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APPENDIX TO "PICARD GROUPS OF ZARISKI SURFACES"

Piotr Blass and Jeffrey Lang

In this appendix we show how to pass from *generic* to *general* and how to prove our conjecture stated in the introduction to [1] and [2] (see Theorem 7(2) below).

We use techniques of P. Samuel and Jeffrey Lang and of course the main result of [1] which was proven with the help of Deligne.

We begin by introducing some notation which is analogous to [3] and [1]. If R is a normal noetherian domain we denote by Cl R its divisor class group.

k algebraically closed field of characteristic $p \geq 5$;

T_{ij} indeterminates; we consider the polynomial ring

$k[T_{ij}]$ $\quad 0 \leq i+j \leq p$ and two polynomials

$F(x, y) = \Sigma T_{ij} X^i Y^j, \quad 0 \leq i+j \leq p$ and

$\Upsilon(x, y) = \Sigma t_{\alpha,\beta} X^\alpha Y^\beta,$

$0 \leq \alpha + \beta \leq p-2$ with $t_{\alpha,\beta}$ also indeterminates over k.

We denote $\nabla = \dfrac{\partial^{2p-2}}{\partial x^{p-1} \partial y^{p-1}}$. We consider the system of equations:

$$(LS)\nabla(F^j \Upsilon) = 0 \quad j = 0, 1, 2, \ldots, p-2$$

$$(PLS)\nabla(F^{p-1}\Upsilon) = \Upsilon^p.$$

We consider the above equations as equalities of polynomials in X, Y. By comparing coefficients of the various monomials in X and Y we get an equivalent system

$$(LSI)\Sigma \underline{p}^{\gamma,\delta}_{\alpha,\beta} t_{\alpha,\beta} = 0$$

$$(PLSI)\Sigma Q^{\gamma,\delta}_{\mu,\nu} t_{\gamma,\delta} = t^p_{\mu,\nu}$$

$\underline{p}, Q \in k[T_{ij}]$.

If we specialize the indeterminates T_{ij} to have values $(c_{ij}) \in \operatorname{Spec}k[T_{ij}]$, a closed point, then we denote the corresponding system $LSI(c_{ij}) + PLSI(c_{ij})$.

We use the following facts.

THEOREM 1 (J. Lang): *Let A be any algebraically closed field of characteristic $p > 0$. Let $G(x, y) \in A[x, y]$ be a polynomial such that $\partial G/\partial x$ and $\partial G/\partial y$ are relatively prime polynomials in $A[x, y]$. Then the surface S: $z^p = G(x, y)$ is normal and its divisor class group Cl S is isomorphic to the set of polynomial solutions $t(x, y) \in A[x. y]$ of degree $\deg t \leq \deg G - 2$ of the following system of equations*:

$$\nabla(G^j t) = 0, \quad j = 0, 1, 2, \ldots, p-2, \text{ and}$$

$$\nabla(G^{p-1} t) = t^p.$$

PROOF: See [3], 2.1, 2.3, 2.9.1.

THEOREM 2 (Blass-Deligne): *The only solution of $(LSI) + (PLSI)$ in $L = \overline{k(T_{ij})}$ i.e. with $t_{\alpha,\beta} \in L$ is the identically zero solution $t_{\alpha,\beta} = 0$.*

PROOF: The set of solutions is isomorphic to Cl $\dfrac{L[X, Y, Z]}{(Z^p - \Sigma T_{ij} X^i Y^j)}$ by above theorem, but the latter group is shown to be zero in [1]. From now on Σ means $\sum_{0 \leq i+j=p}$.

The following is simple.

LEMMA 3. *If $q = (c_{ij}) \in \operatorname{Spec}k[T_{ij}]$, then the set of solutions $(t_{\alpha,\beta})$ of $LS(c_{ij}) + PLS(c_{ij})$ is finite.*

PROOF: Lang [3], proof of Lemma 2.8.

In what follows, Let H be the subscheme of Spec $k[T_{ij}] \times$ Spec $k[t_{\alpha,\beta}]$ defined by $(LS + PLS)$ or equivalently by $(LSI$ and $PLSI)$.

Consider the projection $H_{\text{red}} \to H \to$ Spec $k[T_{ij}]$. We denote by $\kappa^{-1}(q)$ the set (group) of closed points of H_{red} that map to q.

REMARK 4. We point out that if $q = (c_{ij}) \in$ Spec $k[T_{ij}]$ is a closed point, then $\kappa^{-1}(q)$ is in one to one correspondence with the solution set of equations $LSI(c_{ij}) + PLSI(c_{ij})$.

PROPOSITION 5: *There exists an open and dense subset $\mathcal{O}p$ of* Spec $k[T_{ij}]$ *such that for $q \in \mathcal{O}p$, $\kappa^{-1}(q)$ consists of a single point with coordinates $t_{\alpha,\beta} = 0$ for all α, β.*

LEMMA 6: *Let $Z \subset H_{\text{red}}$ be the subset of* Spec $k[T_{ij}] \times$ Spec $k[t_{\alpha,\beta}]$ *defined by $t_{\alpha,\beta} = 0$ (all α, β). Let C by any irreducible component of H_{red} whose image $\kappa(C)$ is dense in* Spec $k[T_{ij}]$. *Then $C = Z$.*

PROOF: First of all, dim $C =$ dim $Z =$ dim $k[T_{ij}]$ because of Lemma 3. Consider the diagram

$$\begin{array}{ccc} \mathcal{O}(C) & \leftarrow & k[T_{ij}] \\ & \searrow^{m} & \downarrow \\ & & \overline{k(T_{ij})} = L. \end{array}$$

Let $[t_{\alpha,\beta}]$ be the class of $t_{\alpha,\beta}$ in $\mathcal{O}(C)$. $\mathcal{O}(C)$ has fraction field which is finite algebraic over $k[t_{ij}]$. Hence we get an injective map m.

Suppose that for some α, β, $[t_{\alpha,\beta}] \neq 0$ in $\mathcal{O}(C)$ then $m([t_{\alpha,\beta}]) \neq 0$ and we would get a non-trivial solution of $(LS)+(PLS)$ in L which contradicts the Blass-Deligne theorem. Thus $[t_{\alpha,\beta}] = 0$ in $\mathcal{O}(C)$ for all α, β, i.e. $C \subseteq Z$ and consequently $C = Z$ since Z is irreducible.

PROOF OF PROPOSITION 5. Let $H_{\text{red}} = Z \cup C_1 \ldots \cup C_s$ be a decomposition of H_{red} into irreducible components.

We have $\overline{\kappa(C_j)} \subset$ Spec $k[T_{ij}]$ by Lemma 6. Thus set $\mathcal{O}\boldsymbol{p} =$ Spec $k[T_{ij}] - \bigcup_{j=1}^{s} \overline{\kappa(c_j)}$. For every $q \in \mathcal{O}(\boldsymbol{p})$, $\kappa^{-1}(q)$ is a single point of Z. Q.E.D.

REMARK 6. There exists an open and dense subset of Spec $k[T_{ij}]$, for example the subset V defined in [1] (0.2), such that if $q = (c_{ij})$ belongs to it, then

$$\kappa^{-1}(q) \leftrightarrow \text{Cl}\frac{k[x, y, z]}{\left(z^p - \sum c_{ij} x^i y^j\right)}.$$

PROOF: For $q \in V$, $q = (c_{ij})$, the polynomials$\partial(\sum c_{ij} x^i y^i)/\partial x$ and $\partial(\sum c_{ij} x^i y^i)/\partial y$ are relatively prime. Thus Remark 6 follows from Remark 4 and Theorem 1. Q.E.D.

THEOREM 6. *There exists an open and dense subset $D \subset$* Spec $k[T_{ij}]$ *such that for every closed point $q = (c_{ij}) \in D$,*

(1) *$\kappa^{-1}(q)$ consists of the single point,*

(2) Spec $\dfrac{k[x, y, z]}{\left(z^p - \sum c_{ij} x^i y^j\right)}$ *is a* UFD,

(3) Cl *of the above ring in* (2) *is the zero group*

(4) *The system $SLI(c_{ij}) + PSLI(c_{ij})$ has only the zero solution.*

PROOF: Set $D = V \cap \mathcal{O}\boldsymbol{p}$. Then (1) follows from Proposition 5 and we deduce (3) and (2) from Remark 6. Finally (4) follows because the closed points of $\kappa^{-1}(q)$ are in one-to-one correspondence with the solutions of the system $SLI(c_{ij}) + PLSI(c_{ij})$. Q.E.D.

References

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[2] P. Blass: Groupes de Picard des surfaces de Zariski. *Comptes Rendus des Seances de l'Academie des Sciences*, I-315, 19 septembre 1983.
[3] Jeffrey Lang: The divisor class group of the surface $z^{p^n} = G(x, y)$ over fields of characteristic $p > 0$. *Journal of Algebra* (2) 1983.

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