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ALEXANDRU DIMCA

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FUNCTION GERMS DEFINED ON ISOLATED HYPERSURFACE SINGULARITIES

Alexandru Dimca

The study of analytic function germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ under $\mathscr{R}$ (right) and $\mathscr{K}$ (contact) equivalence relations is a central point in singularity theory and the information we have in this direction is far-reaching (see for instance the papers of Arnold [1], Siersma [13], Wall [14]).

In particular, it is well known that the $\mathscr{R}$-simple and the $\mathscr{K}$-simple functions are the same and their classification runs as follows (modulo addition of a nondegenerate quadratic form in some other variables):

$$A_k : x^{k+1} \quad k \geqslant 1; \qquad D_k : x^2y + y^{k-1} \quad k \geqslant 4$$

$$E_6 : x^3 + y^4; \quad E_7 : x^3 + xy^3; \quad E_8 : x^3 + y^5.$$

In this paper we start a similar study for analytic function germs defined on an isolated hypersurface singularity X.

Some of the results (for instance those about finite determinacy in par. 1) are the exact analogues of the corresponding results when X is smooth. But there are also definitely new phenomena. We show that on an isolated hypersurface singularity X there are $\mathscr{R}$-simple functions iff X is an A_k-singularity for some $k \geqslant 1$.

The class of singularities X on which there are $\mathscr{K}$-simple functions is larger, but nevertheless very restricted. To describe it precisely we use a Lie algebra of derivations associated to X, whose basic properties were established by Scheja and Wiebe [12].

On the other hand, on a fixed hypersurface X there are usually much more $\mathscr{K}$-simple functions than $\mathscr{R}$-simple ones.

The paper contains the classification of all $\mathscr{R}$-simple functions as well as the classification of $\mathscr{K}$-simple functions defined on an A_k-curve singularity.

Using the Milnor fibration introduced by Hamm [7] we define in the last section a Milnor number $\mu(f)$ for any finitely determined function f on an isolated hypersurface singularity X and note that this number is a topological invariant.

When dim $X > 1$ the computation of this number shows that the

$\mathscr{R}$-classification of $\mathscr{R}$-simple functions coincides with their topological classification. The same result is true when $\dim X = 1$, but the proof this time depends on the existence of Puiseux parametrizations of the plane curve X.

In the end of this introduction we note the following interesting facts.

Goriunov has undertaken a study of a similar situation, namely the study of diagrams

$$(Y, 0) \subset (\mathbb{C}^n, 0) \xrightarrow{p} (\mathbb{C}, 0)$$

where Y is an analytic space germ and p is a projection [5]. His equivalence relation is slightly different from ours, but surprisingly his list of simple diagrams coincides essentially with our list of $\mathscr{R}$-simple functions (if one omits the case Y smooth).

Moreover, our list of $\mathscr{R}$-simple functions in the case $\dim X > 1$ has a striking (formal) resemblance to Arnold list of simple function singularities on a manifold with boundary, associated to the simple Lie groups B_k, C_k and F_4 [2].

The author will try in a subsequent paper to explain some reasons for these coincidences.

It is his great pleasure to thank Professor V.I. Arnold for a very stimulating discussion.

§1. First definitions and finite determinacy

Let A be the local $\mathbb{C}$-algebra of germs of analytic functions defined in a neighbourhood of $0 \in \mathbb{C}^{n+1}$ $(n \geqslant 1)$ and let $m \subset A$ be its maximal ideal.

If $f \in m$ and $X: f = 0$ is an isolated hypersurface singularity, we denote by $A_X = A/(f)$ the local ring of X and by $m_X \subset A_X$ the maximal ideal.

DEFINITION 1.1: Two functions $f, g \in m_X$ are called $\mathscr{R}$-equivalent (resp. $\mathscr{K}$-equivalent) if there is an automorphism u of the local algebra A_X such that $u(f) = g$ [resp. $(u(f)) = (g)$, where (a) means the ideal generated by a in A_X].

In order to study these equivalence relations it is useful to introduce in this new situation the language of group actions, jet spaces, finite determinacy and so on from standard singularity theory (the books of Gibson [4] and Martinet [9] are an excellent reference for this part).

Let L be the group of germs of analytic isomorphisms $h: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$, T the group of invertible lower triangular 2×2 matrices M over A and $S \subset T$ the subgroup of such matrices $M = (m_{ij})$ with $m_{22} = 1$.

We define two groups

$$G_{\mathscr{R}} = L \times S \quad \text{and} \quad G_{\mathscr{K}} = L \times T$$

and two actions

$$\mu_*: G_* \times B \to B, \quad (h, M)\cdot F = M\cdot(F \circ h)$$

where $* = \mathscr{R}, \mathscr{K}$; $B = mA^2$ and we think here of $F \in B$ as a column vector $\binom{f_1}{f_2}$.

The connection between the above equivalence relations and these actions is the following

LEMMA 1.2: *Suppose $X: f = 0$ is a hypersurface singularity and $g_1, g_2 \in m_X$. Then $g_1 \overset{*}{\sim} g_2$ iff $F_1 = (f, g_1)$ and $F_2 = (f, g_2)$ are in the same G_*-orbit, where $* = \mathscr{R}, \mathscr{K}$.*

PROOF: Obvious, using the fact that any automorphism of A_X is induced by some $h \in L$ with $(f \circ h) = (f)$. Note moreover that here and in what follows we identify a function $g \in A_X$ with some representative of it in A. □

In analogy with the $\mathscr{K}$-tangent space of a map germ we define the following tangent spaces for a map germ $F = (f_1, f_2) \in B$.

$$T_{\mathscr{R}}F = mJ(F) + (f_1)e_1 + (f_1)e_2 \subset B$$

$$T_{\mathscr{K}}F = mJ(F) + (f_1)e_1 + (f_1, f_2)e_2 \subset B$$

where $J(F)$ is the A-submodule in B generated by $\dfrac{\partial F}{\partial x_i}$ and $e_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\end{pmatrix}$.

If $X: f_1 = 0$ is a hypersurface and $f_2 \in m_X$, then we define

$$* - \operatorname{codim} f_2 = \dim B/T_* F, \quad \text{where} \quad * = \mathscr{R}, \mathscr{K}.$$

By passing to jets, each action μ_* induces actions

$$G^k{}_* \times J^k(n+1, 2) \to J^k(n+1, 2)$$

and we have the following relation between tangent spaces:

$$T_{j^k F}(G^k{}_* \cdot j^k_F) = j^k(T_* F).$$

DEFINITION 1.3: The function $f \in m_X$ is called k-$*$-determined if $f \overset{*}{\sim} f + f'$, for any $f' \in m_X^{k+1}$. The function f is called finitely $*$-determined if f is k-$*$-determined for some k. ($* = \mathscr{R}, \mathscr{K}$).

We have the following result, in perfect analogy with the smooth case.

PROPOSITION 1.4: *Let X be an isolated hypersurface singularity and $f \in m_X$. Then the following are equivalent.*

i. f is finitely $\mathcal{R}$-determined.

ii. f is finitely $\mathcal{K}$-determined.

iii. $\mathcal{R}$-codim $f < \infty$

iv. $\mathcal{K}$-codim $f < \infty$

v. f has an isolated singular point at $0 \in X$ (i.e. f is nonsingular at every point of $\tilde{X} - 0$, where $\tilde{X}$ is a small enough representative of X).

PROOF: It is clear that i $\Leftrightarrow$ iii, ii $\Leftrightarrow$ iv and i $\Rightarrow$ ii. To show that iv $\Rightarrow$ v suppose X is given by $g = 0$.

If v does not hold, then there is a curve germ $c: (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that:

$$g \circ c = 0, \quad \frac{\partial g}{\partial x_i} \circ c \neq 0 \quad \text{for some } i \text{ and}$$

$$\frac{\partial f}{\partial x_j} \circ c = \lambda\left(\frac{\partial g}{\partial x_j} \circ c\right)$$

for some Laurent series λ and any j. It follows that $f \circ c = 0$ and, if $ae_1 + be_2 \in T_{\mathcal{K}}F$, where $F = (g, f)$, then $b \circ c = \lambda(a \circ c)$. In particular $a = 0$ implies $b \circ c = 0$ and this contradicts iv. This argument shows also that v is equivalent to: $X_0 = \{f = g = 0\}$ is a complete intersection with an isolated singular point at 0.

To prove v $\Rightarrow$ iii we have just to note that

$$\Delta_{ij} e_2 \in T_{\mathcal{R}}F \quad \text{where}$$

$$\Delta_{ij} = \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i} \qquad \text{for any} \quad i, j.$$

Moreover, by v, (Δ_{ij}, g) is a m-primary ideal in A and this ends the proof. □

As to the order of $*$-determinacy we have the following obvious analogue of (1.7) and (1.8) in [13]:

PROPOSITION 1.5: *Suppose X: $g = 0$ is an isolated hypersurface singularity, $f \in m_X$ and $F = (g, f) \in B$. Then ($* = \mathcal{R}, \mathcal{K}$): $T_* F \supset m^r e_2 \Rightarrow f$ is r-$*$-determined $\Rightarrow T_* F \supset m^{r+1} e_2$.* □

Explicit examples will be given in par. 3.

Now we turn to the definition of simple functions.

Let X: $g = 0$ be an isolated hypersurface singularity, $f \in m_X$ and T a neighbourhood of $0 \in \mathbb{C}^p$. A deformation of f is an analytic function germ $F: X \times T \to \mathbb{C}$ such that $F_0 = f$ and $F_t \in m_X$ for any $t \in T$, where $F_t(x) = F(x, t)$.

DEFINITION 1.6: The function $f \in m_X$ is $*$-simple if

i. f has an isolated singular point at $0 \in X$.

ii. For any deformation F of the funtion f, the family of functions $(F_t)_{t \in T}$ intersects only finitely many $*$-equivalence classes (for small enough T).

We can reformulate this definition as follows. Note that we have a natural projection

$$p_1: J^k(n+1, 2) \to J^k(n+1, 1), \quad p_1(j^kF) = j^kg$$

where $F = (g, f)$ as above.

If K^kg is the contact orbit of j^kg, then $p_1^{-1}(K^kg)$ is a union of $G^k{}_*$-orbits.

DEFINITION 1.6′: The function $f \in m_X$ is $*$-simple if

i. f is finitely $*$-determined

ii. For all $k \gg 0$, j^kF has a neighbourhood in $p_1^{-1}(K^kg)$ which intersects only a finite number of $G^k{}_*$-orbits. ($* = \mathcal{R}, \mathcal{K}$).

§2. The hypersurfaces X on which there are simple functions

In this section we describe the isolated hypersurface singularities X on which there are $*$-simple functions.

To do this we need some facts about the Lie algebra Der A_X of $\mathbb{C}$-derivations of the local algebra A_X.

Assume for the moment that $X: f = 0$ is an isolated hypersurface singularity of multiplicity $e(X) = \operatorname{ord}(f) \geqslant 3$.

Let D_X be the image of Der A_X in the Lie algebra Der (A_X/m_X^2). There is one case when it is very easy to compute this Lie algebra D_X.

LEMMA 2.1: *If f is a weighted homogeneous polynomial, then* $\dim D_X = 1$ *and as a generator can be taken the Euler derivation.*

PROOF: The Lie algebra Der A_X consists of all derivations

$$D = \sum_{i=1}^{n+1} \alpha_i \frac{\partial}{\partial x_i} \quad \text{with} \quad \alpha_i \in m$$

such that there is an element $\beta \in A$ satisfying

$$Df = \beta \cdot f. \tag{2.2}$$

Moreover, if f is weighted homogeneous we have the Euler relation

$$\sum_{i=1}^{n+1} r_i x_i \frac{\partial f}{\partial x_i} = f \qquad \text{for some} \quad r_i \in \mathbb{Q}.$$

This combined with (2.2) gives

$$\sum_{i=1}^{n+1} (\alpha_i - r_i x_i \beta) \frac{\partial f}{\partial x_i} = 0 \tag{2.3}$$

Since $\{\frac{\partial f}{\partial x_i}\}_{i=1,n+1}$ is a regular sequence in A it follows that (2.3) is an A-linear combination of trivial relations (see [11], pg. 135)

$$\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_i} = 0$$

Hence $j^1\alpha_i = r_i x_i \beta(0)$ since $e = \operatorname{ord} f \geqslant 3$. □

We shall say that X is a weighted homogeneous singularity if it can be defined (in some coordinate system) by a weighted homogeneous polynomial.

If this is not the case, then we learn from [12] that the Lie algebra D_X is nilpotent.

Note that there is an obvious action ρ_X of D_X on the vector space $V_X = m_X/m_X^2$ and that the nilpotency condition implies that

$$\dim D_X \cdot v \leqslant \dim V_X - 1 = n$$

for any vector $v \in V_X$.

DEFINITION 2.4: We say that the action ρ_X of the nilpotent Lie algebra D_X on V_X is nearly transitive if there is a vector $v \in V_X$ such that

$$\dim D_X \cdot v = \dim V_X - 1.$$

Now we come back to the problem in hand. Let $X\colon g = 0$ be an isolated hypersurface singularity. Using the Splitting Lemma we can assume that

$$g(x) = \bar{g}(x_1, \ldots, x_c) + x_{c+1}^2 + \ldots + x_{n+1}^2 \quad \text{with} \quad \operatorname{ord} \bar{g} \geqslant 3.$$

If $c \geqslant 2$ we denote by $\bar{X}$ the hypersurface singularity defined by $\bar{g} = 0$ in $\mathbb{C}^c$. Recall that c is called the corank of X.

With these notations we can state our results.

PROPOSITION 2.5: *There are $\mathscr{R}$-simple functions on X iff X is an A_k-singularity for some $k \geqslant 1$.*

There are $\mathscr{K}$-simple functions on X iff one of the following disjoint situations occurs:

i. X is an A_k-singularity for some $k \geqslant 1$.

ii. corank $X = 2$ *and $\bar{X}$ is a weighted homogeneous singularity such that $wt(x_1) \neq wt(x_2)$.*

iii. $\bar{X}$ is not weighted homogeneous and the action $\rho_{\bar{X}}$ is nearly transitive.

PROOF: For $k \gg 0$ consider the contact orbit $Y = K^k g$ and let Z denote $p_1^{-1}(Y)$. Then on X there is a $*$-simple function iff Z contains an open $G^k{}_*$-orbit.

Such an open orbit should correspond to a function $f \in m_X$ with a generic linear part. Let $F = (g, f)$ and note that the $G^k{}_*$-orbit of $j^k F$ is open in Z for any $k \gg 0$ iff

$$T_* F = p_1^{-1}(TKg) \quad \text{where}$$

$TKg = mJ(g) + (g)$ is the contact tangent space [9].

This is equivalent to $T_* F \supset me_2$ and such an inclusion depends on how many derivations D exists, satisfying (2.2) with f replaced by g and not all of α_i in m^2.

Assume that $j^1 f = a_1 x_1 + \ldots + a_{n+1} x_{n+1}$ for some $a_i \in \mathbb{C}^*$. If we take $\alpha_1 = 2x_j$ for some $j > c$ and $\alpha_j = -\dfrac{\partial g}{\partial x_1}$ we get

$$x_j \in T_* F + m^2 e_2.$$

In order to get linear forms in the first c coordinates we should consider a relation similar to (2.2), in which f is replaced by $\bar{g}$ and all the functions α_i, β depend only on x_j, where $i, j = 1, \ldots, c$. Then the derivation D induces a derivation in $A_{\bar{X}}$ and hence an element $\bar{D}$ and $D_{\bar{X}}$.

If $v = j^1 \bar{f} \in V_{\bar{X}}$ where $\bar{f} = f(x_1, \ldots, x_c, 0)$, then $\bar{D} \cdot v$ is precisely the new linear form we get from D in $T_* F + m^2 e_2$.

There are two cases to discuss.

Case 1. X is weighted homogeneous.
Then we can take $\bar{g}$ to be weighted homogeneous and hence dim $D_{\bar{X}} = 1$ by (2.1).

It follows that on X there are $\mathscr{R}$-simple functions iff $c \leqslant 1$.

When we pass to $\mathscr{K}$-simple functions, we have a new element in $T_{\mathscr{K}} F$, namely $f \cdot e_2$.

If $c = 2$ and dim $\langle v, D_{\bar{X}} v \rangle = 2$, where $\langle \ldots \rangle$ means the vector space spanned by $\ldots$, then on X there are $\mathscr{K}$-simple functions. This happens precisely when $wt(x_1) \neq wt(x_2)$, since $D_{\bar{X}}$ is generated by the Euler derivation.

For $c > 2$ there are no $\mathscr{K}$-simple functions on X.

Case 2. X is not weighted homogeneous.
Then $D_{\overline{X}}$ is a nilpotent Lie algebra and hence dim $D_{\overline{X}} \cdot v \leqslant c-1$ for any $v \in V_{\overline{X}}$.

It follows that there are no $\mathscr{R}$-simple functions on X.

There are $\mathscr{K}$-simple functions on X iff there is a vector $v \in V_{\overline{X}}$ such that dim $\langle v, D_{\overline{X}}v \rangle = c$ and this happens precisely when the action $\rho_{\overline{X}}$ is nearly transitive. □

It is natural to ask whether the case iii. above really occurs. The condition on $\rho_{\overline{X}}$ is very restrictive. For instance, using the above notations, we have the following result.

LEMMA 2.6: *If $c = 2, 3$ and $\rho_{\overline{X}}$ is a (nilpotent), nearly transitive action, then in suitable linear coordinates $G = j^e\bar{g}$ is independent of x_c, where $e =$* ord $\bar{g}$.

PROOF: For $c = 2$, using suitable coordinates $D_{\overline{X}}$ is spanned by $D = x_1\partial/\partial x_2$ and $DG = 0$ implies G is independent of x_2. If $c = 3$, we can assume that $D_{\overline{X}}$ is spanned by some derivations $D = ax_1(\partial/\partial x_2) + (bx_1 + cx_2)\partial/\partial x_3$. We can write the homogeneous polynomial G in the form

$$G = \sum_{i=0}^{e} a_i(x_1, x_2)x_3^{e-i}$$

and let a_j be the first nonzero coefficient.

Then $DG = 0$ implies the following. Either $a = 0$ and then $D = 0$ or $a \neq 0$ and

$$ax_1\frac{\partial a_{j+1}}{\partial x_2} + (bx_1 + cx_2)(e-j)a_j = 0$$

For $j \neq e$ this relation determines the triple (a, b, c) up to a multiplicative constant and hence dim $D_{\overline{X}} \leqslant 1$, which is a contradiction. It follows $j = e$. □

EXAMPLES 2.7: When $c = 2$ the first examples of singularities $\overline{X}$: $\bar{g} = 0$ as in (2.5.iii) are provided by three of Arnold's exceptional unimodal singularities [1]

$$E_{12}: x^3 + axy^5 + y^7, \quad E_{13}: x^3 + xy^5 + ay^8$$

and $E_{14}: x^3 + axy^6 + y^8$ for $a \neq 0$.

Explicit computations show that $\bar{g}$ can be neither W_{12} nor W_{13}, in spite of the fact that these two exceptional singularities satisfy the conclusion of (2.6).

When $c = 3$ it is easy to show that $j^3\bar{g}$ should be 0, x_1^3 or $x_1^2x_2$ and then the results in Wall's paper on $\mathscr{K}$-unimodals [14] prove that there is no such g with $\mathscr{K}$-modality $\leqslant 1$.

REMARK 2.8: It follows from the proof of (2.5) and the wellknown Mather's Lemma ([10], (3.1)) that on an isolated hypersurface singularity X there are $*$-simple functions iff there is an open set $U \subset V_X$ such that all the functions $f \in m_X$ with $j^1 f \in U$ are $*$-equivalent.

It is natural to call such functions (local) generic projections.

§3. Some classification lists

In this section we derive the list of all $\mathscr{R}$-simple functions and of $\mathscr{K}$-simple functions defined on an A_k-plane curve singularity.

The normal forms given in the following tables are obtained using the method of complete transversals in some jet space $J^k(n+1, 2)$ (for details see [3] where the similar case of contact classification is treated).

PROPOSITION 3.1: *The classification of $\mathscr{R}$-simple functions f on an A_k-hypersurface singularity X with* dim $X > 1$ *is given by the following table.*

TABLE 1

Type of X	Normal form for f		$d_{\mathscr{R}}(f)$	$\mu(f)$
$A_1: x_1x_2 + q$	$C_{p+1}: x_1 + x_2^p$	$p \geqslant 1$	p	p
$A_2: x_1^3 + q$	$B_3: x_1$		1	1
	$F_4: x_2$		1	2
$A_k: x_1^{k+1} + q\ (k \geqslant 3)$	$B_{k+1}: x_1$		1	1

Here q is a nondegenerate quadratic form in the rest of variables, $d_{\mathscr{R}}(f)$ is the order of $\mathscr{R}$-determinacy and $\mu(f)$ is the Milnor number of f to be explained in the next section.

The symbols B, C, F are used because of the formal (for the moment!) resemblance of the above normal forms with the normal forms of simple functions defined on manifolds with boundary in the sense of Arnold (see [2], par. 1).

PROOF: The case A_1 being completely similar to the computations in [3], we give some details only when $X = A_k$ for $k \geqslant 2$. Let $g = 0$ be an equation for X and suppose first that $j^1 f \neq 0$.

If $F = (g, f)$, using the action of $G^2_{\mathscr{R}}$ we can put j^2F in one of the following three forms:

$$\text{i. } (x_2^2 + q, x_1); \quad \text{ii. } (x_2^2 + q, x_2); \quad \text{iii. } (x_2x_3 + q, x_2)$$

The case i corresponds to generic projections and we get the series B_p, $p \geqslant 3$.

In case ii using a complete transversal we obtain

$$F = \left(x_1^{k+1} + x_2^2 + q,\ x_2 + x_1^p a(x_1)\right)$$

for some $p \geqslant 2$. When $k = 2$ it follows that we can omit the last term and we get F_4.

For $k > 2$ it is easy to see that the family

$$F_\lambda = \left(x_1^{k+1} + x_2^2 + q,\ x_2 + \lambda x_1^2\right)$$

produces an infinity of orbits, since $x_1^2 e_2 \notin T_{\mathscr{R}} F_\lambda$ for any λ. Hence in this case we do not obtain simple functions.

In the same way in case iii we do not get simple germs at all.

Now we shall prove that if $j^1 f = 0$ then f cannot be simple. Indeed, for any such f let $r = j^2 f$ be the corresponding quadratic form and assume $g = x_1^{k+1} + q$ $(k \geqslant 1)$.

It is clear that $E = j^2(T_{\mathscr{R}} F \cap m e_2)$ is spanned by the vectors $j^2\Delta_{ij}$ (see the proof of (1.4)), $2x_1 \dfrac{\partial r}{\partial x_1} + (k+1) \sum_{i=2}^{n+1} x_i \dfrac{\partial r}{\partial x_i}$ and $j^2 g$. Hence $\dim E \leqslant \binom{n+1}{2} + 2$ where $n = \dim X$.

The space Q of all quadratic forms in $(n+1)$-variables has dimension $\binom{n+2}{2}$ and hence $\dim Q > \dim E$ for $n > 1$. This means that for any f as above we can find a quadratic form r_0 such that the family $f_t = f + t r_0$ intersects infinitely many orbits.

Indeed, it is enough that $r_0 e_2 \notin j^2(T_{\mathscr{R}} F_t \cap m e_2)$ for any t, where $F_t = (g, f_t)$. And this is possible by the above argument. □

Next we treat the case $\dim X = 1$.

PROPOSITION 3.2: *The classification of $\mathscr{R}$-simple functions f on an A_k-curve singularity X is given by the following table. Moreover, any finitely determined function f on the node (A_1) or on the cusp (A_2) is $\mathscr{R}$-simple.*

TABLE 2

Type of X	Normal form for f		$d_{\mathscr{R}}(f)$	$\mu(f)$
$A_1: xy = 0$	$x^p + y^q$	$p \geqslant q \geqslant 1$	p	$p+q-1$
$A_2: x^3 + y^2 = 0$	x^p	$p \geqslant 1$	p	$2p-1$
	$x^p y$	$p \geqslant 0$	$p+1$	$2p+2$
$A_k: x^{k+1} + y^2 = 0$, $k \geqslant 3$	x		1	1

The proof of this result is similar (and simpler) to the proof of (3.1) and we omit the details.

On a given hypersurface X there are, in general, much more $\mathcal{K}$-simple functions than $\mathcal{R}$-simple ones. To illustrate this fact, we give the following result.

PROPOSITION 3.3: *The classification of $\mathcal{K}$-simple functions f on an A_k-curve singularity X: $x^{k+1}+y^2=0$ is given by the following table.*

TABLE 3

Type of X	Normal form for f		$d_{\mathcal{K}}(f)$	$\mu(f)$
A_k, $k>2$ odd	x^p+y	$0<2p<k$	p	$2p-1$
	x^py+x^q	$0<p<q$, $0<2q<k$	q	$2q-1$
A_k, $k\geqslant 2$ even	x^p	$p\geqslant 1$	p	$2p-1$
	x^py	$p\geqslant 0$	$p+1$	$2p+k$
	x^p+y	$1<p<k$	p	$\min(k,2p-1)$
	x^py+x^q	$p+1<q<p+k$	q	$2p+k$

Here $d_{\mathcal{K}}(f)$ is the order of $\mathcal{K}$-determinacy of f. Moreover any finitely determined function f defined on X is $\mathcal{K}$-simple iff X is irreducible or X is a node A_1. (Note that the case $X=A_1$ follows from (3.2)). The proof of this result is very similar to the proofs in [3] par. 2 and hence we give no more details here.

§4. Topological classification

We start with the following

DEFINITION 4.1: Let X be an isolated hypersurface singularity and $f, g\in m_X$. We say that f and g are $\mathcal{R}$-top equivalent if there is a germ of homeomorphism h making the following diagram commutative.

$$\begin{tikzcd} (X,0) \arrow[rr, "h"] \arrow[dr, "f"'] & & (X,0) \arrow[dl, "g"] \\ & (\mathbb{C},0) & \end{tikzcd}$$

It is easy to detect the $\mathcal{R}$-top equivalence class of a function $f\in m_X$ when $\dim X=1$.

Indeed, let $X_1, \ldots, X_n$ be the irreducible components of X and consider the following intersection multiplicities

$$m_i(f)=I(X_i,\quad \{f=0\})\leqslant\infty.$$

Then we have the following result.

PROPOSITION 4.2: *When* dim $X = 1$, *two functions* f, $g \in m_X$ *are* $\mathcal{R}$*-top equivalent iff up to a permutation*

$$m_i(f) = m_i(g) \qquad \text{for} \quad i = 1, \ldots, n.$$

PROOF: Note that $m_i(f) = \infty$ iff $f | X_i = 0$. Next, if $m_i(f) = p$ we have a commutative diagram of punctured germs:

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "a"] \arrow[dr, "b"'] & X_i \setminus 0 \arrow[d, "f"] \\ & \mathbb{C}^* \end{tikzcd}$$

where a is a homeomorphism induced by a suitable Puiseux parametrization of X_i and $b(x) = x^p$. □

COROLLARY 4.3: *When* dim $X = 1$

$$\overset{\mathcal{K}}{\sim} \Rightarrow \overset{\mathcal{R}\text{-top}}{\sim}.$$

PROOF: Suppose that f, $g \in m_X$ and u is an automorphism of A_X such that $(u(f)) = (g)$. Note that u lifts to an automorphism $\tilde{u}$ of A and if $f_i = 0$ is an equation for the branch X_i of X, then $\tilde{u}(f_i) = 0$ is also an equation for a branch (say X_j) of X.

We have the following obvious equalities

$$m_i(f) = \dim \frac{A}{(f_i, f)} = \dim \frac{A}{(\tilde{u}(f_i), g)} = m_j(g).$$

□

The situation is much more complicated when dim $X > 1$. Let $f \in m_X$ be a function with an isolated singular point at $0 \in X$.

For $\epsilon > 0$ small and $\delta > 0$ sufficiently small with respect to ϵ we consider the following spaces

$$\begin{aligned} X' &= \{x \in X; \|x\| < \epsilon, \qquad 0 < |f(x)| \leqslant \delta\} \\ X_0 &= \{x \in X; f(x) = 0\}, \quad D' = \{t \in \mathbb{C}; 0 < |t| \leqslant \delta\}. \end{aligned}$$

Then the restriction of f induces a locally trivial smooth fibration $f' : X' \to D'$. Its fibres are smooth paralelizable manifolds which are homotopy equivalent to a bouquet of $(n-1)$-spheres where $n = \dim X$ [7]. The number of spheres in this bouquet is called the Milnor number $\mu(f)$ of the function f.

LEMMA 4.4: *The Milnor number $\mu(f)$ is a $\mathcal{K}$ and $\mathcal{R}$-top invariant.*

PROOF: It follows from the work of Greuel that $\mu(f)$ depends only on the complete intersection X_0 with an isolated singularity at the origin [6]. This is enough for showing $\mathcal{K}$-invariance.

The proof of $\mathcal{R}$-top invariance is the same as the corresponding proof in the case of Milnor numbers of isolated hypersurface singularities in $\mathbb{C}^{n+1}$ and we send for details to [8]. □

Using the $\mathcal{K}$-invariance, we have computed the values of μ(f) in Table 1 without difficulty. Indeed, in each of these cases X_0 is an A_k-hypersurface singularity and it is wellknown that $\mu(A_k) = k$.

As a consequence of this computation and of (4.2) we obtain the following result.

COROLLARY 4.5: *The $\mathcal{R}$-top classification of $\mathcal{R}$-simple functions defined on an isolated hypersurface singularity coincides with the $\mathcal{R}$-classification.*

It is perhaps interesting to note that the similar result about the $\mathcal{R}$-top classification of $\mathcal{K}$-simple functions defined on an isolated hypersurface singularity X is false in general even when dim $X = 1$ (use (3.3), (4.2) and the following Remark).

REMARK 4.6: When dim $X = 1$ one clearly has $\mu(f) = \dim Q(F) - 1$, where $Q(F) = A/(f, g)$. If the components f, g of F are weighted homogeneous of degrees d_1, d_2 with respect to the weights w_1, w_2, then dim $Q(F) = d_1 d_2 / w_1 w_2$.

This equality holds also for quasiweighted homogeneous map germs F, i.e. $F = F_0 + F_1$ where F_0 is a nondegenerate weighted homogeneous germ and F_1 contains only terms of higher orders.

This fact gives us the values for $\mu(f)$ in Table 2 and Table 3.

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Department of Mathematics
National Institute for Scientific and Technical Creation
INCREST, Bd. Păcii 220
79622 Bucharest,
Romania

Note added in proof

Most of the results in this paper can be extended to the case of function germs defined on isolated singularities of complete intersections. An application of such an extension can be found in our paper "Are the isolated singularities of complete intersections determined by their singular subspaces?" [*Math. Ann.* 267 (1984) 461–472].