & \end{array}$$

where G acts on $F_b(D_U)$, $F_b(E_U)$ via the map $G \to \pi_1(U, b) \to (e)$. Theorem G2 follows immediately from Theorem (4.2).

PROOF OF THEOREM G2′ (1.5): Follows immediately from G1, G2.

PROOF OF DRL (1.6): Since we have already proven $G2'$ it is enough to prove the implication $G2' \to$ DRL. The proof is somewhat tedious thus we only sketch it here.

SKETCH OF PROOF THAT G2′ → DRL: Let σ be as in $G2'$ then σ induces an automorphism of $\overline{S}$ and it can be shown that there exists a commutative diagram:

$$\begin{array}{ccc} \tilde{S} & \overset{\tilde{\sigma}}{\dashrightarrow} & \tilde{S} \\ \downarrow & & \downarrow \\ \overline{S} & \overset{\sigma}{\dashrightarrow} & \overline{S} \end{array}$$

with $\tilde{\sigma}$ an automorphism. It can be shown that $\tilde{\sigma}$ induces the desired isometry as in DRL. We omit the details of the proof except to point out that $\tilde{S}$ is the result of $\frac{1}{2}(p-1)$ blow ups.

$$
\begin{tikzcd}
S^{\frac{1}{2}(p-1)} = \tilde{S} \arrow[r, dashed, "\tilde{\sigma} = \sigma^{\frac{1}{2}(p-1)}"] \arrow[d] & \tilde{S} = S^{\frac{1}{2}(p-1)} \arrow[d] \\
\vdots & \vdots \\
S^{(2)} \arrow[r, dashed, "\sigma^{(2)}"] \arrow[d] & S^{(2)} \arrow[d] \\
S^{(1)} \arrow[r, dashed, "\sigma^{(1)}"] \arrow[d] & S^{(1)} \arrow[d] \\
\bar{S} \arrow[r, dashed, "\bar{\sigma}"] & \bar{S}
\end{tikzcd}
$$

and in fact we can construct $\sigma^{\frac{1}{2}(p-1)} = \tilde{\sigma}$ by induction. The influence of $\tilde{\sigma}$ on T_α, T_β, T_γ can then be seen if we write out the local equations for the resolution.

Chapter V

Proof of the main theorem

We only need to show the implication:

$$\text{DRL} \Rightarrow \text{MTHG}.$$

We begin with an easy lemma:

Let $\alpha \in \operatorname{Sing} S$. Let P_α be the subgroup of $\operatorname{Pic}(\tilde{S})$ generated by the curves C_i^α, $i = 1, 2, \ldots p-1$. P_α inherits the intersection form from $\operatorname{Pic}(\tilde{S})$.

LEMMA (5.1): *Let $x_\alpha \in P_\alpha$ have the property that $x_\alpha \cdot w \equiv o(p)$ for every $w \in P_\alpha$. Then $x_\alpha = n_\alpha D_\alpha + pz$ where $0 \leqslant n_\alpha \leqslant p-1$, $z \in P_\alpha$.*

PROOF: Let $\bar{P}_\alpha = P_\alpha/pP_\alpha$. This is a Z/pZ vector space. Let u: $P_\alpha \to P_\alpha = P_\alpha/pP_\alpha$ be the natural map. We have a commutative diagram:

$$
\begin{tikzcd}
P_\alpha \times P_\alpha \arrow[r, dashed, "i"] \arrow[d, "u \times u"'] & Z \arrow[d, "v"] \\
\bar{P}_\alpha \times \bar{P}_\alpha \arrow[r, dashed, "i_1"] & Z/pZ
\end{tikzcd}
$$

where i is the intersection form and i_1 is the intersection form modulo p.

Now i_1 has a one dimensional kernel over Z/pZ because the intersection matrix of i_1 with respect to the basis $u(C_1^\alpha)$, $u(C_2^\alpha), \dots u(C_{p-1}^\alpha)$ which is

$$\begin{bmatrix} -2 & 1 & & & & 0 \\ 1 & -2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ 0 & & & & 1 & -2 \end{bmatrix}$$

of size $(p-1)x(p-1)$, has rank $p-2$ over Z/pZ. $u(D_\alpha)$ is in the kernel and is $\neq 0$ in $\bar{P}_\alpha$. Thus we have $u(x_\alpha) = n_\alpha u(D_\alpha)$ for some $0 \leqslant n_\alpha \leqslant p-1$. Consequently, $x_\alpha - n_\alpha D_\alpha = pz$ for some $z \in P_\alpha$ as asserted. Q.E.D.

COROLLARY (5.2): *Any element $x \in \mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$ which satisfies the condition that $x \cdot C_j^\alpha \equiv o(p)$ for all $\alpha \in \mathrm{Sing}\ S$ and $1 \leqslant j \leqslant p-1$ can be written in the form*

$$x = n_l l + \sum m_\gamma D_\gamma + py$$

where $y \in \mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$ and $0 \leqslant n_l$, $m_\gamma \leqslant p-1$ and $\sum$ is taken over all the singularities of S.

PROOF OF THE MAIN THEOREM (1.1), (1.2): We only need to show that

$$p\ \mathrm{Pic}(\tilde{S}) \subset p\ \mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$$

since there is no torsion.

Let $x \in p\ \mathrm{Pic}(\tilde{S})$, then $x \in \mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$ because of (0.5.1). Write

$$x = n_0 l + \sum n_\alpha D_\alpha + py$$

$$\alpha \in \mathrm{Sing}\ S$$

where $0 \leqslant n_0$, $n_\alpha \leqslant p-1$, $y \in \mathrm{Pic}^{\mathrm{ob}}(\tilde{S})$. This representation is possible because of (5.2). We only need to show that $n_\alpha = 0$ for all $\alpha \in \mathrm{Sing}\ S$ and that $n_0 = 0$.

Pick any pair of singularities $\alpha \neq \beta$ and apply the DRL, thus we get a $\sigma \in G$ and if $i(\sigma)$ is the induced isometry of $\mathrm{Pic}(\tilde{S})$ we have

$$\begin{aligned} x - i(\sigma)(x) &= n_\alpha(D_\alpha - D_\alpha^{op}) + n_\beta(D_\beta - D_\beta^{op}) + p(y - i(\sigma)(y)) \\ &= 2n_\alpha D_\alpha + 2n_\beta D_\beta - pn_\alpha \sum C_i^\alpha \\ &\quad - pn_\beta \sum C_j^\beta + p(y - i(\sigma)(y)) \end{aligned}$$

in any case since $p \neq 2$ we conclude that

$$n_\alpha D_\alpha + n_\beta D_\beta \in p \operatorname{Pic}(\tilde{S})$$

and then p^2 must divide

$$\left(n_\alpha D_\alpha + n_\beta D_\beta\right)^2 = \left(n_\alpha^2 + n_\beta^2\right) p(1-p)$$

by (0.4.5) so that p must divide $n_\alpha^2 + n_\beta^2$, thus for any pair of singularities α, β $\alpha \neq \beta$ we have shown that $n_\alpha^2 = -n_\beta^2$ modulo p. But there are at least 3 singularities and we conclude that in fact this is only possible if $n_\alpha = 0$ for all $\alpha \in \operatorname{Sing} S$.

Thus $x - py = n_0 l$ belongs to $p \operatorname{Pic}(\tilde{S})$. Squaring again we get that p^2 divides $(n_0 l)^2 = pn_0^2$ so that $n_0 = 0$. Q.E.D.

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Mathematics Division
Department of Mathematical Sciences
301 Science-Engineering Building
Fayetteville, AR 72701
USA