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MONODROMY OF FUNCTIONS DEFINED ON ISOLATED SINGULARITIES OF COMPLETE INTERSECTIONS

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A basic tool in the study of an analytic function germ $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ with an isolated singularity at the origin (or of the corresponding hypersurface germ $Y = f^{-1}(0)$) is the wellknown local monodromy group ([4], [8], [12]).

This widely studied monodromy group can be defined in two equivalent ways:

- (i) Using a morsification of the function f .
- (ii) Using a line in the base space B of a versal deformation for Y , in general position with respect to the discriminant hypersurfaces $\Delta \subset B$.

In this paper we extend the construction (i) above to function germs $f: (X, 0) \rightarrow (\mathbb{C}, 0)$ defined on a complete intersection $(X, 0) \subset (\mathbb{C}^{n+p}, 0)$ with an isolated singular point at the origin and such that $X_0 = f^{-1}(0)$ is also a complete intersection with an isolated singularity at 0 (here $n = \dim X > 0$).

In this way we obtain an action of a fundamental group $\pi = \pi_1(\text{disc} \setminus \{s \text{ points}\})$ on the exact sequence of the pair (\tilde{X}, \tilde{X}_0) in homology (with \mathbb{Z} -coefficients):

$$0 \rightarrow H_n(\tilde{X}) \rightarrow H_n(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} H_{n-1}(\tilde{X}_0) \rightarrow 0 \quad (*)$$

where \tilde{X}, \tilde{X}_0 are the Milnor fibers of X and X_0 ([5]) chosen such that $\tilde{X}_0 \subset \tilde{X}$ and $s = \mu(X) + \mu(X_0)$ is the sum of their Milnor numbers.

More precisely, the action of π on $H_n(\tilde{X})$ is trivial, while the actions on the other two homology groups can be described in terms of Picard-Lefschetz formulas with respect to thimbles $\Delta_k \in H_n(\tilde{X}, \tilde{X}_0)$ and corresponding vanishing cycles $\delta_k = \partial\Delta_k \in H_{n-1}(\tilde{X}_0)$.

The π -exact sequence (*) is proved to be a contact invariant of the function f i.e. it depends only on the isomorphism class (in a natural sense) of the pair of complete intersections (X, X_0) . This fact, as well as the independence of the sequence (*) on the choice of the morsification for f is obtained by a simple application of the Thom-Mather Second Isotopy Lemma.

To give some explicit examples, we compute next the π -sequence (*) for all the \mathcal{R} -simple functions f defined on an isolated hypersurface singularity X of dimension $n > 1$, as listed in [1].

Note that the π -sequence (*) gives us in particular two monodromy groups

$$G_0(f) = \text{Im}\{\pi \rightarrow \text{Aut } H_{n-1}(\tilde{X}_0)\}$$

$$G(f) = \text{Im}\{\pi \rightarrow \text{Aut } H_n(\tilde{X}, \tilde{X}_0)\}.$$

We prove that $G_0(f)$ is precisely the monodromy group of the complete intersection X_0 defined as in (ii). In fact the morsification process used above gives rise to a line in the base space B of a (suitable chosen) versal deformation of X_0 , whose direction depends on the function f and is *not* generic with respect to the discriminant $\Delta \subset B$.

That is why we need a slightly modified version of a result of Hamm-Lê on the fundamental group $\pi_1(B \setminus \Delta)$ (see Lemma 3.5).

Then we show that the other monodromy group $G(f)$ is a semidirect product of $G_0(f)$ with a free abelian group \mathbb{Z}^α and we also give some estimates for the rank α .

Finally we remark that constructions similar to some of ours (i.e. morsifications and connections with versal deformations) have been used many a time before (e.g. by Iomdin [7] and Lê [10]) but always with different aims in view, as far as we know.

We would like to express our deep gratitude to Professor V.I. Arnold for a very stimulating discussion.

§1. Morsifications and monodromy map of pairs

Let $X: g_1 = \dots = g_p = 0$ be an analytic complete intersection in a neighbourhood of the origin of \mathbb{C}^{n+p} , with an isolated singular point at 0. ($n \geq 1, p \geq 0$). Consider also an analytic function germ

$$f: (\mathbb{C}^{n+p}, 0) \rightarrow (\mathbb{C}, 0)$$

such that $X_0 = f^{-1}(0) \cap X$ is again a complete intersection with an isolated singularity at 0.

For $\epsilon \gg \delta > 0$ chosen sufficiently small, it is known that the Milnor fiber of X

$$X_\epsilon = \{x \in B_\epsilon; g(x) = r\}$$

is a compact C^∞ -manifold with boundary for any $r \in \mathbb{C}^p$ sufficiently general with $0 < |r| \leq \delta$, where

$$B_\epsilon = \{x \in \mathbb{C}^{n+p}; |x| \leq \epsilon\}. \quad [5]$$

The space X_r (denoted in the introduction by \tilde{X}) has the homotopy type of a bouquet of n -spheres, the number of which is by definition the Milnor number $\mu(X)$ of the complete intersection X .

For r small enough, it is easy to see that $f' = f|_{\text{Int } X_r}$ has only a finite number of critical points a_1, \dots, a_k and moreover $a_i \rightarrow 0$ when $r \rightarrow 0$ for any $i = 1, \dots, k$.

Let us denote by $\mu(f', a_i)$ the Milnor number of the function f' at the critical point a_i .

One has the following property, in analogy with a result of Lê ([10], (3.6.4)).

PROPOSITION 1.1:

$$\sum_{i=1, k} \mu(f', a_i) = \mu(X) + \mu(X_0).$$

PROOF: Let D_δ denote the open disc $\{z \in \mathbb{C}; |z| < \delta\}$. For ϵ , δ and r suitable chosen, the inclusion

$$E = X_r \cap f^{-1}(D_\delta) \hookrightarrow X_r \tag{1.2}$$

is a homotopy equivalence (see for instance [10] (3.5)) and moreover the restriction

$$f|_{\partial E}: \partial E \rightarrow D_\delta \quad \text{where} \quad \partial E = \partial X_r \cap f^{-1}(D_\delta) \tag{1.3}$$

is a submersion.

Let $b \in D_\delta$ be a regular value of $\tilde{f} = f|_E$ and let $c_i = f(a_i) \in D_\delta$ be the (not necessarily distinct) critical values of \tilde{f} .

Then $F = \tilde{f}^{-1}(b)$ is the Milnor fiber of the complete intersection X_0 and the exact sequence of the pair (E, F) shows that $H_n(E, F)$ is a free abelian group of rank $s = \mu(X) + \mu(X_0)$. (\mathbb{Z} -coefficients for homology are used throughout in this paper).

We compute now this group in a different way, following ([9], §5).

Choose small disjoint closed discs D_i centered at the critical values c_i and fix some points $b_i \in \partial D_i$.

For each i , take a C^∞ -embedded interval l_i from b to b_i such that $l = \cup l_i$ can be contracted within itself to b and D_δ can be contracted to $C = \cup D_i \cup l$.

Since \tilde{f} induces a (proper) locally trivial fibration

$$E \setminus f^{-1}\{c_i\}_i \rightarrow D_\delta \setminus \{c_i\}_i$$

these retractions can be lifted to the corresponding subsets of E and we

get the following isomorphisms

$$H_n(E, F) \xleftarrow{\sim} H_n(\tilde{f}^{-1}(C), F) \xrightarrow{\sim} H_n(\tilde{f}^{-1}(C), \tilde{f}^{-1}(l)).$$

By excision, the last group is equal to

$$\bigoplus_i H_n(\tilde{f}^{-1}(D_i), \tilde{f}^{-1}(b_i))$$

Assume that $a_{i,1}, \dots, a_{i,m}$ are the critical points of \tilde{f} in the fiber over c_i . Let B_j be the intersection of a small closed ball centered at $a_{i,j}$ with $\tilde{f}^{-1}(D_i)$ and denote with F_i the fiber $\tilde{f}^{-1}(b_i)$.

It follows that

$$H_n(\tilde{f}^{-1}(D_i), F_i) \simeq H_n\left(\bigcup_{j=1}^m B_j \cup F_i, F_i\right) \simeq \bigoplus_{j=1}^m H_n(B_j, B_j \cap F_i).$$

Moreover

$$H_n(B_j, B_j \cap F_i) \xrightarrow[\sim]{\partial} H_{n-1}(B_j \cap F_i)$$

is a free abelian group of rank $\mu(f', a_{i,j})$ by the definition of the Milnor numbers of f' , if the discs D_i and the balls B_j are chosen small enough. \square

We consider now the problem of the existence of morsifications of the function $f': X_r \rightarrow \mathbb{C}$, i.e. small deformations of f' having only nondegenerate critical points with distinct critical values.

If P denotes the vector space of polynomials in x_1, \dots, x_{n+p} of degree ≤ 3 , it is easy to show by standard transversality arguments that there is a Zariski open subset $U \subset P$ such that the function

$$f_q = (f + q)|_{X_r}$$

is a Morse function for any $q \in U$.

Moreover, if we have chosen already $\epsilon \gg \delta > 0$ such that (1.2) and (1.3) hold true for any generic $r \in \mathbb{C}^p$ with $|r| \leq \delta$, then there is an $\eta > 0$ such that $|q| < \eta$ implies similar properties for f_q .

Suppose now we have two polynomials $q_0, q_1 \in U$ such that $|q_i| < \eta$. We can find a C^∞ -path q_t in U such that $q_t = q_0$ for $0 \leq t \leq a$, $q_t = q_1$ for $1 - a \leq t \leq 1$ and $|q_t| < \eta$ for any $t \in [0, 1]$, where $a \in (0, 1/3)$.

Consider the spaces

$$\tilde{D} = D_\delta \times (0, 1) \quad \text{and} \quad \tilde{E} = \{(x, t) \in X_r \times (0, 1); f_{q_t}(x) \in D_\delta\}$$

and the proper map

$$\varphi: \tilde{E} \rightarrow \tilde{D}, \quad \varphi(x, t) = (f_{q_t}(x), t).$$

If $a_i(t)$ (resp. $c_i(t)$) denote the critical points (resp. critical values) of f_{q_t} , for $i = 1, \dots, s = \mu(X) + \mu(X_0)$, then we can stratify the map φ as follows ([2], Chap. I). The strata in \tilde{D} are given by

$$\tilde{D}_1 = \{(c_i(t), t); t \in (0, 1), i = 1, \dots, s\} \quad \text{and} \quad \tilde{D}_3 = \tilde{D} \setminus \tilde{D}_1.$$

The strata in \tilde{E} are given by

$$\tilde{E}_1 = \{(a_i(t), t); t \in (0, 1), i = 1, \dots, s\}$$

$$\tilde{E}_{2n-2} = \{(x, t); t \in (0, 1), x \in (f_{q_t})^{-1}(c_i(t)) \cap \partial X_r, i = 1, \dots, s\}$$

$$\tilde{E}_{2n-1} = \{(x, t); t \in (0, 1), x \in (f_{q_t})^{-1}(c_i(t)) \cap \text{Int } X_r, i = 1, \dots, s\}$$

$$\tilde{E}_{2n} = (\partial X_r \times (0, 1)) \cap (\tilde{E} \setminus \tilde{E}_{2n-2})$$

$$\tilde{E}_{2n+1} = \tilde{E} \setminus \text{the union of the other strata } \tilde{E}_k \text{ defined above.}$$

The lower index gives the real dimension of the stratum. (These definitions work for $n \geq 2$. The simpler case $n = 1$ is left to the reader.)

The Whitney-Thom regularity conditions are obviously satisfied for any pair of strata.

By Thom-Mather Second Isotopy Lemma ([2], II, (5.8)) we obtain a commutative diagram

$$\begin{array}{ccc} \varphi^{-1}(D_\delta \times \alpha) & \xrightarrow{H} & \varphi^{-1}(D_\delta \times (1 - \alpha)) \\ \downarrow f_{q_0} & & \downarrow f_{q_1} \\ D_\delta \times \alpha & \xrightarrow{h} & D_\delta \times (1 - \alpha) \end{array}$$

where $\alpha \in (0, a)$ and H, h are homomorphisms compatible with the induced stratifications.

In particular we get the following result.

LEMMA 1.4: *The topological type of the map of pairs*

$$f_q: (f_q^{-1}(D_\delta), f_q^{-1}(C)) \rightarrow (D_\delta, C)$$

where C is the set of critical values of the function f_q is independent of the polynomial $q \in U$, $|q| < \eta$.

It is also clear the independence of the topological type of the map above of the choice of (suitable) ϵ , δ and r . Moreover, if we change the function f to a function $f_1 = f + k$, where k is a function in the ideal (g_1, \dots, g_p) of the complete intersection X , note that the distance $\|f_1 - f\|_X$, can be made as small as we want by taking r small enough.

Using a stratification argument as above it follows that the topological type of the map of pairs in (1.4) depends only on the restriction $f|_X$ i.e. on a function in $m_X = m/(g_1, \dots, g_p)$, where $m \subset \mathcal{O}_{n+p}$ is the maximal ideal.

(We shall consider throughout in this paper only functions $f \in m_X$ such that $X_0 = f^{-1}(0)$ is a complete intersection with an isolated singularity at 0).

The discussion below will also imply independence from the defining equations $g_i = 0$ of X , and hence we can give the following.

DEFINITION 1.5: The topological type of the map of pairs in (1.4) will be called the *monodromy map of pairs* of the function $f \in m_X$ and will be denoted simply by

$$f^*: (E^*, E_c^*) \rightarrow (D, C).$$

This topological object is constant in μ -constant families in the following precise sense (compare to [12], §9).

Let $(X_t, 0) \subset (\mathbb{C}^{n+p}, 0)$ be a smooth family of complete intersections with isolated singular points at the origin such that $\dim X_t = n$ and $\mu(X_t) = \text{const.}$ for $t \in [0, 1]$. Assume that $f_t \in m_{X_t}$ is a smooth family of function germs such that $\mu(f_t^{-1}(0)) = \text{const.}$

Using the construction of morsifications and stratification arguments as above, one can then show that the monodromy map of pairs of the function f_t is independent of t .

A special case of this situation is the following.

DEFINITION 1.6 [1]: We say that two function germs $f_1, f_2 \in m_X$ defined on the complete intersection $(X, 0)$ are \mathcal{X} -(contact)-equivalent if there is an automorphism u of the local \mathbb{C} -algebra \mathcal{O}_X such that $(u(f_1)) = (f_2)$, where (a) means the ideal generated by a in \mathcal{O}_X .

Since the complete intersections X and $X_{0_i} = f_i^{-1}(0)$ $i = 1, 2$ have isolated singularities at the origin, the question of \mathcal{X} -equivalence of f_1 and f_2 can be settled in a jet space $J^k(n+p, p+1)$, via the action of a connected algebraic group $G_{\mathcal{X}}^k$ (the particular case when X is a hypersurface is treated in detail in [1]).

It follows that (X, f_1) and (X, f_2) can be connected by a μ -constant family (X_t, f_t) as above and we get thus the following.

COROLLARY 1.7: *If two function germs $f_1, f_2 \in m_X$ are \mathcal{K} -equivalent then their associated monodromy maps f_1^* and f_2^* are the same.*

§2. Monodromy exact sequence. Examples

Let $f^*: (E^*, E_c^*) \rightarrow (D, C)$ be the monodromy map of pairs of a function $f \in m_X$ as in §1.

If $b \in D \setminus C$ and $F = (f^*)^{-1}(b)$, then the locally trivial fibration $E^* \setminus E_c^* \rightarrow D \setminus C$ defines in the usual way an action of the fundamental group $\pi = \pi_1(D \setminus C)$ on the middle homology group $H_{n-1}(F)$ of the fiber.

Moreover, for any homotopy class $w \in \pi$ there is a well defined homomorphism

$$\tau_w: H_{n-1}(F) \rightarrow H_n(E^*, F)$$

called *the extension along the path w* . For a detailed construction and the main properties of τ_w we send to ([9], (6.4)).

We can define an action of the fundamental group π on the homology group $H_n(E^*, F)$ by the formula

$$w \cdot x = x + (-1)^{n-1} \tau_w(\partial x) \quad (2.1)$$

where ∂ is the connecting homomorphism in the exact sequence of the pair (E^*, F)

$$0 \rightarrow H_n(E^*) \xrightarrow{i} H_n(E^*, F) \xrightarrow{\partial} H_{n-1}(F) \rightarrow 0. \quad (2.2)$$

If we consider the trivial action of π on $H_n(E^*)$, then this exact sequence is a π -exact sequence, i.e. the homomorphisms i and ∂ are π -equivariant.

Let \tilde{X} (say equal to X_r in §1) and \tilde{X}_0 (say equal to $X_r \cap f^{-1}(b)$) denote the associated Milnor fibers of the complete intersections X and X_0 .

The corresponding exact sequence

$$0 \rightarrow H_n(\tilde{X}) \rightarrow H_n(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} H_{n-1}(\tilde{X}_0) \rightarrow 0 \quad (2.3)$$

is isomorphic to the exact sequence (2.2) and via this isomorphism we can transfer the π -actions on the homology groups in (2.3).

DEFINITION 2.4: The π -exact sequence (2.3) constructed as above is called *the monodromy exact sequence* of the function f .

EXAMPLE 2.5: If the complete intersection X is smooth, then the sequence (2.3) becomes

$$0 \rightarrow 0 \rightarrow H_n(\tilde{X}, \tilde{X}_0) \xrightarrow[\sim]{\partial} H_{n-1}(\tilde{X}_0) \rightarrow 0$$

and hence it contains the same information as the action of π on $H_{n-1}(\tilde{X}_0)$ i.e. the classical monodromy action for the hypersurface X_0 . \square

Put again $s = \mu(X) + \mu(X_0) = rkH_n(\tilde{X}, \tilde{X}_0)$ and let $C = \{c_1, \dots, c_s\}$. We denote by $w_k \in \pi$ the elementary path encircling c_k ([9] (6.1)) and chose the order of these paths such that

$$w_s \cdot \dots \cdot w_1 = w_0$$

where w_0 is the class of the path $w_0(t) = b \cdot e^{2\pi it}$, $0 \leq t \leq 1$ (we assume here $|b| > |c_k|$ for any $k = 1, \dots, s$).

We recall from the proof of (1.1) the isomorphisms

$$H_n(\tilde{X}, \tilde{X}_0) \simeq H_n(E^*, F) \simeq \bigoplus_{k=1}^s H_n((f^*)^{-1}(D_k), (f^*)^{-1}(b_k))$$

Since f^* is a morsification, each of the last homology groups is free abelian of rank one.

We shall denote by $\Delta_1, \dots, \Delta_s$ the corresponding generators of the group $H_n(\tilde{X}, \tilde{X}_0)$, which are precisely the *thimbles* of Lefschetz ([9] (6.2)).

With these notations, the π -actions in the exact sequence (2.3) can be described in terms of Picard-Lefschetz formulas.

LEMMA 2.6:

$$\text{For } x \in H_n(\tilde{X}, \tilde{X}_0): w_k \cdot x = x + (-1)^{n(n+1)/2}(\partial x, \partial\Delta_k)\Delta_k$$

$$\text{For } x \in H_{n-1}(\tilde{X}_0): w_k \cdot x = x + (-1)^{n(n+1)/2}(x, \partial\Delta_k)\partial\Delta_k$$

where $(,)$ denotes the intersection form on $H_{n-1}(\tilde{X}_0)$ and $k = 1, \dots, s$.

PROOF. The second formula is the usual Picard-Lefschetz formula (see for instance ([8], §5)). The first one follows from (2.1) and the formula for τ_w given in ([9], (6.7.1)). \square

It follows that in order to determine the monodromy exact sequence it is enough to fix a basis $\{\delta_k\}$ of the group $H_{n-1}(\tilde{X}_0)$ and to compute with respect to it the *vanishing cycles* $\partial\Delta_i$ and the intersection form.

As examples of this method, we give the description of the monodromy exact sequences of the \mathcal{R} -simple functions defined on an isolated hypersurface singularity X with $\dim X > 1$ which were classified in ([1], §3).

In all these cases X_0 is an isolated hypersurface singularity of type A_k for some k and we can choose a distinguished basis of vanishing cycles $\{\delta_i\}$ for $H_{n-1}(\tilde{X}_0)$ corresponding to a Dynkin diagram of type A_k ([4], (2.4)).

Moreover, using the stabilization of singularities (i.e. addition of a sum of squares to the given equation of X_0 as described in [4] (2.3)), we can assume $n = 1$ when we compute $\partial\Delta_i$.

The results are given below, without these tedious computations.

PROPOSITION 2.7: *For the simple function of type B_m ($m \geq 2$) given by $X: x_1^m + x_2^2 + \dots + x_{n+1}^2 = 0$ and $f = x_1$ there is a basis of thimbles $\Delta_1, \dots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a vanishing cycle δ which generates $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_k = \delta$ for any $k = 1, \dots, m$.*

PROPOSITION 2.8: *For the simple function of type C_{m+1} ($m \geq 1$) given by $X: x_1x_2 + x_3^2 + \dots + x_{n+1}^2 = 0$ and $f = x_1 + x_2^m$ there is a basis of thimbles $\Delta_0, \dots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles $\delta_1, \dots, \delta_m$ of $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_0 = \delta_1 + \dots + \delta_m$ and $\partial\Delta_k = \delta_k$ for any $k = 1, \dots, m$. (Note that $C_2 \equiv B_2$).*

PROPOSITION 2.9: *For the simple function of type F_4 given by $X: x_1^3 + x_2^2 + \dots + x_{n+1}^2 = 0$ and $f = x_2$ there is a basis of thimbles $\Delta_1, \dots, \Delta_4$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles δ_1, δ_2 of $H_{n-1}(\tilde{X}_0)$ such that*

$$\partial\Delta_1 = \delta_1, \quad \partial\Delta_3 = \delta_2, \quad \partial\Delta_2 = \partial\Delta_4 = \delta_1 + \delta_2.$$

REMARK 2.10: It will follow from the results in the next section, that for $n \equiv 3 \pmod{4}$ the monodromy group $G_0(f)$ (defined in the introduction) is a symmetric group for any \mathcal{R} -simple function f . More precisely

$$G_0(B_m) = S_2, \quad G_0(C_m) = S_m, \quad G_0(F_4) = S_3.$$

On the other hand, in these cases the monodromy groups $G(f)$ are all infinite (see 3.7 ii).

Therefore one cannot establish a simple connection between these monodromy groups and the Weyl groups associated to the root systems of type B_m , C_m and F_4 .

REMARK 2.11: It is easy to see that the action of the path w_0 on $H_{n-1}(\tilde{X}_0)$ is precisely the dual of the monodromy operator in cohomology h^* introduced in [5].

3. The monodromy groups $G_0(f)$ and $G(f)$

Let $(X_0, 0) \subset (Y, 0) \xrightarrow{F} (B, 0)$ be a versal deformation of the complete intersection X_0 , with a smooth base space B and let us denote by $\Delta \subset B$ the discriminant hypersurface of F [3].

For a base point $b \in B \setminus \Delta$, the fundamental group $\pi_1(B \setminus \Delta, b)$ acts on the homology of the smooth fiber $F^{-1}(b) \sim \tilde{X}_0$ and we obtain in this way the *monodromy group* of X_0

$$G(X_0) = \text{Im}\{ \pi_1(B \setminus \Delta, b) \rightarrow \text{Aut } H_{n-1}(\tilde{X}_0) \}.$$

This group is independent of the choice of the versal deformation F and of the base point b (provided we take B to be a small enough open ball in some \mathbb{C}^N).

Suppose we fix a morsification $f_q: X_r \rightarrow \mathbb{C}$ of the given function f as in (1.4). Then there is a versal deformation F of X_0 as above and a line l in the base space B such that after a natural identification $l \simeq \mathbb{C}$ we have a commutative diagram

$$\begin{array}{ccc}
 f_q^{-1}(D_\delta) \simeq F^{-1}(D_\delta) & & \\
 \swarrow f_q & & \searrow F \\
 & D_\delta &
 \end{array} \tag{3.1}$$

To obtain such a versal deformation F it is enough to take a system of generators of the \mathbb{C} -vector space $\mathcal{O}_{X_0}^{p+1} / \partial G / \partial x_1 \cdot \mathcal{O}_{X_0} + \dots + \partial G / \partial x_{n+p} \cdot \mathcal{O}_{X_0}$ (where $\partial G / \partial x_i = (\partial g_1 / \partial x_i, \dots, \partial g_p / \partial x_i, \partial f / \partial x_i)$) including the constant vectors e_1, \dots, e_{p+1} and the vector $(0, \dots, 0, q)$.

The set C of critical values of f_q corresponds via (3.1) to the intersection $l \cap \Delta$ and since f_q is a Morse function it follows that all the points $c_k \in l \cap \Delta$ are simple points on Δ and that the intersection $l \cap \Delta$ is transverse (situation denoted in the sequel by $l \pitchfork \Delta$). ([3], 1.3.i).

The number s of intersection points in $l \cap \Delta$ is equal to the intersection multiplicity $(\Delta, l_0)_0$, where l_0 is the line through $0 \in B$ with the same direction as l [10].

EXAMPLE 3.2: For the simple function of type B_m introduced in (2.7) one can take $F: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^2, 0)$

$$F(x) = (x_1^m + x_2^2 + \dots + x_{n+1}^2, x_1)$$

Then the discriminant Δ is given by the equation $y_1 = y_2^m$ and the morsification $f_0 = x_1: X_r \rightarrow \mathbb{C}$ corresponds to the line $l: y_1 = r$. Hence in this case $s = m$, though Δ is smooth at 0. It follows that the direction $l_0: y_1 = 0$ is not generic with respect to the discriminant, as mentioned in the introduction. □

The main result of this section is the following.

PROPOSITION 3.3:

$$G_0(f) = G(X_0).$$

PROOF: Suppose that B is an open neighbourhood of 0 in \mathbb{C}^N for some $N \geq 2$ and let $h = 0$ be the equation of the discriminant hypersurface Δ in B .

We denote here by B_ρ the closed ball of radius ρ centered at 0 in \mathbb{C}^N and by d_a the line determined by a direction $d \in P(\mathbb{C}^N)$ and a point $a \in B$.

The results of Hamm-Lê [6] prove the existence of a Zariski open set $U \subset P(\mathbb{C}^N)$ such that for any $d \in U$ there is a $\rho_0 = \rho(d) > 0$ with the property that for any ρ with $0 < \rho \leq \rho_0$ there is a $\theta_\rho > 0$ such that the homomorphism

$$\pi_1((B_\rho \setminus \Delta) \cap d_a, b) \rightarrow \pi_1(B_\rho \setminus \Delta, b) \tag{3.4}$$

induced by the inclusion is an epimorphism for any point a with $0 < |a| \leq \theta_\rho$ and $b \in (B_\rho \setminus \Delta) \cap d_a$.

We cannot apply this result to the line l in our construction above, since l is not in general position with respect to the discriminant Δ (3.2).

That is why we need the following.

LEMMA 3.5: *Suppose that the direction $d \in P(\mathbb{C}^N)$ is chosen such that $d_0 \not\subset \Delta$. Then there is $\rho, \delta > 0$ such that (3.4) is an epimorphism for any point a with $|a| \leq \delta$ and $d_a \pitchfork \Delta$.*

PROOF: Let $\rho > 0$ be chosen such that

- (i) $B_\rho \cap d_0 \cap \Delta = \{0\}$.
- (ii) Inside the ball B_ρ we have a conical topological structure for Δ , i.e.

$$(B_\rho, \Delta \cap B_\rho) \simeq C(S_\rho, K)$$

where $S_\rho = \partial B_\rho$, $K = \Delta \cap S_\rho$ as in [11] (2.10).

There is a connected open neighbourhood V of d in $P(\mathbb{C}^N)$ such that $d' \in \bar{V}$ implies $d'_0 \cap K = \emptyset$.

We choose $\delta > 0$ small enough, such that $d'_a \cap K = \emptyset$ for any $d' \in \bar{V}$ and any point a with $|a| \leq \delta$.

Take now a point a with $|a| \leq \delta$ and $d_a \pitchfork \Delta$. Using a linear parametrization $\gamma: (\mathbb{C}, 0) \rightarrow (d_a, a)$, we define the function $\varphi = h\gamma$.

Then φ is defined on a neighbourhood of $0 \in \mathbb{C}$ which contains the disc $D = d_a \cap B_\rho$ (if ρ and δ are chosen small enough) and $\varphi^{-1}(0) = \{x_1, \dots, x_s\}$ where the roots x_i are all in D and have multiplicity one.

We choose now a direction $d' \in V \cap U$ such that

$$(d'_0, \Delta)_0 = m(\Delta)$$

where $m(\Delta)$ is the multiplicity of the discriminant Δ at the origin. An explicit formula for $m(\Delta)$ can be found in [3], [10] and it follows that $m(\Delta) \geq \mu(X_0)$ with equality iff X_0 is a hypersurface singularity.

Note that a path connecting d with d' within V gives rise to a homotopy $\varphi_t: D \rightarrow \mathbb{C}$, $0 \leq t \leq 1$ of $\varphi = \varphi_0$ with φ_1 , the function defined as above with respect to d'_a .

Since the direction d' is in U , there is a $\rho' > 0$ and a $\theta' > 0$ such that, for any a' with $0 < |a'| \leq \theta'$, the corresponding homomorphism (3.4) is an epimorphism.

Choose a path $a(t)$ $1 \leq t \leq 2$ in B_δ such that $a(1) = a$, $a(2) = a'$ with $0 < |a'| \leq \theta'$ and $d'_{a(t)} \pitchfork \Delta$ for any t . This gives rise as above to a homotopy $\varphi_t: D \rightarrow \mathbb{C}$ $1 \leq t \leq 2$. Since all the functions φ_t have only simple roots $x_k(t)$ in $\text{Int } D$, we obtain in this way s paths $x_1(t), \dots, x_s(t)$ for $0 \leq t \leq 2$.

We choose the order on the paths such that $x_1(2), \dots, x_m(2)$ are precisely the end points within the disc $B_{\rho'} \cap d'_{a'} \subset D$, where $m = m(\Delta)$ (Note the identification $D \simeq d'_{a(t)} \cap B_\rho$ for any t).

Consider the following commutative diagram.

$$\begin{CD} \pi_1((B_\rho \setminus \Delta) \cap d_a, b) @>i_\#>> \pi_1(B_\rho \setminus \Delta, b) \\ @V\tilde{\varphi}VV @VVc_*V \\ \pi_1((B_\rho \setminus \Delta) \cap d'_{a'}, b') @>>> \pi_1(B_\rho \setminus \Delta, b') \\ @V{i_\#}VV @VV{i_\#}V \\ \pi_1((B_{\rho'} \setminus \Delta) \cap d'_{a'}, b') @>i_\#>> \pi_1(B_{\rho'} \setminus \Delta, b') \end{CD}$$

The isomorphism c_* is induced by a path in $B_\rho \setminus \Delta$ from b to b' and $\tilde{\varphi}$ is obtain via the homotopy φ_t .

If we denote by w_k (resp. w'_k) the elementary path in $D \setminus \{x_1(t), \dots, x_s(t)\}$ encircling the point $x_k(t)$ for $t = 0$ (resp. $t = 2$), then the left hand side of the diagram corresponds to

$$F(w'_1, \dots, w'_m) \xrightarrow{i_\#} F(w'_1, \dots, w'_s) \xrightarrow[\sim]{\tilde{\varphi}} F(w_1, \dots, w_s)$$

where $F(a_1, \dots, a_p)$ denotes the free group generated by a_1, \dots, a_p .

This ends the proof of (3.5) and hence of (3.3). \square

COROLLARY 3.6: *Suppose X_0 is a hypersurface singularity and let $m = m(\Delta) = \mu(X_0)$. Then in the monodromy exact sequence (2.3) of the function f (up to a change of indexes) the vanishing cycles $\delta_k = \partial\Delta_k$ ($k = 1, \dots, m$) form a basis of $H_{n-1}(\tilde{X}_0)$ and the Picard-Lefschetz transformations associated to the elementary paths w_k ($k = 1, \dots, m$) generate the group $G_0(f)$.*

PROOF: The proof of (3.5) implies that (up to a change of indexes) the images of w_1, \dots, w_m generate the group $G_0(f) = G(X_0)$.

The monodromy group $G(X_0)$ acts transitively on the set of vanishing cycles in $H_{n-1}(\tilde{X}_0)$ [4], (2.58).

Hence for any such cycle δ there is an element $g \in G_0(f)$ such that $\delta = \pm g \cdot \delta_1$.

Since g is a product of Picard-Lefschetz transformations associated to w_1, \dots, w_m , it follows that

$$\delta \in \mathbb{Z}\langle \delta_1, \dots, \delta_m \rangle$$

i.e. $\delta_1, \dots, \delta_m$ form a basis of $H_{n-1}(\tilde{X}_0)$. \square

Finally we give some information about the other monodromy group of f , namely $G(f)$.

PROPOSITION 3.7:

(i) *There is an exact sequence of groups*

$$0 \rightarrow \mathbb{Z}^\alpha \rightarrow G(f) \rightarrow G_0(f) \rightarrow 1$$

for some $\alpha \in \mathbb{N}$ with $0 \leq \alpha \leq \mu(X) \cdot \mu(X_0)$.

(ii) *Suppose that X_0 is a hypersurface singularity and the intersection form on $H_{n-1}(\tilde{X}_0)$ is nondegenerate.*

Then $\alpha \geq \mu(X)$.

If moreover the action of $G_0(f)$ on $H_{n-1}(\tilde{X}_0) \otimes \mathbb{C}$ is irreducible, then $\alpha = \mu(X) \cdot \mu(X_0)$.

PROOF: Put $m = \mu(X_0)$, $m' = \mu(X)$ and $s = m + m'$. Assume that $\{\Delta_i\}$ is a basis of $H_n(\tilde{X}, \tilde{X}_0)$ (made of thimbles only in the proof of (ii)!) such that $\delta_k = \partial\Delta_k$ for $k = 1, \dots, m$ form a basis for $H_{n-1}(\tilde{X}_0)$.

Then for any $k > m$ there is a combination

$$v_k = \Delta_k + \sum_{i=1}^m a_{ki} \Delta_i \quad \text{such that} \quad \partial v_k = 0.$$

In the basis $v_{m+1}, \dots, v_s, \Delta_1, \dots, \Delta_m$ the action of w_k on $H_n(\tilde{X}, \tilde{X}_0)$ is given by a matrix

$$T_k = \begin{pmatrix} 1 & A_k \\ 0 & B_k \end{pmatrix}$$

We define an epimorphism $\rho: G(f) \rightarrow G_0(f)$ by associating to an $s \times s$ matrix as above the $m \times m$ matrix in the lower right corner. We get thus an exact sequence

$$1 \rightarrow \ker \rho \rightarrow G(f) \xrightarrow{\rho} G_0(f) \rightarrow 1$$

where $\ker \rho$ is a subgroup in the (abelian!) multiplicative group of all the matrices

$$M = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$$

It follows that $\ker \rho \subset \mathbb{Z}^{m \cdot m'}$ and this gives us (i). To prove (ii) we assume the basis δ_k chosen as in (3.6). Note that the matrix A_k defined above is zero for $k \leq m$ and has a single nonzero row (that corresponding to the vector v_k) for $m < k \leq s$ if the intersection form is nondegenerate. This proves the first part of (ii).

Moreover, note that if

$$\begin{pmatrix} \vdots & 0 \\ 1 & \dots u \dots \\ \vdots & 0 \\ 0 & \vdots 1 \end{pmatrix} \in \ker \rho$$

for some row vector $u \neq 0$, then the same is true for the vector $u \cdot B$ for any $B \in G_0(f)$.

If the action of $G_0(f)$ on the homology group $H_{n-1}(\tilde{X}_0; \mathbb{C})$ is irreducible, then it follows that

$$\dim \mathbb{C}\langle u \cdot B; B \in G_0(f) \rangle = m$$

Hence $\ker \rho$ contains in this case $m \cdot m'$ \mathbb{C} -linearly independent vectors and this implies the result in the second part of (ii). □

REMARKS 3.8:

a. The condition about the intersection form in (3.7.ii) is necessary. For instance, if f is a simple function of type B_k and n is even, it follows from (2.7) that $G_0(f) = G(f) = 0$.

On the other hand, note that both assumptions in (3.7.ii) hold when X_0 is one of Arnold simple hypersurface singularities A_n , D_n , E_6 , E_7 or E_8 and $n \equiv 3 \pmod{4}$ ([12], §8).

b. In general the subgroup $\ker \rho \subset \mathbb{Z}^{mm'}$ is not the whole group, even when they have the same rank.

For instance, for a function of type B_k and n odd, $\ker \rho = 2 \cdot \mathbb{Z}^{k-1} \subset \mathbb{Z}^{k-1}$.

References

- [1] A. DIMCA: Function germs defined on isolated hypersurface singularities, *Comp. Math.* 53 (1984) 245–258.
- [2] C.G. GIBSON et al.: Topological stability of smooth mappings, *Lecture Notes in Maths.* 552, Springer Verlag, Berlin, (1976).
- [3] G.-M. GREUEL; LÊ DŨNG TRĂNG: Spitzen, Doppelpunkte und vertikale Tangenten in der Diskriminante verseller Deformationen von vollständigen Durchschnitten, *Math. Ann.* 222 (1976) 71–88.
- [4] C.M. GUSEIN ZAIDE: Monodromy groups of isolated hypersurface singularities, *Uspekhi Mat. Nauk*, 32 (2) (1977) 23–65 (Russian).
- [5] H. HAMM: Lokal topologische Eigenschaften komplexer Räume, *Math. Ann.* 191 (1971) 235–252.
- [6] H. HAMM and LÊ DŨNG TRĂNG: Un théorème de Zariski du type de Lefschetz, *Ann. Sci. Ec. Norm. Sup.* 6 (1973) 317–366.
- [7] I.N. IOMDINE: Computation of the relative monodromy of a complex hypersurface, *Funct. Analiz. i priloz.* 9 (1) (1975) 67–68 (Russian).
- [8] K. LAMOTKE: Die Homologie isolierter Singularitäten, *Math. Z.* 143 (1975) 27–44.
- [9] K. LAMOTKE: The topology of complex projective varieties after S. LEFSCHETZ, *Topology* 20 (1981) 15–51.
- [10] LÊ DŨNG TRĂNG: Calculation of Milnor number of an isolated singularity of complete intersection, *Funct. Analiz. i priloz.* 8 (1974) 45–49 (Russian).
- [11] J. MILNOR: Singular points of complex hypersurfaces, *Ann. of Math. Stud.* 61 (1968).
- [12] D. SIERSMA: Classification and deformation of singularities. Thesis. Univ. of Amsterdam (1974).

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