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ORE SETS IN ENVELOPING ALGEBRAS

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1. Introduction

Let $\boldsymbol{g}$ be a finite dimensional Lie algebra over $\mathbb{C}$, with enveloping algebra $\mathscr{U}(\boldsymbol{g})$. In the study of $\mathscr{U}(\boldsymbol{g})$ and its representation theory, localisation has proved to be the most useful weapon from the armoury of non-commutative ring theory. In this paper we sharpen that weapon by describing, for a prime ideal P of the enveloping algebra of a solvable subalgebra $\boldsymbol{p}$ of $\boldsymbol{g}$, the largest Ore set $\mathscr{S}(P)$ contained in $\mathscr{C}_{\mathscr{U}(\boldsymbol{p})}(P)=\mathscr{U}(\boldsymbol{p})\backslash P$, (5.3) and (7.1). (A subset $\mathscr{S}$ of a ring R is a *right Ore set*, or satisfies the right Ore condition if for all $s\in\mathscr{S}$ and $r\in$ R there exist $d\in\mathscr{S}$ and $u\in R$ such that $su=rd$. In this case one can localise at $\mathscr{S}$ to form the ring $R_{\mathscr{S}}=\{\bar{a}\bar{c}^{-1}: a\in R, c\in\mathscr{S}\}$, where $\bar{\ }$ denotes images in R/I, $I=\{r\in R$: $rs=0, s\in\mathscr{S}\}$. Similar definitions apply on the left. An *Ore set* is a right and left Ore set.)

In §§2, 5 and 6, we deal with the case $\boldsymbol{p}=\boldsymbol{g}$. In §2 we derive a necessary condition on $\mathscr{S}(P)$ in terms of certain subsemigroups of $(\boldsymbol{g}/[\boldsymbol{g},\boldsymbol{g}])^*$ arising from the Jordan-Hölder values of $\boldsymbol{g}$ (2.3). In §5 it is shown that the largest subset $\mathscr{C}(P)$ satisfying this necessary condition is, in fact, an Ore set (5.3); analogous results are obtained for the maximal right, and left, Ore sets, $\mathscr{S}'(P)$ and $'\mathscr{S}(P)$, in $\mathscr{C}(P)$. Sections 3 and 4 contain results needed for the proof of (5.3) which may be of independent interest to non-commutative ring theorists.

The structure of the ring $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ is examined in §6, (again here $\boldsymbol{g}=\boldsymbol{p}$ is solvable). We prove that *$\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ is a Noetherian domain with only countably many maximal right or left ideals, each of which is a twosided ideal generated by the image of P under a suitable winding automorphism of $\mathscr{U}(\boldsymbol{g})$. Let M be one such ideal. Then $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}/M$ is isomorphic to the division ring of quotients of $\mathscr{U}(\boldsymbol{g})/P$, and M has a regular normalising set of t generators,* (see (5.3) for the definition), *where t, the height of P and of M, is the global dimension and the Krull dimension of $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$. Every primitive ideal of $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ is maximal, $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ has stable range 1,* (see (6.2) for the definition), *and every projective module over $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ is free*. When P is localisable, that is, when $\mathscr{S}(P)=\mathscr{C}(P)$, then of course $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ is a "local ring". In view of the above properties, one might feel that the tranditional generalisation of the definition of "local ring" from the commutative to the non-commutative context has been too restrictive; we discuss this further in (6.3).

By localising at $\mathscr{S}'(P)$ instead of $\mathscr{S}(P)$, one can obtain interesting right Noetherian left Ore domains; see (6.4).

In the next section we consider the general case where $\boldsymbol{p} \subseteq \boldsymbol{g}$. A sufficient condition for an Ore subset of $\mathscr{U}(\boldsymbol{p})$ to be Ore in $\mathscr{U}(\boldsymbol{g})$ was obtained in [5, 4.5]. Theorem 7.1 shows that, with the addition of a mild technical assumption on the Ore sets in question, this condition is also necessary. One can thus describe the maximal Ore subset of $\mathscr{C}_{\mathscr{U}(\boldsymbol{p})}(P)$ in $\mathscr{U}(\boldsymbol{g})$, for a prime ideal P of $\mathscr{U}(\boldsymbol{p})$. In Theorem 8.2 we specialise this description to the case where $\boldsymbol{p}$ is a Borel subalgebra $\boldsymbol{b}$ of a semisimple Lie algebra $\boldsymbol{g}$. The statement is particularly neat where P is the annihilator of a 1-dimensional $\mathscr{U}(\boldsymbol{b})$-module, say $P = l(\mathbb{C}_\mu)$ where $\mu \in \boldsymbol{h}^*$, $\boldsymbol{h}$ a Cartan subalgebra in $\boldsymbol{b}$; here, $\mathscr{S}(P)$, the largest Ore subset of $\mathscr{C}_{\mathscr{U}(\boldsymbol{b})}(P)$ in $\mathscr{U}(\boldsymbol{g})$, is $\bigcap\{\mathscr{C}_{\mathscr{U}(\boldsymbol{b})}(l(\mathbb{C}_\lambda))\colon \lambda \in \mu + \mathbb{Z}R\}$, where R is the set of roots of $\boldsymbol{g}$ relative to $\boldsymbol{h}$, (8.2(iii)). The rest of §8 consists of some observations on the structure of $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ for this choice of $\boldsymbol{g}$ and P.

A word about coefficient fields. While we have assumed throughout that Lie algebras are complex, much of [7], and hence, much of this paper can be routinely generalised to a completely solvable Lie algebra over an arbitrary field. The results on Ore sets described here can be applied to enveloping algebras over other coefficient fields using the following results.

1.1. Lemma: *Let $K \subseteq F$ be fields and $\boldsymbol{g}$ a Lie algebra, finite dimensional over K. Then a subset $\mathscr{S}$ of $\mathscr{U}(\boldsymbol{g})$ satisfies the Ore condition if and only if $\mathscr{S}$ satisfies the Ore condition in $\mathscr{U}(\boldsymbol{g}) \otimes_K F$.*

1.2. Lemma: *Let $K \subseteq F$ be fields and let U be a K-algebra with the ascending chain condition on ideals. Let $\mathscr{P}$ be a set of completely prime ideals of $U \otimes_K F$. Suppose that $|\mathscr{P}| < |K|$ and that $\mathscr{S} = \bigcap_{P \in \mathscr{P}} \mathscr{C}(P)$ is an Ore set in $U \otimes F$. Then $\mathscr{S} \cap U$ is an Ore set in U.*

The proof of 1.1 is an easy exercise; the proof of 1.2 is sketched in 4.7.

In the hope of stimulating applications of this work to the further study of enveloping algebras, we have tried to make this paper readable by the non-specialist in Noetherian ring theory; but it may be as well to note that the approach to localisation described here originates in work of A.V. Jategaonkar, recent accounts of which may be found in [12], [14]. The fact that Jategonkar's theory, much of which applied originally only to FBN rings, could be made to apply to certain other classes of Noetherian rings, was observed independently by Jategaonkar and the author [13], [6]. The key idea from this theory which underlies the present paper is that of a *prime link*, which is implicit in 2.1.

The nature of the "prime links" for enveloping algebras of complex solvable Lie algebras was determined in [7], but for this paper we in effect need to know only the connected components of the graph of links, rather than the exact form of the edge set. It is an interesting but apparently difficult question for Noetherian ring theory, whether the Ore sets obtained in this paper, as intersections determined by the graph of links between primes, exist in a more general, more abstract, setting. The only other results obtained to date in this direction apply to certain *P.I.* rings [18]. *

2. Maximal possible Ore set

In this section $\mathfrak{g}$ denotes a finite dimensional complex solvable Lie algebra. Let P be a prime ideal of $\mathscr{U}(\mathfrak{g})$. Our target is a necessary condition (2.3) for a subset of $\mathscr{C}(P)$ to satisfy a one-sided Ore condition. (We denote the set of regular elements modulo an ideal I of a ring S by $\mathscr{C}(I)$, or $\mathscr{C}_S(I)$.)

2.1. Lemma: *Let R be a Noetherian ring with prime ideals P and Q, and let $\mathscr{S} \subseteq \mathscr{C}(P)$ be a left Ore set. Let $A \subseteq B$ be ideals of R with $l(B/A) = P$ and $r(B/A) = Q$. Then $\mathscr{S} \subseteq \mathscr{C}(Q)$.*

Proof: The case where B/A is R/P- and R-Q-torsion free is given by [14, 2.3], (or see [7, 1.4]). The required result reduces easily to this case, noting that if I is a prime containing Q, and $\mathscr{S} \subseteq \mathscr{C}(I)$, then $\mathscr{S} \subseteq \mathscr{C}(Q)$.

2.2. Let $0 = \mathfrak{g}_0 \subset \ldots \subset \mathfrak{g}_n = \mathfrak{g}$ be a composition series of $\mathfrak{g}$, so $\mathfrak{g}_i \lhd \mathfrak{g}$ and $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ has dimension one, $1 \leq i \leq n$, [4, 6.6]. Let $\lambda_i \in \mathfrak{g}^*$ be the eigenvalue of ad $\mathfrak{g}$ on $\mathfrak{g}_i/\mathfrak{g}_{i-1}$, $1 \leq i \leq n$, so $\lambda_1, \ldots, \lambda_n$ are the Jordan-Hölder values of $\mathfrak{g}$.

For $i = 1, \ldots, n$, fix $x_i \in \mathfrak{g}_i \setminus \mathfrak{g}_{i-1}$. For a prime ideal P of $\mathscr{U}(\mathfrak{g})$, let $P_i = P \cap \mathscr{U}(\mathfrak{g}_i)$, $0 \leq i \leq n$, and $\mathscr{I} \subseteq \{1, \ldots, n\}$ consist of those integers for which $P_i \neq P_{i-1}\mathscr{U}(\mathfrak{g}_i)$. For $i \in \mathscr{I}$, let m_i be the minimal degree in x_i of an element of $P_i \setminus P_{i-1}\mathscr{U}(\mathfrak{g}_i)$.

Let the subsemigroups of $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ generated by $\{m_i\lambda_i : i \in \mathscr{I}\}$, $\{-m_i\lambda_i : i \in \mathscr{I}\}$, and $\{\pm m_i\lambda_i : i \in \mathscr{I}\}$ be denoted by $L(P)$, $R(P)$ and $O(P)$ respectively. (This notation is apparently defective, since $L(P)$, etc. seem to depend on our choice of composition series for $\mathfrak{g}$. But, as will become apparent from 5.3, these subsemigroups are independent of this choice.) Recall that for $\mu \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ the *winding automorphism* τ_μ of $\mathscr{U}(\mathfrak{g})$ is defined by $\tau(x) = x + \mu(x)$ for $x \in \mathfrak{g}$, [4, 10.2].

* *Added in proof*: For recent developments, consult the published version of [12].

2.3. THEOREM: *Let P be a prime ideal of $\mathscr{U}(\boldsymbol{g})$, where $\boldsymbol{g}$ is a finite dimensional solvable complex Lie algebra. Define the subsemigroups $L(P)$, $R(P)$ and $O(P)$ of $(\boldsymbol{g}/[\boldsymbol{g},\boldsymbol{g}])^*$ as in 2.2. Let $\mathscr{S} \subseteq \mathscr{C}(P)$ be a* (i) *left Ore,* (ii) *right Ore, or* (iii) *Ore set. Then*

$$\text{(i)}\ \mathscr{S} \subseteq \mathscr{C}(P) \cap \bigcap_{\lambda \in L(P)} \mathscr{C}(\tau_\lambda(P)),$$

$$\text{(ii)}\ \mathscr{S} \subseteq \mathscr{C}(P) \cap \bigcap_{\lambda \in R(P)} \mathscr{C}(\tau_\lambda(P)),$$

$$\text{(iii)}\ \mathscr{S} \subseteq \mathscr{C}(P) \cap \bigcap_{\lambda \in O(P)} \mathscr{C}(\tau_\lambda(P)), \quad \textit{respectively.}$$

PROOF: Let $\boldsymbol{n}$ be the nilpotent radical of $\boldsymbol{g}$. If $i \in \mathscr{I}$ and $\lambda_i \neq 0$, then $\boldsymbol{g}_i \subseteq \boldsymbol{n}$ since $\boldsymbol{n}$ acts trivially on $\boldsymbol{g}_i/\boldsymbol{g}_{i-1}$. Hence, If $\lambda \in (\boldsymbol{g}/[\boldsymbol{g},\boldsymbol{g}])^*$,

$$L(P) = L(\tau_\lambda(P)). \tag{1}$$

Fix $i \in \mathscr{I}$ with $\lambda_i \neq 0$. The image $\mathscr{C}$ of the subset $\mathscr{C}_{\mathscr{U}(\boldsymbol{g}_{i-1})}(P_{i-1})$ of $\mathscr{U}(\boldsymbol{g})$ in the factor ring $\overline{\mathscr{U}(\boldsymbol{g})} = \mathscr{U}(\boldsymbol{g})/P_{i-1}\mathscr{U}(\boldsymbol{g})$ satisfies the Ore condition [4, 4.4]. The ring $R = \overline{\mathscr{U}(\boldsymbol{g})}_{\mathscr{C}}$ contains a $\boldsymbol{g}$-invariant subring $(\mathscr{U}(\boldsymbol{g}_i)/P_{i-1}\mathscr{U}(\boldsymbol{g}_i))_{\mathscr{C}} \cong K[X;\ D]$, where K is the quotient division ring of $\mathscr{U}(\boldsymbol{g}_{i-1})/P_{i-1}$, x_i maps onto X, and D is the derivation of K induced by x_i [4, 4.4]. The (non-zero) image of P_i in $K[X;\ D]$ generates an ideal I of $K[X;\ D]$ which is (left and right) principal, generated by $\boldsymbol{g}$-eigenvector with eigenvalue $m_i\lambda_i$ [7, 2.1].

Now put $B = IR$ and $A = \bar{P}IR$, ideals of R. Since $B = \alpha R = R\alpha$ is a free left R-module, $l(B/A) = \bar{P}R$. If $y \in \boldsymbol{g}$, then $y\alpha = \alpha(y + m_i\lambda_i(y))$ (modulo A); hence, $r(B/A) = \tau_{m_i\lambda_i}(\bar{P})R$, (where, abusing notation, $\tau_{m_i\lambda_i}$ denotes the automorphism of R induced by the corresponding winding automorphism of $\mathscr{U}(g)$). Intersecting ideals of R with $\overline{\mathscr{U}(\boldsymbol{g})}$ and taking inverse images in $\mathscr{U}(\boldsymbol{g})$, we obtain ideals $A_0 \subseteq B_0$ of $\mathscr{U}(\boldsymbol{g})$ with $\ell(B_0/A_0) = P$, $r(B_0/A_0) = \tau_{m_i\lambda_i}(P)$. Under hypothesis (i), 2.1 now shows that $\mathscr{S} \subseteq \mathscr{C}(\tau_{m_i\lambda_i}(P))$, for $i \in \mathscr{I}$.

In view of (1), we can repeat the above argument with $\tau_{m_j\lambda_j}(P)$ in place of P. Since $\tau_\lambda\tau_\mu = \tau_{\lambda+\mu}$ for λ, $\mu \in (\boldsymbol{g}/[\boldsymbol{g},\boldsymbol{g}])^*$, (i) follows. Similar arguments (ii) and (iii), using the analogue of 2.1 for right Ore sets, and the appropriate analogues of (1).

3. A lifting theorem for Ore sets

3.1. A result of P.F. Smith gives a criterion for a prime ideal with a normalising set of generators to be localisable, expressed in terms of the generating set [20, Theorem 2.4]. We need the following slight generalisation of his result:

THEOREM: *Let S be a right Noetherian ring and I an ideal of S with a normalising set of generators $\{y_1, \ldots, y_n\}$. Let $\mathscr{E}$ be a non-empty multiplicatively closed subset of $\mathscr{C}(I)$, such that*

(i) *the image of $\mathscr{E}$ in S/I satisfies the right Ore condition in S/I;*

(ii) *for all $c \in \mathscr{C}$, for all $s \in S$ and for all $i = 1, \ldots, n$ there exists $u \in \sum_{j=1}^{i-1} y_j S$ such that $c + x_i s + u \in \mathscr{E}$;*

(iii) *for all $c \in \mathscr{E}$ and for all $i = 1, \ldots, n$ there exists $d_i \in \mathscr{E}$ and $u_i \in S$ such that*

$$y_i d_i - c u_i \in \begin{cases} \sum_{j=1}^{i-1} y_j S & \text{if} \quad i > 1, \\ 0 & \text{if} \quad i = 1. \end{cases}$$

Then $\mathscr{E}$ satisfies the right Ore condition in S.

3.2. To prove 3.1 we need the following refinement of [20, Proposition 2.1 (iv) ⇒ (i)], which can be proved by making trivial adjustments to Goldie's proof of the parent result [20, page 43]. The right AR property is defined in [20, page 39].

LEMMA: *Let I be an ideal of the right Noetherian ring S, and let $\mathscr{E}$ be a non-empty multiplicatively closed subset of $\mathscr{C}(I)$. Suppose that*

(i) *I has the right AR property;*

(ii) *if $c \in \mathscr{E}$ and $r \in I$, then $c + r \in \mathscr{E}$;*

(iii) *the image of $\mathscr{E}$ in S/I^n satisfies Ore condition, for all $n \geq 1$.*

Then $\mathscr{E}$ satisfies the right Ore condition in S.

3.3. PROOF OF 3.1.: Since passage to a factor ring preserves all the hypotheses, we can assume by a Noetherian induction that the image of $\mathscr{E}$ in S/y_1S is right Ore. Let $c \in \mathscr{E}$ and $s \in S$, so there exist $c_1 \in \mathscr{E}$ and $r_1 \in S$ such that

$$cr_1 - sc_1 \in y_1 S. \tag{1}$$

We claim that for all $t \geq 1$ there exist $r_t \in S$ and $c_t \in \mathscr{E}$ such that

$$cr_t - sc_t \in y_1^t S. \tag{2}$$

The claim is proved by induction on t, $t = 1$ being (1). Assume (2) holds some $t \geq 1$. By hypothesis (iii), there exists $d \in \mathscr{E}$ and $u \in S$ such that

$$y_1^t d = cu. \tag{3}$$

By (2), there exists $a \in S$ such that

$$cr_t - sc_t = y_1^t a. \tag{4}$$

Since $\mathscr{E}$ is right Ore modulo $y_1 S$, there exists $d' \in \mathscr{E}$ and $f, e \in S$ such that

$$ad' - de = y_1 f. \tag{5}$$

From (3), (4) and (5),

$$\begin{aligned} sc_t d' - c(r_t d' - ue) &= -y_1^t ad' + cue \\ &= -y_1^t(ad' - de) \\ &= -y_1^{t+1} f. \end{aligned}$$

Thus (2) holds for $(t+1)$ with $c_{t+1} = c_t d'$ and $r_{t+1} = r_t d' - ue$.

Now the ideal $y_1 S$ has the right AR property, by [17, Lemma 8], and (ii) of 3.2 is implied by (ii) of 3.1. Hence 3.1 follows from 3.2.

4. Intersections of Ore sets

4.1. Let R be a Noetherian ring and $\{\mathscr{S}_i : i \in \mathscr{I}\}$ a collection of Ore sets in R. Let $\mathscr{S} = \bigcap_i \mathscr{S}_i$. Under what circumstances is $\mathscr{S}$ an Ore set? This is an interesting and – it would appear – difficult problem, which has received little or no attention up to now. Even in the case of particular interest to us, where R is a ring of polynomials over a division ring, and each $\mathscr{S}_i$ is $\mathscr{C}(P_i)$ for a prime ideal P_i, we have been unable to obtain a general solution. However, we offer here two partial solutions, either of which suffice for our purposes – the first is included because of an important corollary (6.1 (vi)), the second because of its generality, and because essentially the same argument proves 1.2.

4.2. We only require the first part of the following lemma, but its proof is very similar to, and in fact modelled on, that of (ii). Part (ii) is an unpublished result of J.T. Stafford, whom we thank for permission to include it here.

LEMMA: *Let D be a division ring with centre k, such that $D \otimes_k K$ is a division ring for all finite field extensions K of k. Let $R = D[X_1, \ldots, X_n]$ for some $n \geq 1$, and let $\mathscr{P}$ be a set of maximal ideals of R. Let $a, b \in R$ be such that $I = aR + bR$ satisfies $I \cap \mathscr{C}(M) \neq \emptyset$ for all $M \in \mathscr{P}$. Suppose that either* (i) $|\mathscr{P}| < |D|$ *or* (ii) *D is not algebraic over k. Then there exist $d \in D$ and a positive integer m such that for all $k \geq m$, $a + bd^k \in \bigcap_{M \in \mathscr{P}} \mathscr{C}(M)$.*

NOTE: If R arises as a partial quotient ring of a factor ring of $\mathscr{U}(\boldsymbol{g})$, for a solvable Lie algebra $\boldsymbol{g}$, then D satisfies the hypothesis of the lemma, essentially because all the prime ideals of $\mathscr{U}(\boldsymbol{g}\otimes_{\mathbf{C}}K)$ are completely prime; see [4, 5.2].

PROOF: (i) The ideals of R are generated by elements of the centre, $k[X_1, \ldots, X_n]$. Let F be the subfield of k generated by the coefficients of a set of generators (of cardinality $|\mathscr{P}|$) of the ideals in $\mathscr{P}$. By hypothesis (i), there exists $d \in D$ transcendental over F. Let L be the algebraic closure of F and K the subfield of the algebraic closure of k generated by k and L. Since $D \otimes_k K$ is a division ring, for each $M \in \mathscr{P}$ there are finitely many maximal ideals $M_1, \ldots, M_t$ of $R \otimes K$, (with t depending on M), with $\bigcap_i M_i = M \otimes K$. Each M_i is of the form $\sum_{j=1}^{n} (X_j - \alpha_j)(R \otimes K)$, where $\alpha_j \in L$.

Let V be the vector space spanned over L by 1 and the coefficients in D of a and b. Now d is transcendental over L, and $W = VL[d]$ is a finitely generated torsion free $L[d]$-module, so that $\bigcap_i Wd^i = 0$. Since V is finite dimensional, there exists $m \geq 1$ sucht that, for all $k \geq m$,

$$V \cap Vd^k \subseteq V \cap Wd^k = 0. \tag{2}$$

Let S be $R \otimes K$. We claim that for all $k \geq m$

$$a + bd^k \in \bigcap_{M \in \mathscr{P}} \bigcap_{i=1}^{t(M)} \mathscr{C}_S(M_i). \tag{3}$$

Clearly, (3) implies (i). Fix $M \in \mathscr{P}$ and $i \in \{1, \ldots, t(M)\}$. Then $a \in M_i + V$ and $b \in M_i + V$. Let $k \geq m$ suppose $a + bd^k \in M_i$. Now $bd^k \in M_i + Vd^k$, and so

$$a = (a + bd^k) - bd^k \in M_i + Vd^k.$$

But $a \in M_i + V$, so

$$a \in (M_i + V) \cap (M_i + Vd^k) = M_i + (V \cap Vd^k) = M_i,$$

by (2). Hence $bd^k \in M_i$ and so $I \subseteq M_i \cap R = M$, a contradiction. Thus $a + bd^k \notin M_i$. Since M_i is completely prime, by hypothesis, this proves (3).

(ii) The proof is similar to (i) – one simply chooses d to be an element of D transcendental over k.

4.3. Suppose that the hypotheses of Lemma 4.2 hold, with the exception that I is an arbitrary right ideal of R such that $I \cap \mathscr{C}(M) \neq \emptyset$ for all $M \in \mathscr{P}$, say $I = \sum_{i=1}^{t} a_i R$. If $t > 1$, the argument used in 4.2 yields $d \in D$ and a positive integer m such that for all $k \geq m$ and all $M \in \mathscr{P}$ there exists i, $1 \leq i \leq t-1$, such that $a_i + a_t d^k \in \mathscr{C}(M)$. Repeating the argument with $I' = \sum_{i=1}^{t-1} (a_i + a_t d^m)R$, we conclude that $I \cap \bigcap_{M \in \mathscr{P}} \mathscr{C}(M) \neq \emptyset$ in this case too. It is now easy to deduce the following corollary.

4.4. Corollary: Under the hypotheses and notation of 4.2. (i) or (ii) $\mathscr{C} = \bigcap_{M \in \mathscr{P}} \mathscr{C}(\mathrm{M})$ is an Ore set.

Proof: Let $d \in \mathscr{C}$ and $r \in R$, and let K be the kernel of the canonical homomorphism from R to $rR + dR/dR$. Each $M \in \mathscr{P}$, being centrally generated, is localisable [20, Theorem 2.2], so that $K \cap \mathscr{C}(M) \neq \emptyset$ for all $M \in \mathscr{P}$. We conclude from the remarks above that $K \cap \mathscr{C} \neq \emptyset$.

4.5. The following lemma follows immediately from the fact that a straight line cuts a hyperplane in at most one point.

Lemma: *Let D be an infinite division ring and V a finite dimensional D-vector space. Let $\mathscr{I}$ be an index set with $|\mathscr{I}| < |D|$, $\{v_i : i \in \mathscr{I}\}$ a collection of non-zero vectors in V, $\{V_i : i \in \mathscr{I}\}$ a collection of proper subspaces of V. Let $0 \neq v \in V$. Then there exists $w \in vD$ such that $w \notin v_i + V_i$ for all $i \in \mathscr{I}$.*

4.6. Since the ideals of polynomial rings over division rings are centrally generated, another approach to Corollary 4.4 is via the following

Theorem: *Let R be a ring with the maximal condition on ideals, D a division subring of R. Let $\mathscr{P}$ be a set of completely prime right localisable ideals of R, with $|\mathscr{P}| < |D|$. Let $\mathscr{S} = \bigcap_{P \in \mathscr{P}} \mathscr{C}(P)$. Then $\mathscr{S}$ is a right Ore set.*

Proof: Note that $D \subseteq \mathscr{S}$, so $\mathscr{S}$ is non-empty. Clearly, we may assume that D is infinite. Let $r \in R$ and $c \in \mathscr{S}$. Fix $P \in \mathscr{P}$. By hypothesis, there exist $u \in R$ and $\gamma \in \mathscr{C}(P)$ such that

$$r\gamma = cu. \tag{4}$$

Set $\mathscr{P}_\gamma = \{Q \in \mathscr{P} : \gamma \in Q\}$, $\mathscr{D}_\gamma = \mathscr{P} \backslash \mathscr{P}_\gamma$, and $I_\gamma = R \cap \bigcap_{Q \in \mathscr{P}_\gamma} Q$. Since $\gamma \in I_\gamma$, $I_\gamma \neq 0$. Clearly

$$\mathscr{P}_\gamma = \{Q \in \mathscr{P} :: I_\gamma \subseteq Q\}. \tag{5}$$

Using the ascending chain hypothesis, choose $\gamma \in \mathscr{C}(P)$ so that I_γ is as large as possible. We claim that in this case $I_\gamma = R$. Suppose not, so that $\mathscr{P}_\gamma \neq \emptyset$; choose $M \in \mathscr{P}_\gamma$. There exist $\delta \in \mathscr{C}(M)$ and $\omega \in R$ such that

$$r\delta = c\omega. \tag{6}$$

Let V be a finite dimensional D-subspace of R with γ, $\delta \in V$. Let $\mathscr{I}$ be a suitable index set, and define $V_i = V \cap Q_i$ for $Q_i \in \mathscr{D}_\gamma$, $i \in \mathscr{I}$. By 4.4, there exists $d \in D$ such that, for all $i \in \mathscr{I}$,

$$\delta d \notin \gamma + V_i. \tag{7}$$

Now put $\gamma' = \gamma - \delta d$. By (4) and (6),

$$r\gamma' = c(u - \omega d).$$

Since $\gamma \in I_\gamma \subseteq M$, $\gamma' \in \mathscr{C}(M)$. Let $Q \in \mathscr{D}_\gamma$; then $\gamma' \notin Q$ by (7), so $\gamma' \in \mathscr{C}(Q)$. Hence $\mathscr{P}_{\gamma'} \subsetneqq \mathscr{P}_\gamma$, and so, by (5), $I_\gamma \subsetneqq I_{\gamma'}$. By maximality of I_γ, this is impossible. Hence $I_\gamma = R$ and $\mathscr{P}_\gamma = \emptyset$, as required.

4.7. Let us sketch how an argument similar to the above proves 1.2. We are given a K-algebra U, with the ascending chain condition on ideals, a field extension F of K, a set $\mathscr{P}$ of completely prime ideals P of $U' =: \mathscr{U} \otimes_K F$, with $|\mathscr{P}| < |K|$, and we suppose that $\mathscr{S} = \bigcap_{P \in \mathscr{P}} \mathscr{C}(P)$ is right Ore. The assertion to be proved is that $\mathscr{S} \cap U$ is right Ore in U. Let $s \in \mathscr{S} \cap U$ and $u \in U$, so there exist $d \in \mathscr{S}$ and $b \in U'$ with $sb = ud$. Let $\{e_i\}$ be a basis for F over K, so that $b = \sum_{i=1}^{n} b_i e_i$, $d = \sum_{i=1}^{n} d_i e_i$, say, and we have $sb_i = ud_i$ for $i = 1, \ldots, n$. For each $P \in \mathscr{P}$ there exists i such that $d_i \notin P$. Thus an argument like that in 4.6 can be used to adjust an initial choice of d_i to obtain an elemtn $d' \in \mathscr{S} \cap U$, with $sa = ud'$ for some $a \in U$.

5. Affirmation of the Ore condition

5.1. In this section $\boldsymbol{g}$ denotes a finite dimensional complex solvable Lie algebra. We retain the notation introduced in 2.2; in particular, P denotes a fixed prime ideal of $\mathscr{U}(\boldsymbol{g})$. Define $\mathscr{S}'(P)$ [resp. $'\mathscr{S}(P)$] [resp. $\mathscr{S}(P)$] to be $\mathscr{C}(P) \cap \bigcap_{\lambda \in R(P)} \mathscr{C}(\tau_\lambda(P))$ [resp. $\mathscr{C}(P) \cap \bigcap_{\lambda \in L(P)} \mathscr{C}(\tau_\lambda(P))$] [resp. $\mathscr{C}(P) \cap \bigcap_{\lambda \in 0(P)} \mathscr{C}(\tau_\lambda(\mathrm{P}))$]. We shall show that these sets are right Ore [resp. left Ore] [resp. Ore] (5.3) – this is the converse of 2.3.

5.2. Put $\boldsymbol{t} = \bigcap \{\ker \lambda_i : i \in \mathscr{I}\}$, so $\boldsymbol{t}$ is an ideal of $\boldsymbol{g}$ containing the nilpotent radical. Let $\boldsymbol{h}$ be an ideal of $\boldsymbol{g}$ chosen maximal such that $P \cap \mathscr{U}(\boldsymbol{h}) = P \cap$

$\mathscr{U}(\boldsymbol{t}))\mathscr{U}(\boldsymbol{h})$. By [7, 2.9] $P\cap\mathscr{U}(\boldsymbol{h})$ is a localisable ideal of $\mathscr{U}(\boldsymbol{h})$. Put $\mathscr{S}=\mathscr{U}(\boldsymbol{h})\backslash(P\cap\mathscr{U}(\boldsymbol{h}))$; by [5, 4.5] $\mathscr{S}$ is actually an Ore set in $\mathscr{U}(\boldsymbol{g})$. Let $S=\mathscr{U}(\boldsymbol{g})_{\mathscr{S}}$ and let D be the quotient division ring of $\mathscr{U}(\boldsymbol{h})/P\cap\mathscr{U}(\boldsymbol{h})$. Let $m=\dim \boldsymbol{g}/\boldsymbol{h}$.

LEMMA: *$S/(P\cap\mathscr{U}(\boldsymbol{h}))S$ is isomorphic to $D[X_1, \ldots, X_m]$, the polynomial algebra over D.*

PROOF: Let $x\in \boldsymbol{g}\backslash\boldsymbol{h}$ and put $\mathbf{1}=\boldsymbol{h}\oplus\mathbb{C}x$. Thus

$$P\cap\mathscr{U}(\mathbf{1})\neq(P\cap\mathscr{U}(\boldsymbol{h}))\mathscr{U}(\mathbf{1}), \tag{1}$$

by the maximality of $\boldsymbol{h}$. Let $x_1, \ldots, x_m$ be a basis for a complement to $\boldsymbol{h}$ in $\boldsymbol{g}$. From (1) with $x=x_i$ we see that for $i=1, \ldots, m$ the skew polynomial ring $D[x_i]$ is split, in the sense of [4, 4.7]. Hence, there exists $d_i\in D$ such that $D[x_i]=D[X_i]$, where $X_i=x_i-d_i$ and X_i is central in $D[X_i]$. Thus

$$\begin{aligned} S/(P\cap\mathscr{U}(\boldsymbol{h}))S &\simeq D[x_1, \ldots, x_m] \\ &= D[X_1, \ldots, X_m]. \end{aligned}$$

Let $1\leq i, j\leq m$. Then, writing $Z(D)$ for the centre of D,

$$\begin{aligned} [X_i, X_j] &= [x_i, x_j]-[x_i, d_j]-[d_i, x_j]+[d_i, d_j] \\ &= [x_i, x_j]-[d_i, d_j]\in Z(D), \end{aligned}$$

since $[x_i, \quad]=[d_i, \quad]$ on D. Therefore, if $[X_i, X_j]\neq 0$ then $D[X_i, X_j]$ is $A_1(D)$, the first Weyl algebra over D, which is impossible since $D[X_i, X_j]$ has at least one proper ideal, namely that generated by the image of the intersection of P with $\mathscr{U}(\boldsymbol{h}\oplus\mathbb{C}x_i\oplus\mathbb{C}x_j)$. Hence $[X_i, X_j]=0$ and the lemma is proved.

5.3. Recall that an ideal I of a ring R has a *regular normalising set of generators* $\{x_1, \ldots, x_t\}$ if $\sum_{i=1}^{j} x_iR$ is an ideal for all $j=1, \ldots, t$, x_j is not a zero divisor modulo $\sum_{i=1}^{j-1} x_iR$, $j=1, \ldots, t$, and $I=\sum_{i=1}^{t} x_iR$.

THEOREM: *Let $\boldsymbol{g}$ be a finite dimensional complex solvable Lie algebra, let P be a prime ideal of $\mathscr{U}(\boldsymbol{g})$, and define $\mathscr{S}'(P)$, $'\mathscr{S}(P)$, and $\mathscr{S}(P)$ as in* 5.1.

(i) *If $\mathscr{S}\subseteq\mathscr{C}(P)$ is a right Ore set [resp. left Ore set] [resp. Ore set], then $\mathscr{S}\subseteq\mathscr{S}'(P)$ [resp. $\mathscr{S}\subseteq{}'\mathscr{S}(P)$] [resp. $\mathscr{S}\subseteq\mathscr{S}(P)$].*

(ii) *$\mathscr{S}'(P)$ is a right Ore set, $'\mathscr{S}(P)$ is a left Ore set, and $\mathscr{S}(P)$ is an Ore Set.*

PROOF:

(i) is a restatement of 2.3.

(ii) We shall prove that $\mathscr{S}'(P)$ satisfies the right Ore condition; the proofs for $'\mathscr{S}(P)$ and $\mathscr{S}(P)$ are similar. Fix $\boldsymbol{h}$ as in 5.2, and note that $\mathscr{S}= \mathscr{U}(\boldsymbol{h})\backslash(P\cap\mathscr{U}(\boldsymbol{h}))$ lies in $\mathscr{S}'(P)$. As observed in 5.2, $\mathscr{S}$ is an Ore set. Let $S = \mathscr{U}(\boldsymbol{g})_{\mathscr{S}}$, so that, by 5.2, $S/(P\cap\mathscr{U}(\boldsymbol{h}))S$ is a polynomial ring $\bar{S}$ over the quotient division ring D of $\mathscr{U}(\boldsymbol{h})/(P\cap\mathscr{U}(\boldsymbol{h}))$. Since $\mathbb{C} \subseteq D$, D is uncountable. Now $\mathscr{S}'(P) = \bigcap_{\lambda\in R(P)} \mathscr{C}(\tau_\lambda(P))$, and $R(P)$ is countable. Every prime ideal of $\bar{S}$ is centrally generated, and so localisable [20, Theorem 2.2]. Thus the hypotheses of 4.6 hold for the image $\overline{\mathscr{S}'(P)}$ of $\mathscr{S}'(P)$ in $\bar{S}$, and so $\overline{\mathscr{S}'(P)}$ is a Ore set in $\bar{S}$.

As shown in [7, §4 and 2.1], the ideal $(P\cap\mathscr{U}(\boldsymbol{h}))S$ of S has a regular normalising set of generators $\{y_i : i\in\mathscr{I}\}$, ($\mathscr{I}$ as in 2.2). Here, the element $y_i + (\sum_{j<i} y_jS)/\sum_{j<i} y_jS$ of $S/\sum_{j<i} y_jS$ is a $\boldsymbol{g}$-eigenvector with eigenvalue $m_i\lambda_i$. (This is shown in the second paragraph of the proof of 2.3.) Hence if $u\in\mathscr{U}(\boldsymbol{g})$ and $i\in\mathscr{I}$,

$$uy_i - y_i\tau_{m_i\lambda_i}(u)\in \sum_{j<i} y_jS. \tag{1}$$

Now $R(P)=\langle -m_i\lambda_i : i\in\mathscr{I}\rangle$ and $\mathscr{S}'(P)= \bigcap_{\lambda\in R(P)} \mathscr{C}(\tau_\lambda(P))$, so that $\tau_{m_i\lambda_i}(\mathscr{S})\subseteq\mathscr{S}$ for $i\in\mathscr{I}$. Therefore, by (1) and 3.1, $\mathscr{S}'(P)$ is a right Ore set in S. Since $\mathscr{S}\subseteq\mathscr{S}'(P)$, $\mathscr{S}'(P)$ is a right Ore set in $\mathscr{U}(\boldsymbol{g})$.

5.4. Continue to assume the hypotheses of 5.3. The above proof shows that if E is any finitely generated subsemigroup of $(\boldsymbol{g}/[\boldsymbol{g}, \boldsymbol{g}])^*$ containg $R(P)$, and $\mathscr{S}=: \bigcap_{E\cup\{0\}} \mathscr{C}(\tau_\lambda(P))$, then $\mathscr{S}$ is a right Ore subset of $\mathscr{C}(P)$. Similar remarks apply to $L(P)$ and $O(P)$.

5.5. The symmetry inherent in Theorem 5.3 may be better appreciated in the light of the following result. (A subset $\mathscr{S}$ of a ring R is *saturated* if $ab\in\mathscr{S}$ implies $a\in\mathscr{S}$ and $b\in\mathscr{S}$.)

THEOREM: *Let $\boldsymbol{g}$ be solvable and P a prime ideal of $\mathscr{U}(\boldsymbol{g})$. Let $\mathscr{S}(P)= \bigcap\{\mathscr{C}(\tau_\lambda(P)) : \lambda\in O(P)\}$ as before, and $\mathscr{O}(P)= \mathscr{C}(\bigcap\{\tau_\lambda(P) : \lambda\in O(P)\})$, so that $\mathscr{S}(P)\subseteq\mathscr{C}(P)\subseteq\mathscr{O}(P)$.*

(i) *$\mathscr{O}(P)= \mathscr{C}((P\cap\mathscr{U}(\boldsymbol{h}))\mathscr{U}(\boldsymbol{g}))$, where $\boldsymbol{h}$ is defined as in* 5.2.

(ii) *$\mathscr{O}(P)$ is the smallest saturated Ore set [resp. right Ore set] [resp. left Ore set] containing $\mathscr{C}(P)$.*

(iii) *P is localisable if and only if $\mathscr{S}(P)= \mathscr{C}(P)= \mathscr{O}(P)$.*

PROOF:

(i) It follows easily from the definition of $\boldsymbol{h}$ in [4, proof of 11.3] that $(P\cap\mathcal{U}(\boldsymbol{h}))\mathcal{U}(\boldsymbol{g})=\bigcap\{\tau_\lambda(P):\lambda\in O(P)\}$.

(ii) $\mathcal{O}(P)$ is an Ore set by [7, 2.9(iv)].
To prove that $\mathcal{O}(P)$ is the smallest saturated right Ore set containing $\mathcal{C}(P)$, one first inverts the elements of $\mathcal{C}_{\mathcal{U}(\boldsymbol{h})}(P\cap\mathcal{U}(\boldsymbol{h}))$ in $\mathcal{U}(\boldsymbol{g})$, (which is possible by [7, 2.9] and [5, 4.5]), to obtain a ring R, say. As noted in the proof of 5.3, $(P\cap\mathcal{U}(\boldsymbol{h}))R$ has a regular normalising set of generators $\{x_1, \ldots, x_t\}$ such that for each i, $rx_i=x_i\tau_\lambda(r)$ modulo $(\sum_{j<i} x_jR)$, where $\lambda=m_i\lambda_i$.

Put $x=x_i$ for some i, and $\lambda=m_i\lambda_i$. We claim that if $\mathcal{C}(P)\subseteq\mathcal{S}$ and $\mathcal{S}$ is a saturated right Ore set, then

$$\tau_{-\lambda}(\mathcal{S})\subseteq\mathcal{S}. \tag{2}$$

Given (2), Then $\tau_{-\lambda}(\mathcal{C}(P))=\mathcal{C}(\tau_{-\lambda}(P))\subseteq\mathcal{S}$; repeating the argument (a) with $\tau_{-\lambda}(P)$ in place of P and (b) for all $i=1, \ldots, t$, we see that

$$\bigcup\{\mathcal{C}(\tau_\lambda(P):\lambda\in R(P)\}\subseteq\mathcal{S}. \tag{3}$$

But $\bigcap\{\tau_\lambda(P):\lambda\in R(P)\}=(P\cap\mathcal{U}(\boldsymbol{h}))\mathcal{U}(\boldsymbol{g})$ as in (ii), so (3) implies $\mathcal{C}((P\cap\mathcal{U}(\boldsymbol{h}))\mathcal{U}(\boldsymbol{g}))\subseteq\mathcal{S}$. Thus (ii) follows from (i); (iii) is a consequence of (ii) and 5.3. It remains to prove (2); in doing so we may clearly factor by $\sum_{j=1}^{i-1} x_jR$, and so arrange that x is a normal clement. Then (2) follows from 5.6.

5.6. LEMMA: *Let $\{x=x_1, \ldots, x_t\}$ be a regular normalising set of generators of the prime ideal P of a ring R, and let $\mathcal{S}$ be a saturated right Ore set containing $\mathcal{C}(P)$. Let $c\in\mathcal{C}(P)$, so there exists (a unique) $d\in R$ such that $cx=xd$. Then $d\in\mathcal{S}$.*

PROOF: An easy induction on t shows that $\mathcal{C}(P)\subseteq\mathcal{C}(xR)$. Since $\mathcal{S}$ is a right Ore set and $c\in\mathcal{S}$, there exist $y\in R$, $v\in\mathcal{S}$ such that $cy=xv$. Since $c\in\mathcal{C}(P)$, $y\in xR$; say $y=xu$. Thus

$$xv=cy=cxu=xdu.$$

Since $x\in\mathcal{C}(0)$, $v=du$. Thus $d\in\mathcal{S}$, because $v\in\mathcal{S}$ and $\mathcal{S}$ is saturated.

6. Structure of the localised rings

6.1. Once again, $\boldsymbol{g}$ denotes a finite dimensional complex solvable Lie algebra and P is a prime ideal of its enveloping algebra $\mathcal{U}(\boldsymbol{g})$. Write $\mathcal{S}'$,

$'\mathcal{S}$ and $\mathcal{S}$ for $\mathcal{S}'(P)$, $'\mathcal{S}(P)$ and $\mathcal{S}(P)$ respectively. It is not at first glance obvious that $\mathcal{S}$ need contain many elements; or equivalently, that the localisation of $\mathcal{U}(\mathbf{g})$ at $\mathcal{S}$ is markedly different from $\mathcal{U}(\mathbf{g})$ itself. We dispel these doubts with the following result; some of the terminology used in its statement is explained in 6.2.

THEOREM: *Let R be the ring* (a) $\mathcal{U}(\mathbf{g})_{\mathcal{S}'}$, (b) $\mathcal{U}(\mathbf{g})_{'\mathcal{S}}$, (c) $\mathcal{U}(\mathbf{g})_{\mathcal{S}}$.

(i) *R is a domain and is* (a) *right Noetherian*, (b) *Left Noetherian* (c) *Noetherian.*

(ii) *R has countably many maximal one-sided ideals, namely the (two-sided) ideals* (a) $\tau_\lambda(P)R, \lambda \in \{0\} \cup R(P)$, (b) $R\tau_\lambda(P), \lambda \in \{0\} \cup L(P)$, (c) $R\tau_\lambda(P), \lambda \in \{0\} \cup O(P)$.

(iii) *Let M be a maximal ideal of R. Then R/M is isomorphic to the quotient division ring of* $\mathcal{U}(\mathbf{g})/P$, *and M has a regular normalising set of generators containing t elements, say.*

(iv) *In* (iii)

$$t = \text{height } P = \text{height } M = \text{gl. dim. } (R) = k - \text{dim. } (R) = \text{grade } (M).$$

(*Here, the dimensions are taken on the right in case* (a) *and the left in case* (b). *In case* (c), *dimensions can be taken on either side. The grade of an ideal I of R is* $\inf\{i : \text{Ext}^i_R(R/I, R) \neq 0\}$, *the modules R/I and R being* (a) *right*, (b) *left*, (c) *right or left.*)

(v) *The Jacobson radical J(R) is given by*

$$J(R) = (P \cap \mathcal{U}(\mathbf{t}))R, \textit{ where } \mathbf{t} = \bigcap\{\ker \lambda : \lambda \in O(P)\}.$$

(vi) *R has right stable range and left stable range one.*

(vii) *Every projective right or left R-module is free.*

(viii) *The graph of links of the maximal ideals of R is connected. In fact there is a directed path of* (a) *right* [(b) *left*] [(c) *right or left*] *links connecting PR to each maximal ideal of R.*

6.2. A ring S has right stable one if, given $a, b \in S$ such that $S = aS + bS$, there exsits $d \in S$ such that $S = (a + bd)S$ [22, p. 199]. With reference to 6.1 (viii), the graph of links is discussed and defined in [18, pp. 235–236]. In (iii), k-dim( ) denotes the Krull dimension in the sense of Gabriel and Renschler, and gl. dim.( ) denotes the global dimension.

6.3. Theorem 6.1 seems to be the appropriate generalisation of the result of P.F. Smith [21] that the localisation of the enveloping algebra of a *nilpotent* Lie algebra at a prime ideal yields a *"non-commutative regular local ring"* in the sense of R. Walker [24]. Note that this result, and more generally the case of a localisable prime ideal of an enveloping algebra of a solvable Lie algebra treated in [7, §4], are included in 6.1; they correspond to the case where $\boldsymbol{h} = \boldsymbol{g}$, in the notation of 5.2.

Indeed, we venture to suggest that, in the light of 6.1, the conventional definition of a non-commutative Noetherian local ring S, which requires that $S/J(S)$ be simple Artinian, is too restrictive a generalisation of the commutative definition. The stimulating idea that Noetherian rings might exist possessing the properties listed in 6.1, and that such rings could justifiably be regarded as "local", was first suggested to the author by A.V. Jategaonkar in 1979.

6.4. Let us illustrate some of 6.1 by examining the augmentation ideal $P = \langle x, y\rangle$ of the enveloping algebra $\mathcal{U}$ of the 2-dimensional non-abelian Lie algebra $\boldsymbol{g} = \langle x, y : [y, x] = x\rangle$. In this case $\mathcal{S} = \{1\}$, $m_1 = 1$ and $\lambda_1 \in \boldsymbol{g}^*$ is given by $\lambda_1(y) = 1$, $\lambda_1(x) = 0$. Thus $\mathcal{S}'(P) = \bigcap_{n\geq 0} \mathcal{C}(\langle x, y - n\rangle)$, $'\mathcal{S}(P) \bigcap_{n\geq 0} \mathcal{C}(\langle x, y + n\rangle)$ and $\mathcal{S}(P) = \bigcap_{n\in \mathbf{Z}} \mathcal{C}(\langle x, y + n\rangle)$.. If $R = \mathcal{U}(\boldsymbol{g})_{\mathcal{S}'}$, $\mathcal{U}(\boldsymbol{g})'_{\mathcal{S}}$ or $\mathcal{U}(\boldsymbol{g})_{\mathcal{S}}$ then $J(R) = xR$. The chains of links between maximal ideals of R described in 6.1 (viii) are

$$\langle x,y\rangle \leftarrow \langle x,y-1\rangle \leftarrow \langle x,y-2\rangle \leftarrow \langle x,y-3\rangle \cdots \quad \text{for } \mathcal{U}(\boldsymbol{g})_{\mathcal{S}'},$$

$$\langle x,y\rangle \rightarrow \langle x,y+1\rangle \rightarrow \langle x,y+2\rangle \rightarrow \langle x,y+3\rangle \cdots \quad \text{for } \mathcal{U}(\boldsymbol{g})_{'\mathcal{S}},$$

and

$$\cdots \rightarrow \langle x,y-2\rangle \rightarrow \langle x,y-1\rangle \rightarrow \langle x,y\rangle \rightarrow \langle x,y+1\rangle \rightarrow \langle x,y+2\rangle \cdots \quad \text{for } \mathcal{U}(\boldsymbol{g})_{\mathcal{S}}.$$

Let $R = \mathcal{U}(\boldsymbol{g})'_{\mathcal{S}}$. Then it is not hard to show that the right ideal $\sum_{j\geq 1} (y-1)^{-j}xR$ of R is *not* finitely generated; hence no improvement is possible in 6.1(i). Note that in this case R is a left Noetherian right Ore domain which is not right Noetherian; the existence or otherwise of such rings was raised as a question by J. Cozzens and C. Faith [8]; the first such example was constructed by D.A. Jordan [15]. That localisations of $\mathcal{U}(\boldsymbol{g})$ gave such examples was pointed out to me by I.N. Musson.

6.5. Note the following generalisation of 6.1:— As in 5.4, let $E \leq (\boldsymbol{g}/[\boldsymbol{g}, \boldsymbol{g}])^*$ be an arbitrary finitely generated semigroup containing $R(P)$ and let $\mathcal{E} = \bigcap_{\lambda\in E\cup\{0\}} \mathcal{C}(\tau_\lambda(P))$, a right Ore set as noted in 5.4, and put $R = \mathcal{U}(\boldsymbol{g})_{\mathcal{E}}$. Then *the statements and proofs of 6.1 (i), …, (vii) apply equally well to R* (Except that now $\boldsymbol{t} = \bigcap\{\ker \lambda : \lambda \in E\}$ in (v)). And the generalisation of (viii) is: *The connected components of the graph of links of* max spec (R) *are in* $1-1$ *correspondence with the cosets of* $R(P)$ *in* E. Similar remarks apply to $L(P)$ and $O(P)$.

The proof of 6.1 occupies the next four paragraphs. We shall work throughout with $'\mathcal{S}(P)$; the other cases are similar.

6.6. PROOF OF 6.1 (i), (ii), (v) AND (iii): Since R is a partial left quotient ring of the Noetherian domain $\mathscr{U}(\mathbf{g})$, (i) is clear. Let I be a maximal right or left ideal of R. By 5.2, $R/(P\cap\mathscr{U}(\mathbf{t}))R$ is a polynomial algebra over a division algebra D satisfying the hypothesis of Lemma 4.2, as noted in 4.2. If $\gamma\in\mathscr{U}(\mathbf{g})$ and $\gamma\equiv 1$ (modulo $(P\cap\mathscr{U}(\mathbf{t}))\mathscr{U}(\mathbf{g})$), then $\gamma\in{}'\mathscr{S}$ by definition of $\mathbf{t}$. Hence

$$(P\cap\mathscr{U}(\mathbf{t}))R\subseteq J(R), \tag{1}$$

and so $(P\cap\mathscr{U}(\mathbf{t}))R\subseteq I$. Since the elements of $'\mathscr{S}$ are invertible in R (and thus in $R/(P\cap\mathscr{U}(\mathbf{t}))R$), 4.2(i) now shows that $I=R\tau_\lambda(R)$ for some $\lambda\in L(P)\cup\{0\}$. (Here, we need also remark 4.3.) This proves (ii).

Now $(P\cap\mathscr{U}(\mathbf{t}))R=\bigcap\{R\tau_\lambda(P):\lambda\in L(P)\cup\{0\}\}$, by [4, proof of 11.3], so $J(R)\subseteq(P\cap\mathscr{U}(\mathbf{t}))R$. Coupled with (1), this proves (v). Let E be the quotient division ring of $\mathscr{U}(\mathbf{g})/P$, (so that E has the form $D\otimes_k K$ for a finite extension K of the centre k of D). As already noted, R/PR is a simple factor ring of a polynomial algebra over D, so $R/PR\cong E$. Since every maximal ideal of R is the image of RP under an automorphism τ_λ of R, the first part of (iii) follows.

We have already observed, in the proof of 5.3(ii), that the ideal $(P\cap\mathscr{U}(\mathbf{t}))R$ has a regular normalising set of generators. As shown above, if M is a maximal ideal of R, then $(P\cap\mathscr{U}(\mathbf{t}))R=J(R)\subseteq M$, and $M/J(R)$ is a localisation of a maximal ideal of a polynomial algebra over D. This proves the remainder of (iii).

6.7. PROOF OF (iv): Since $R/J(R)\cong D[X_1,\ \ldots,\ X_m]_{'\bar{\mathscr{S}}}$, where $'\bar{\mathscr{S}}$ is the image of $'\mathscr{S}$,

$$t=\text{gl. dim. }(R)=k\text{-dim}(R) \tag{2}$$

by [24, 1.4 and 1.9]. Clearly, height $M\geq t$; since height $M\leq k$-dim(R), $t=$ height M by (2). Also, height $P=$ height $PR=$ height M since $M=\tau_\lambda(PR)$.

Trivially, grade $(M)\leq$ gl. dim. (R), so it only remains to prove

$$t\geq\text{grade }(M). \tag{3}$$

Let $M=QR$, so Q is a prime ideal of $\mathscr{U}(\mathbf{g})$ of height t. By [3, 2.7.1] and [23, 2.5 and 2.6],

$$\text{grade }(Q)=t. \tag{4}$$

It follows from (4) that

$$\text{Ext}^t_{\mathscr{U}(\mathbf{g})}(\mathscr{U}(\mathbf{g})/Q,\mathscr{U}(\mathbf{g}))=0 \tag{5}$$

for $0 \leq i \leq t-1$, (where we take *left* modules in the arguments). Take an injective resolution of ${}_{\mathscr{U}(g)|}\mathscr{U}(g)$, and apply $\mathscr{U}(g)'_{\mathscr{S}}\otimes$ – to it, to get an injective resolution of R. From (5) and (2) we see that that injective hull of R/M occurs only in the tth term of this resolution. In other words, (3) is proved.

6.8. The argument in 6.7 also shows that $\mathscr{U}(g)/P$ occurs in the tth term, and *only* in the tth term of a minimal injective resolution of $\mathscr{U}(g)$. This result is due to M.P. Malliavin [26, §4]; her proof is completely different.

6.9 PROOFS OF (vi) AND (vii): The ring $R/J(R)$ has right and left stable range 1 by 4.2. The same is thus true of R since $1+J(R)$ consists of units.

Since $R/J(R)$ is a localisation of a polynomial algebra over a division ring, $K_0(R/J(R)) \simeq \mathbb{Z}$ by Grothendieck's theorem [2, Ch. XII, (3.1)]. Hence $K_0 R) \cong \mathbb{Z}$ by [2, Ch. IX, (1.3)], so if M is a finitely generated projective right or left R-module there exist $r, t \in \mathbb{N}$ such that $M \oplus R^{(r)} \cong R^{(t)}$ [22, II.1.10]. By (vi) and the proof of [22, 12.4], M is free.

Finally, let M be a projective R-module which is *not* finitely generated. Since M is a direct sum of countably generated modules [16], without loss we may assume M is countably generated. By Nakayama's lemma $M/J(R)M$ is not finitely generated, so $M/J(R)M$ is a free $R/J(R)$-module [1]. In particular, M/PM is not finitely generated, for each maximal ideal P of R. Hence, M is free by [1], since R clearly has no proper idempotent ideals.

6.10. PROOF OF 6.1(viii): Let M be any maximal ideal of R, so $M = QR$ for some prime ideal Q of $\mathscr{U}(g)$, and there exists a chain of prime ideals $P = P_1, \ldots, P_t = Q$ of $\mathscr{U}(g)$ such that for $i = 1, \ldots, t-1$ there are ideals $A_i \subseteq B_i$ of $\mathscr{U}(g)$ with $l(B_i/A_i) = P_i$ and $r(B_i/A_i) = P_{i+1}$, by definition of $'\mathscr{S}(P)$ and the proof of Theorem 2.3. Be enlarging A_i if necessary, $\mathscr{U}(g)/A_i$ can be chosen to be left P_i-primary; (see [7, §6]). By [7, 6.3(ii)], P_{i+1}/A_i is a minimal prime of $\mathscr{U}(g)/A_i$, and there is a path in the graph of left links $\mathscr{U}(g)$ connecting P_i to P_{i+1}. Since $'\mathscr{S}(P) \subseteq \mathscr{C}(P_i) \cap \mathscr{C}(P_{i+1})$, $P_i R$ and $P_{i+1} R$ are proper prime ideals of R and $P_i R$ is connected by a directed path in the graph of left links of R to $P_{i+1} R$. Joining these paths together, we get a directed path from PR to M.

7. Ore sets and subalgebras

7.1. In [5, 4.5] W. Borho and R. Rentschler obtained a sufficient condition for a right Ore set of $\mathscr{U}(p)$ to be a right Ore set in $\mathscr{U}(g)$, where p is a solvable subalgebra of the Lie algebra g. The following result shows that, given a mild and natural additional hypothesis, their condition is also necessary.

THEOREM: *Let $\boldsymbol{g}$ be a finite dimensional complex Lie algebra and let $\boldsymbol{p}$ be a solvable subalgebra of $\boldsymbol{g}$. Let $\mathcal{S}$ be a saturated right [resp. left] Ore set in $\mathcal{U}(\boldsymbol{p})$. Then $\mathcal{S}$ satisfies the right [resp. left] Ore condition in $\mathcal{U}(\boldsymbol{g})$ if and only if $\tau_\lambda(\mathcal{S}) \subseteq \mathcal{S}$ [resp. $\tau_{-\lambda}(\mathcal{S}) \subseteq \mathcal{S}$] for every eigenvalue $\lambda \in \boldsymbol{p}^*$ occurring in a composition series of the $\boldsymbol{p}$-module $\boldsymbol{g}/\boldsymbol{p}$.*

The maps τ_λ are the winding automorphisms of [4, 10.2].

7.2. We isolate a special case of 7.1: –

LEMMA: *Let $\boldsymbol{b}$ be a subalgebra of codimension one in the Lie algebra $\boldsymbol{g}$. Let $\mathcal{S}$ be a multiplicatively closed and saturated subset of $\mathcal{U}(\boldsymbol{g})$ which is right [resp. left] Ore in $\mathcal{U}(\boldsymbol{g})$. Let $\lambda \in \boldsymbol{b}^*$ be the eigenvalue of the $\boldsymbol{b}$-module $\boldsymbol{g}/\boldsymbol{b}$. Then $\tau_\lambda(\mathcal{S}) \subseteq \mathcal{S}$ [resp. $\tau_{-\lambda}(\mathcal{S}) \subseteq \mathcal{S}$].*

PROOF: We prove the result for $\mathcal{S}$ right Ore. Fix $x \in \boldsymbol{g}$ such that $\boldsymbol{g} = \boldsymbol{b} \oplus \mathbb{C}x$. If $y \in \boldsymbol{b}$, then $yx = x(y + \lambda(y)) + \sigma(y)$, where $\sigma: \boldsymbol{b} \to \boldsymbol{b}$. It follows easily that there is a map σ: $\mathcal{U}(\boldsymbol{b}) \to \mathcal{U}(\boldsymbol{b})$ such that

$$ux = x\tau_\lambda(u) + \sigma(u)$$

for all $u \in \mathcal{U}(\boldsymbol{b})$. Let $c \in \mathcal{S}$. Then there exists $a \in \mathcal{U}(\boldsymbol{g})$ and $d \in \mathcal{S}$ such that

$$ca = xd.$$

Now $x\mathcal{U}(\boldsymbol{b}) = \mathcal{U}(\boldsymbol{b})x + \mathcal{U}(\boldsymbol{b})$, so $a = a_1x + a_2$, say, where $a_1, a_2 \in \mathcal{U}(\boldsymbol{b})$. Hence

$$\begin{aligned} xd &= ca_1 + ca_2x \\ &= ca_1 + cx\tau_\lambda(a_2) + c\sigma(a_2) \\ &= c(a_1 + \sigma(a_2)) + \sigma(c)\tau_\lambda(a_2) + x\tau_\lambda(ca_2). \end{aligned}$$

Since the first two terms of the final expression above are in $\mathcal{U}(\boldsymbol{b})$, using the Poincaré-Birkhoff-Witt Theorem we deduce that

$$d = \tau_\lambda(ca_2) = \tau_\lambda(c)\tau_\lambda(a_2). \tag{1}$$

Because $d \in \mathcal{S}$ and $\mathcal{S}$ is saturated, (1) implies $\tau_\lambda(c) \in \mathcal{S}$.

7.3. PROOF OF 7.1: We prove the right hand version. If $\tau_\lambda(\mathcal{S}) \subseteq \mathcal{S}$ for all the relevant $\lambda \in \boldsymbol{p}^*$, then $\mathcal{S}$ is right Ore in $\mathcal{U}(\boldsymbol{g})$ by [5, 4.5]. Suppose for the converse that $\mathcal{S}$ satisfies the right Ore condition in $\mathcal{U}(\boldsymbol{g})$. Let $\boldsymbol{r}$ be the solvable radical of $\boldsymbol{g}$ and $\boldsymbol{b}/\boldsymbol{r}$ a Borel subalgebra of $\boldsymbol{g}/\boldsymbol{r}$ containing

$\boldsymbol{p}+\boldsymbol{r}/\boldsymbol{r}$. There is a chain of subalgebras

$$\boldsymbol{p}=\boldsymbol{p}_0 \subset \boldsymbol{p}_1 \subset \ldots \subset \boldsymbol{p}_t=\boldsymbol{b}$$

with $\dim_{\mathbb{C}} \boldsymbol{p}_{i+1}/\boldsymbol{p}_i = 1$ for all i, since $\boldsymbol{b}$ is completely solvable. Now t applications of 7.2 show that $\tau_\lambda(\mathcal{S}) \subseteq \mathcal{S}$ for all Jordan-Holder values λ of $\boldsymbol{p}$ on $\boldsymbol{b}/\boldsymbol{p}$.

Fix a Cartan subalgebra $\boldsymbol{h}/\boldsymbol{r}$ in $\boldsymbol{b}/\boldsymbol{r}$. There exists a basis Δ of the root system of $\boldsymbol{g}/\boldsymbol{r}$ relative to $\boldsymbol{h}/\boldsymbol{r}$ such that $\boldsymbol{b}/\boldsymbol{r}=\boldsymbol{h}/\boldsymbol{r} \oplus \sum_{\alpha<0} (\boldsymbol{g}/\boldsymbol{r})_\alpha$ [11, page 86]. Let $\alpha \in \Delta$; if β is any positive root, then $\beta-\alpha$ is not a root, by the defining properties of a basis. Hence, $[(\boldsymbol{g}/\boldsymbol{r})_\beta, (\boldsymbol{g}/\boldsymbol{r})_{-\alpha}] \subseteq \boldsymbol{r}$ and so there exists $x \in \boldsymbol{g}$ such that $\mathbb{C}x+\boldsymbol{b}/\boldsymbol{b}$ is a $\boldsymbol{b}$-module with eigen value $-\alpha$. Thus, by 7.2 again $\tau_{-\alpha}(\mathcal{S}) \subseteq \mathcal{S}$ for all $\alpha \in \Delta$. But every Jordän-Holder value of $\boldsymbol{b}$ on $\boldsymbol{g}/\boldsymbol{b}$ is a positive integral linear combination of $\{-\alpha : \alpha \in \Delta\}$. Since $\tau_\lambda \tau_\mu = \tau_{\lambda+\mu}$ for $\lambda, \mu \in \boldsymbol{b}^*$, this completes the proof of 7.1.

8. The semisimple case

8.1. In this section, $\boldsymbol{g}$ denotes a finite dimensional complex semisimple Lie algebra, $\boldsymbol{h}$ a Cartan subalgebra, $R \subseteq \boldsymbol{h}^*$ a root system for $\boldsymbol{g}$ relative to $\boldsymbol{h}$ with root spaces $\{\boldsymbol{g}_\alpha : \alpha \in R\}$, $R^+ \cup R^- = R$ a decomposition of R into positive and negative roots, with a basis $\Delta \subseteq R^+$, and $\boldsymbol{b}$ the Borel subalgebra containing $\boldsymbol{h}$ with $\boldsymbol{b}=\boldsymbol{h} \oplus \sum^{\oplus}_{\alpha \in R^+} \boldsymbol{g}_\alpha$. (Note that every Borel subalgebra of $\boldsymbol{g}$ arises in this way for some choice of $\boldsymbol{h}$ and R^+, [11, page 86].) Let $\mathbb{Z}R \subseteq \boldsymbol{h}^*$ denote the root lattice [11, page 67], and $\mathbb{N}X$ denote the subsemigroup generated by a subset X of $\boldsymbol{h}^*$. Note that if $\lambda \in \boldsymbol{h}^*$ then $\lambda \in (\boldsymbol{b}/[\boldsymbol{b}, \boldsymbol{b}])^*$ and so λ defines a winding automorphism τ_λ of $\mathcal{U}(\boldsymbol{b})$.

8.2. Combining 5.3. and 7.1 we obtain immediately:

THEOREM: *Let* $\boldsymbol{g}$ *be a semisimple Lie algebra and* $\boldsymbol{b}$ *a Borel subalgebra of* $\boldsymbol{g}$. *Let* $\boldsymbol{h} \subseteq \boldsymbol{b}$ *and* R *be as in* 8.1. *Let* P *be a prime ideal of* $\mathcal{U}(\boldsymbol{b})$. *Define the subsemigroups* $L(P)$ *and* $R(P)$ *as in* 2.2, *so that* $L(P) \subseteq \mathbb{N}R^+$, $R(P) = -L(P) \subseteq \mathbb{N}R^-$. *Define*

$$'\mathcal{S}(P) = \bigcap\{\mathcal{C}_{\mathcal{U}(\boldsymbol{b})}(\tau_\lambda(P)) : \lambda \in \mathbb{N}(L(P) \cup R^-)\},$$

$$\mathcal{S}'(P) = \bigcap\{\mathcal{C}_{\mathcal{U}(\boldsymbol{b})}(\tau_\lambda(P)) : \lambda \in \mathbb{N}(R(P) \cup R^+)\},$$

and

$$\mathcal{S}(P) = \bigcap\{\mathcal{C}_{\mathcal{U}(\boldsymbol{b})}(\tau_\lambda(P)) : \lambda \in \mathbb{Z}R\}.$$

(i) $'\mathcal{S}(P)$ [*resp.* $\mathcal{S}'(P)$] *is the unique largest subset of* $\mathcal{C}_{\mathcal{U}(\boldsymbol{b})}(P)$ *satisfying the left* [*resp. right*] *Ore condition in* $\mathcal{U}(\boldsymbol{g})$.

(ii) $\mathscr{S}(P)$ *is the unique largest Ore subset of* $\mathscr{C}_{\mathscr{U}(\boldsymbol{b})}(P)$ *in* $\mathscr{U}(\boldsymbol{g})$.
(iii) *Suppose that* $\mathscr{U}(\boldsymbol{b})/P$ *is Artinian, so* $P = l(\mathbb{C}_\mu)$, *the annihilator of the* 1-*dimensional* **b**-*module defined by* $\mu \in \boldsymbol{h}^*$. *Then*

$$\mathscr{S}(P) = {}'\mathscr{S}(P) = \mathscr{S}'(P) = \bigcap\left\{ \mathscr{C}_{\mathscr{U}(\boldsymbol{b})}(l(\mathbb{C}\hat{\lambda})) : \lambda \in \mu + \mathbb{Z}R \right\}.$$

8.3. Notation: In case 8.2(iii) we shall denote $\mathscr{S}$(P) by $\mathscr{S}(\mu)$.

8.4. Let $\mu \in \boldsymbol{h}^*$. What can be said of the ring $\mathscr{U}(\boldsymbol{b})_{\mathscr{S}(\mu)}$? In particular, what are its prime ideals? Let P be a prime ideal of $\mathscr{U}(\boldsymbol{g})$. Then P is primitive if and only if $P \cap Z(\boldsymbol{g})$ is a maximal ideal of $Z(\boldsymbol{g})$ (where $Z(\boldsymbol{g})$ is the centre of $\mathscr{U}(\boldsymbol{g})$), and if M is a maximal ideal of $Z(\boldsymbol{g})$, the $M\mathscr{U}(\boldsymbol{g})$ is primitive, [9, 8.5.7, 8.5.8]; the ideal $M\mathscr{U}(\boldsymbol{g})$ is thus called a *minimal primitive ideal*. Every minimal primitive ideal is contained in only finitely many primitive ideals [9, 8.5.7(b)]. Every primitive ideal Q of $\mathscr{U}(\boldsymbol{g})$ is the annihilator of the unique irreducible image $L(\lambda)$ of a Verma module $M(\lambda)$ by Duflo's theorem [10]. Now if $z \in \mathscr{S}(\mu)$, then $z + u \in \mathscr{S}(\mu)$ for all $u \in \boldsymbol{n}^+\mathscr{U}(\boldsymbol{b})$, where $\boldsymbol{n}^+ = \sum^{\oplus}_{\alpha \in R^+} \boldsymbol{g}_\alpha$. Hence,

$$Q \cap \mathscr{S}(\mu) \neq \emptyset \Leftrightarrow (Q \cap \mathscr{U}(\boldsymbol{h})) \cap \mathscr{S}(\mu) \neq \emptyset. \qquad (1)$$

As a $\mathscr{U}(\boldsymbol{h})$-module, $L(\lambda)$ is a direct sum of weight spaces. Each weight $v \in \boldsymbol{h}^*$ defines a 1-dimensional $\mathscr{U}(\boldsymbol{h})$-module $\mathscr{U}(\boldsymbol{h})/I(v)$, say, and so

$$Q \cap \mathscr{U}(\boldsymbol{h}) = \bigcap\{ I(v) : v \text{ a weight of } L(\lambda)\}. \qquad (2)$$

In view of 6.1(ii), an ideal J of $\mathscr{U}(\boldsymbol{h})$ satisfies $J \cap \mathscr{S}(\mu) \neq \emptyset$ if and only if $J \not\subseteq I(v)$ for all $v \in \mu + \mathbb{Z}R$. Therefore, we can deduce from (1) and (2):

THEOREM: *Let* $Q = l(L(\lambda))$ *be a primitive ideal of* $\mathscr{U}(\boldsymbol{g})$. *Let* $\mu \in \boldsymbol{h}^*$. *Then* $Q \cap \mathscr{S}(\mu) = \emptyset$ *if and only if*

$$\mu \in \mathbb{Z}R + \overline{\{v : v \text{ a weight of } L(\lambda)\}},$$

(*the closure being with respect to the Zariski topology on* $\boldsymbol{h}^*$).

8.5. To illustrate 8.4, consider the cases where (i) Q is co-Artinian and (ii) Q is a minimal primitive ideal [9, 8.4.3, 8.4.4]. In case (i), $L(\lambda)$ is finite dimensional; that is $\lambda \in \Lambda$, the lattice of integral weights, and λ is dominant. Here, $\{v : v \text{ a weight of } L(v)\}$ is a finite and hence closed subset of $\lambda + ZR \leq \Lambda$. Thus

$$Q \cap \mathscr{S}(\mu) = \emptyset \Leftrightarrow \mu \in \lambda + \mathbb{Z}R.$$

When (ii) Q is a minimal primitive ideal we can take $L(\lambda) = M(\lambda)$ [9, 8.4.3] and it follows easily that $\overline{\{v : v \text{ a weight of } L(\lambda)\}} = \boldsymbol{h}^*$. Hence in this case 8.4 yields: – For all $\mu \in \boldsymbol{h}^*$, $Q \cap \mathscr{S}(\mu) = \emptyset$.

8.6. Suppose now that $Q = l(L(\lambda))$ is an arbitrary primitive ideal. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ and let $h_i = h_{\alpha_i}$ be the corresponding basis of $\boldsymbol{h}$. It is not difficult to generalise one direction of the implications in 8.5 to show that:

If $\mu \in \boldsymbol{h}^*$ and there exists $\tau \in \mathbb{Z}R$ such that $(\mu - \lambda)(h_i) = \tau(h_i)$ for all i such that $\lambda(h_i) \in \mathbb{N}$, then $Q \cap \mathscr{S}(\mu) = \emptyset$.

We omit the routine deduction of this result from 8.4.

8.7. Let us sketch some possible directions for further work in the semisimple case. It is easy to show that the semigroup generated by a family of Ore sets of regular elements in a ring is itself an Ore set. In particular, there is a unique maximal Ore subset $\mathscr{S}(P)$ of $\mathscr{C}(P)$, where P is a prime ideal. What is this set when P is primitive ideal of $\mathscr{U}(\boldsymbol{g})$ and $\boldsymbol{g}$ is semisimple? Let P_0 be the minimal primitive ideal contained in P; then $\mathscr{U}(\boldsymbol{g}) \backslash P_0$ and $1 + P_0$ are both Ore sets, since P_0 is centrally generated. Thus if $P = P_0$, $\mathscr{S}(P) = \mathscr{U}(\boldsymbol{g}) \backslash P_0$ since P_0 is completely prime [9, 8.4.3, 8.4.4(i)]. In general, $\mathscr{S}(P)$ contains $1 + P_0$ and the various $\mathscr{S}(\mu)$ supplied by 8.4 as $\boldsymbol{b}$ ranges through the Borel subalgebras of $\boldsymbol{g}$ and λ and μ range through their permitted values. Do we perhaps have $\mathscr{S}(P) = \langle 1 + P_0, \{\mathscr{S}_{\mathscr{S}(\boldsymbol{b})}(\mu) : \boldsymbol{b}$ a Borel subalgebra, μ given by 8.4$\}\rangle$? What are the maximal ideals of $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$? We say that prime ideals M and N of $\mathscr{U}(\boldsymbol{g})$ are *linked* if there are ideals $J \subseteq I$ of $\mathscr{U}(\boldsymbol{g})$ with $l(I/J) = M$ *and* $r(I/J) = N$, and define $\sim$, the equivalence class on Spec $\mathscr{U}(\boldsymbol{g})$ generated by the links [12]. In view of 2.1, if $P \sim Q$ then Q generates a proper prime ideal of $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$. It is tempting to conjecture that the maximal ideals of $\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$ are precisely the ideals $Q\mathscr{U}(\boldsymbol{g})_{\mathscr{S}(P)}$, where $P \sim Q$. It follows from [27, Theorem 2.2] that the ideals in the equivalence class containing P all have (a) the same central character as P and (b) the same associated variety; moreover, if one restricts to *integral* central characters, then the associated variety is always irreducible [25], and then, as noted by Borho in a forthcoming paper, (a) and (b) characterise the equivalence classes under $\sim$.

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