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F. TRICERRI
L. VANHECKE

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NATURALLY REDUCTIVE HOMOGENEOUS SPACES AND GENERALIZED HEISENBERG GROUPS

F. Tricerri and L. Vanhecke

1. Introduction

Among the *homogeneous Riemannian manifolds* the *naturally reductive homogeneous* spaces are the simplest kind and because of this they have been studied extensively (see [10], [6]). Classical examples are the symmetric spaces and more general all the isotropy irreducible homogeneous spaces studied by J. Wolf [17]. See also [6]. Also some nice examples are constructed in the theory of 3-symmetric spaces [7], [11]. But in general it is mostly difficult to see whether a given homogeneous metric g on a Riemannian manifold M is naturally reductive or not because one has to look at *all* the groups of isometries of (M, g) acting transitively on M. Hence it could be useful to have an infinitesimal characterization of such manifolds which at least in some cases makes it possible to come more quickly to a conclusion.

The first purposes of this paper is to provide such a characterization and then to illustrate this by treating some remarkable examples. The main tool to derive our result is a theorem of Ambrose and Slinger [1]. In what follows we always suppose the Riemannian manifolds to be connected, simply-connected and complete. In that case Ambrose and Singer characterize homogeneous Riemannian manifolds by a local condition which has to be satisfied at all points. They use a tensor field T of type (1,2) and derive two necessary and sufficient conditions involving T, the Riemann curvature tensor and their covariant derivatives (see section 2 for more details). As such their theorem is a generalization of Cartan's theorem for symmetric spaces. We shall prove that a Riemannian manifold is a naturally reductive homogeneous space if and only if there exists a tensor field T satisfying the two conditions of Ambrose and Singer and in addition the condition $T_X X = 0$ for all tangent vector fields X. Note that this last condition appears very naturally in our attempt to give a kind of classification for homogeneous spaces based on the Ambrose-Singer theorem [14].

This key result will be used in the study of some nilpotent Lie groups. More specifically, in the second part of this paper we will treat briefly the so-called *generalized Heisenberg groups* or *groups of type H*, studied in [8], [9]. Using the result above we shall give an alternative proof of Kaplan's result: the only naturally reductive groups of type H are the Heisenberg groups and their quaternionic analogs.

The explicit determination of the geodesics and the Killing vector fields gives rise to a remarkable example already discovered by A. Kaplan [9]. It is well-known that the geodesics of a naturally reductive homogeneous space are orbits of one-parameter subgroups of isometries. The 6-dimensional example of Kaplan provides a counterexample for the converse theorem. Hence, by treating this example, we show that there exist connected, simply connected homogeneous manifolds which do not admit a tensor field T satisfying the two conditions of Ambrose and Singer and the additional condition $T_X X = 0$ but all of whose geodesics are still orbits of one-parameter subgroups of isometries. This implies that Theorem 5.4 of [1] has to be modified and we do this in Corollary 1.4 of section 2. Moreover this example shows that the differentiability condition in the paper of Szenthe [13] cannot be removed.

Our study of the naturally reductive spaces and the groups of type H arose during our study of harmonic, commutative and D'Atri spaces (see [15] for a survey). D'Atri spaces are spaces such that all the geodesic symmetries are volume-preserving local diffeomorphisms. It is not difficult to prove that all naturally reductive homogeneous spaces are D'Atri spaces. This is done in [5] but it can also be proved using Jacobi vector fields. A natural question is whether this property characterizes naturally reductive homogeneous spaces. It is proved in [9] that all the groups of type H have this property and hence they provide again a lot of counterexamples. But there is much more. The naturally reductive spaces have the stronger property that *all* the eigenvalues of the matrix of the metric tensor with respect to a normal coordinate system have the antipodal symmetry. (A D'Atri space is characterized by the fact that the product of the eigenvalues has antipodal symmetry.) We shall show that the 6-dimensional group of type H has also this property and so we provide an example of a manifold with this property but which is not naturally reductive.

Finally we show that the 6-dimensional example is a 3- and 4-symmetric space.

2. Naturally reductive homogeneous spaces

Let (M, g) be a connected n-dimensional homogeneous manifold. Further let G be a Lie group acting transitively and effectively on the left on M as a group of isometries and denote by K the isotropy subgroup at some point p of M. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K. Suppose $\mathfrak{m}$ is a vector space complement to $\mathfrak{k}$ in $\mathfrak{g}$ such that $Ad(K)\mathfrak{m} \subseteq \mathfrak{m}$. Then we may identify $\mathfrak{m}$ with T_pM by the map $X \mapsto X_p^*$, where X^*

denotes the Killing vector field on (M, g) generated by the one-parameter subgroup $\{\exp tX\}$ acting on M. We denote by $\langle , \rangle$ the inner product on $\mathfrak{m}$ induced by the metric g.

DEFINITION 2.1.: The manifold (M, g) (or the metric g) is said to be *naturally reductive* if there exists a Lie group G and a subspace $\mathfrak{m}$ with the properties described above and such that

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0, \quad X, Y, Z \in \mathfrak{m}, \tag{2.1}$$

where $[X, Y]_{\mathfrak{m}}$ denotes the projection of $[X, Y]$ on $\mathfrak{m}$.

Geometrically these manifolds may be defined by using the following theorem (see [10, II, chapter X]).

THEOREM 2.2: *The homogeneous manifold (M, g) is naturally reductive if and only if the geodesic through p and tangent to $X \in \mathfrak{m} \simeq T_pM$ is the curve* $(\exp tX)p$, *orbit of the one-parameter subgroup* $\exp tX$ *of G, for all X.*

It is clear that if we want to say that M is naturally reductive we first have to determine all transitive isometry groups of G of M and then to consider all the complements of k in g which are invariant under $Ad(K)$. In most of the cases this is not an easy task. Therefore we shall proof a theorem which makes it in a lot of examples much easier to reach a conclusion. We refer to [14] for a series of examples.

THEOREM 2.3: *Let (M, g) be a connected, simply connected and complete Riemannian manifold. Then (M, g) is a naturally reductive homogeneous space if and only if there exists a tensor field T of type* (1,2) *such that*

$$(\mathrm{AS})\begin{cases} (\mathrm{i}) & g(T_XY, Z) + (Y, T_XZ) = 0, \\ (\mathrm{ii}) & (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_XYZ} - R_{YT_XZ}, \\ (\mathrm{iii}) & (\nabla_X T)_Y = [T_X, T_Y] - T_{T_XY} \end{cases}$$

and

$$T_XY + T_YX = 0 \tag{2.2}$$

where $X, Y, Z \in \mathscr{X}(M)$. ∇ denotes the Levi Civita connection and R is the Riemann curvature tensor.

The conditions (AS) are the Ambrose-Singer conditions and the existence of a tensor T satisfying these conditions is equivalent to the homogeneity of the manifold. Note that with $\tilde{\nabla} = \nabla - T$, (ii) and (iii) are equivalent to $\tilde{\nabla}R = \tilde{\nabla}T = 0$ and (i) means that $\tilde{\nabla}$ is a metric connection. (See [1].) Hence we shall only prove that (2.2) is equivalent to the naturally reductive property.

To do this we need to recall briefly the construction of a transitive and effective group G of isometries acting on M when a tensor T is given. Thereby we concentrate on some facts concerning the Lie algebra $\mathfrak{g}$ of G. The proofs are in [1]. See also [14].

Let $\mathcal{O}(m) \xrightarrow{\pi} M$ be the principal bundle of orthonormal frames of M. (AS(i)) implies that the linear connection $\tilde{\nabla} = \nabla - T$ is metric and hence induces an infinitesimal connection on $\mathcal{O}(M)$. Let $u = (p, u_1, \ldots, u_n)$ be a point of $\mathcal{O}(M)$ and denote by $\tilde{\mathscr{T}}(u)$ the holonomy bundle of $\tilde{\nabla}$ through u. $\tilde{\mathscr{T}}(u)$ is a principal subbundle of $\mathcal{O}(M)$ whose structure group is the holonomy group $\tilde{\psi}(u)$ of $\tilde{\nabla}$. This is a subgroup of $\mathrm{O}(n)$. Ambrose and Singer proved that when T satisfies the conditions (AS), then $G = \tilde{\mathscr{T}}(u)$ has a Lie group structure and acts transitively and effectively on M on the left as a group of isometries.

In what follows we use the same notation as in [10]. A^* denotes the fundamental vertical vector field which corresponds to an element A of the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{O}(n)$. Further $B(\xi)$ denotes the standard horizontal vector field with respect to $\tilde{\nabla}$ and corresponding to the vector ξ of $\mathbb{R}^n$. Then the Lie algebra $\mathfrak{g}$ of G is the subalgebra of $\mathscr{X}(\tilde{\mathscr{T}}(u))$ generated by the restrictions of the vector fields $A_1^*, \ldots, A_r^*$, $B_1, \ldots, B_n$ to $\tilde{\mathscr{T}}(u)$. Here $(A_1, \ldots, A_r)$ is a basis of the Lie algebra of $\tilde{\psi}(u)$ and $B_1, \ldots, B_n$ are the horizontal vector fields corresponding to a natural basis of $\mathbb{R}^n$ (see [10, II, p. 137]). The isotropy subgroup K of the point $p = \pi(u)$ is the connected subgroup of G whose Lie algebra $\mathfrak{k}$ is generated by $A_1^*, \ldots, A_n^*$. Note that $\mathfrak{k}$ is isomorphic to the holonomy algebra of $\tilde{\nabla}$.

Next we write down explicitly the Lie brackets for the basis elements of $\mathfrak{g}$. Therefore we have only to note that (see [10, I, p. 120])

$$[A^*, A'^*] = [A, A']^*, \quad A, A' \in \mathfrak{so}(n), \tag{2.3}$$

$$[A^*, B(\xi)] = B(A(\xi)), \quad A \in \mathfrak{so}(n), \quad \xi \in \mathbb{R}^n. \tag{2.4}$$

Further, for $v = (q, v_1, \ldots, v_n) \in \tilde{\mathscr{T}}(u)$ we have (see [10, I, p. 137]):

$$[B(\xi_1), B(\xi_2)]_v = -B\Big(v^{-1}\Big(\tilde{S}_{v(\xi_1)}v(\xi_2)\Big)\Big) + \Big(v^{-1} \circ \Big(\tilde{R}_{v(\xi_1)v(\xi_2)}\Big) \circ v\Big)^*, \quad \xi_1\xi_2 \in \mathbb{R}^n, \tag{2.5}$$

where v is the isometry

$$v : \mathbb{R}^n \to T_qM : (\xi^1, \ldots, \xi^n) \mapsto \sum_{i=1}^n \xi^i v_i \tag{2.6}$$

and $\mathbb{R}^n$ is equipped with the standard metric. Here $\tilde{R}$ denotes the curvature tensor of $\tilde{\nabla}$ given by

$$\tilde{R}_{XY} = \tilde{\nabla}_{[X,Y]} - [\tilde{\nabla}_X, \tilde{\nabla}_Y], \quad X, Y \in \mathscr{X}(M), \tag{2.7}$$

and the tensor $\tilde{S}$ given by

$$\tilde{S}_X Y = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = T_Y X - T_X Y, \quad X, Y \in \mathscr{X}(M), \tag{2.8}$$

is the torsion tensor of $\tilde{\nabla}$.

Now let $\mathfrak{m}$ denote the vector subspace of $\mathfrak{g}$ generated by $B_1, \ldots, B_n$. Then it follows from (2.4) and from the fact that K is connected that

$$Ad(K)\mathfrak{m} \subseteq \mathfrak{m}. \tag{2.9}$$

This means that the homogeneous Riemannian space is *reductive*.

Finally we note that $(B_1, \ldots, B_n)$ is an orthonormal basis of $\mathfrak{m}$ with respect to the inner product induced from g. Hence the map

$$B \circ v^{-1} : T_q M \to \mathfrak{m} : X \mapsto B(v^{-1}(x)) \tag{2.10}$$

is an isometry for all $v = (q, v_1, \ldots, v_n)$ of $\mathscr{O}(M)$ (see [1]).

Proof of Theorem 2.1: First suppose that the homogeneous space (M, g) is naturally reductive and let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$ be a naturally reductive decomposition of the Lie algebra $\mathfrak{g}$ of the transitive group of isometries G. Let $\tilde{\nabla}$ be the corresponding canonical connection and put $T = \nabla - \tilde{\nabla}$. Then $T_X Y + T_Y X = 0$ and T satisfies the conditions (AS) since $\tilde{R}$ and T are parallel with respect to $\tilde{\nabla}$ (see [10, II, chapter X]).

Conversely, let T be a (1,2)-tensor field which satisfies the conditions of the theorem. Put again $\tilde{\nabla} = \nabla - T$. Then the theorem of Ambrose and Singer implies that $G = \tilde{\mathscr{T}}(u)$ is a transitive group of isometries of M acting effectively on M. Further, as we have already noted, there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}$. An element of $\mathfrak{m}$ is of the form $B(\xi)$, $\xi \in \mathbb{R}^n$ and

$$\begin{aligned}([B(\xi_1), B(\xi_2)]_{\mathfrak{m}})|_v &= -B\Big(v^{-1}\big(\tilde{S}_{v(\xi_1)} v(\xi_2)\big)\Big)\\ &= 2(B \circ v^{-1})\big(T_{v(\xi_1)} v(\xi_2)\big),\end{aligned}$$

where T is evaluated at $q = \pi(v)$. Hence we have

$$\langle [B(\xi_1), B(\xi_2)]_{\mathfrak{m}}, B(\xi_3)] \rangle + \langle B(\xi_2), [B(\xi_1), B(\xi_3)]_{\mathfrak{m}} \rangle$$
$$= 2g_q\Big(T_{v(\xi_1)}v(\xi_2), v(\xi_3)\Big) + 2g_q\Big(v(\xi_2), T_{v(\xi_1)}v(\xi 3)\Big).$$

Since T_X is skew-symmetric for all X we obtain (2.1) and hence M is naturally reductive.

From Theorem 2.2 and from (2.3) we obtain

COROLLARY 2.4: *Let (M, g) be a connected, simply connected homogeneous manifold. Then there exists a tensor field T of type* (1,2) *which satisfies the conditions* (AS) *and such that $T_X Y + T_Y X = 0$ for all $X \in \mathfrak{X}(M)$, if and only if the geodesic tangent to $X \in \mathfrak{m} \cong T_p M$ at p is the curve $(\exp tX)p$ for all X.*

This is Theorem 5.4 of [1] with a slight modification. This is necessary because of the 6-dimensional example of Kaplan which will be discussed in Section 4. Further we note that in their Theorem 5.4 Ambrose and Singer prove that when T satisfies (AS) and (2.2), then the geodesics are orbits of one-parameter subgroups of $G = \tilde{\mathscr{P}}(u)$ with infinitesimal generators in $\mathfrak{m}$. Taking into account Theorem 2.2, this provides another proof of Theorem 2.3.

3. Lie groups of type H

In this section we give a brief survey on some general aspects of groups of type H. We refer to [8], [9] for more details. At the same time we concentrate on the naturally reductive case and we give a different proof of the main theorem using the theory of two-fold vector cross products.

First we start with the definition of such a group. Let V and Z be two real vector spaces of dimension n and m $(m \geqslant 1)$ both equipped with an inner product which we shall denote for both spaces with the same symbol $\langle , \rangle$. Further let $j\colon Z \to \mathrm{End}(V)$ be a linear map such that

$$|j(a)x| = |x|\,|a|, \qquad x \in V, \quad a \in Z, \tag{3.1}$$

$$j(a)^2 = -|a|^2 I, \qquad a \in Z. \tag{3.2}$$

Note that these conditions imply, using polarization:

$$\langle j(a)x, (b)x \rangle = \langle a, b \rangle |x|^2,$$

$$\langle j(a)x, j(a)y \rangle = |a|^2 \langle x, y \rangle,$$

for all $x, y \in V$ and $a, b \in Z$.

Next we define the Lie algebra $\mathfrak{n}$ as the direct sum of V and Z together with the bracket defined by

$$[a+x, b+y] = [x, y] \in Z, \tag{3.3}$$

$$\langle [x, y], a \rangle = \langle j(a)x, y \rangle, \tag{3.4}$$

where $a, b \in Z$ and $x, y \in V$. Then $\mathfrak{n}$ is said to be a *Lie algebra of type H*. It is a 2-step nilpotent Lie algebra with center Z.

The simply connected, connected Lie group N whose Lie algebra is $\mathfrak{n}$ is called a *Lie group of type H* or a *generalized Heisenberg group*. There are infinitely many groups of type H with center of any given dimension.

Note that the Lie algebra $\mathfrak{n}$ has an inner product such that V and Z are orthogonal:

$$\langle a+x, b+y \rangle = \langle a, b \rangle + \langle x, y \rangle,$$

$a, b \in Z$ and $x, y \in V$. Hence the Lie group has a left invariant metric induced by this metric on $\mathfrak{n}$.

In what follows some special Lie algebras of type H will play a fundamental role. These non-Abelian algebras can be obtained using composition algebras W (the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the Cayley numbers Cay) as follows: Let Z be the subspace of W formed by the purely imaginary elements. Further let $V = W^n$, $n \in \mathbb{N}_0$, and put $j: Z \to \mathrm{End}(V)$ for the linear map defined by

$$j(a)x = ax \tag{3.5}$$

where ax denotes the ordinary scalar multiplication of a and x. The corresponding groups are the *Heisenberg groups* or their *quaternionic* and *Cayley analogs*.

Now we look for the naturally reductive groups and we give an alternative proof of a result of Kaplan [9], using Theorem 2.3.

THEOREM 3.1: *The homogeneous manifold* $(N, \langle , \rangle)$ *is naturally reductive if and only if N is a Heisenberg group or a quaternionic analog.*

To prove this we first prove the following

LEMMA 3.2: *If* $(N, \langle , \rangle)$ *is naturally reductive, then* $\dim Z = 1$ *or* 3.

PROOF: It follows from Theorem 2.3 that there exists a tensor T of type (1,2) such that $T_X Y + T_Y X = 0$ and which satisfies the conditions (AS). Further let ρ denote the Ricci tensor of the manifold $(N, \langle , \rangle)$. Then we have from (AS(ii)):

$$(\nabla_X \rho)_{YZ} = -\rho_{T_X Y Z} - \rho_{Y T_X Z}. \tag{3.6}$$

The connection of Levi Civita has been computed in [8]. We have:

$$\begin{cases} \nabla_x y = \frac{1}{2}[x, y], \\ \nabla_a x = \nabla_x a = -\frac{1}{2}j(a)x, \\ \nabla_a b = 0, \end{cases} \tag{3.7}$$

where $x, y \in V$ and $a, b \in Z$. For the Ricci tensor one obtains (see [8]):

$$\begin{cases} \rho_{xy} = -\dfrac{m}{2}\langle x, y\rangle, \\ \rho_{ab} = \dfrac{n}{4}\langle a, b\rangle, \\ \rho_{xa} = 0. \end{cases} \tag{3.8}$$

Then it follows easily from (3.7) and (3.8) that all the components of $\nabla\rho$ vanish except

$$(\nabla_x \rho)_{bz} = -\frac{n+2m}{8}\langle j(b)x, z\rangle. \tag{3.9}$$

Hence (3.6) will be satisfied if and only if

$$\langle T_x z, b\rangle = \tfrac{1}{2}\langle j(b)x, z\rangle,$$

$$\langle T_a z, b\rangle = 0. \tag{3.11}$$

Since T_w is skew-symmetric for all $w \in \mathfrak{n}$ we must have

$$T_a b \in Z, \tag{3.12}$$

$$T_x b = -T_b x = -\tfrac{1}{2}j(b)x. \tag{3.13}$$

Next we put $\tilde{\nabla} = \nabla - T$. Then it follows from (3.7) and (3.13) that

$$\begin{cases} \tilde{\nabla}_a x = -j(a)x, \\ \tilde{\nabla}_a b = -T_a b. \end{cases} \tag{3.14}$$

So we obtain

$$\langle(\tilde{\nabla}_a T)_{yz}, b\rangle$$
$$= \tfrac{1}{2}\{\langle j(b)j(a)y, z\rangle + \langle j(b)y, j(a)z\rangle + \langle j(T_a b)y, z\rangle\}.$$

Since $j(a)$ is skew-symmetric for all $a \in Z$, $\tilde{\nabla}T = 0$ implies

$$j(T_a b) = j(a)j(b) - j(b)j(a). \tag{3.15}$$

Further polarization of (3.2) implies

$$j(a)j(b)+j(b)j(a) = -2\langle a, b\rangle I \tag{3.16}$$

and hence (3.15) becomes

$$j(T_a b) = 2\{ j(a)j(b) + \langle a, b\rangle I \}. \tag{3.17}$$

Finally we have

$$j(T_a b)^2 = -|T_a b|^2 I$$

and from this, using (3.16) and (3.17), we conclude that

$$|T_a b|^2 = 4\big(|a|^2|b|^2 - \langle a, b\rangle^2\big). \tag{3.18}$$

But we also have

$$\langle T_a b, a\rangle = \langle T_a b, b\rangle = 0. \tag{3.19}$$

Hence, with

$$\tau_a b = \tfrac{1}{2} T_a b, \qquad a, b \in Z,$$

we can conclude from (3.18) and (3.19) that τ is a two-fold vector cross product on Z (see [2]). Hence we must have dim $Z = 1$ when $T_a b = 0$ or otherwise dim $Z = 3$ or 7.

It remains to prove that dim $Z = 7$ is not possible. To show this we first recall that with $W = \mathbb{R} \oplus Z$ and the multiplication

$$\begin{cases} 1a = a1 = a, & 1 \in \mathbb{R}, \\ ab = \tau_a b - \langle a, b\rangle 1, & a, b \in Z, \end{cases}$$

we obtain an 8-dimensional composition algebra. The inner product on Z can be extended to W by putting

$$|1| = 1$$

and taking Z to be orthogonal to $\mathbb{R}$. Then (3.17) implies that W is associative. Indeed, let $\tilde{j}: W \to \mathrm{End}(V)$ be the linear map defined by

$$\tilde{j}(1) = I, \qquad \tilde{j}(a) = j(a) \qquad \text{for} \quad a \in Z. \tag{3.20}$$

It is clear that $\tilde{j}$ is injective. Moreover $\tilde{j}(ab) = \tilde{j}(a)\tilde{j}(b)$ since

$$\begin{aligned} \tilde{j}(ab) = \tilde{j}(\tau_a b) - \langle a, b\rangle I &= \tfrac{1}{2}\tilde{j}(T_a b) - \langle a, b\rangle I \\ &= j(a)j(b) \end{aligned}$$

as follows from (3.17) and (3.20). Hence $\tilde{j}$ is a monomorphism between the algebras W and $\mathrm{End}(V)$ and so W is associative. This excludes the case $\dim Z = 7$ since any 8-dimensional composition algebra is not associative. Hence the lemma is proved.

PROOF OF THEOREM 3.1: Using the classification of Clifford modules, Kaplan proved in [9] that for $\dim Z = 1$ the corresponding groups N are the Heisenberg groups and for $\dim Z = 3$ the groups N are the quaternionic analogs.

To finish the proof we have to show that in these cases there exists a tensor T satisfying the required conditions. Therefore, let T be defined as follows:

$$\begin{cases} T_x y = -T_y x = \frac{1}{2}[x, y], \\ T_x a = -T_a x = -\frac{1}{2} j(a) x = -\frac{1}{2} a x, \\ T_a b = 2\{ ab + \langle a, b \rangle 1 \}. \end{cases} \tag{3.21}$$

It is easy to verify that T satisfies the conditions (AS) or equivalently $\tilde{\nabla} R = \tilde{\nabla} T = 0$. The explicit expression for R is given in [8] and the properties of the composition algebras are given in [2].

4. Geodesics and killing vector fields on groups of type H

The main purpose of this section is to give an answer to the question: When are the geodesics on a group $(N, \langle,\rangle)$ of type H orbits of one-parameter subgroups of isometries of $(N, \langle,\rangle)$? To do this we will not use the description of the full group of isometries of $(N, \langle,\rangle)$ given in [8], but we will consider the Killing vector fields.

First we determine a global coordinate system $(v_1, \dots, v_n;\ u_1, \dots, u_m)$ on N. To do this, let $(x_1, \dots, x_n)$ and $(a_1, \dots, a_m)$ be orthonormal frames on V and Z. Then we put for $p \in N$:

$$\begin{cases} v_i(p) = v_i(\exp(x(p) + a(p))) = \langle x(p), x_i \rangle, & i = 1, \dots, n, \\ u_\alpha(p) = u_\alpha(\exp(x(p) + a(p))) = \langle a(p), a_\alpha \rangle, & \alpha = 1, \dots, m. \end{cases} \tag{4.1}$$

Then we have

$$\begin{cases} \dfrac{\partial}{\partial v_i} = x_i - \dfrac{1}{2} \sum\limits_{\alpha, j} A^\alpha_{ji} v_j a_\alpha, \\ \dfrac{\partial}{\partial u_\alpha} = a_\alpha, \end{cases} \tag{4.2}$$

where the A_{ji}^{α} are the structure constants of $\mathfrak{n}$, i.e.

$$[x_i, x_j] = \sum_{\alpha} A_{ij}^{\alpha} a_{\alpha}. \tag{4.3}$$

Next let A, respectively B, be a skew-symmetric endomorphism of V, respectively Z, such that

$$Aj(a) - j(a)A = j(B(a)), \quad a \in Z, \tag{4.4}$$

and put

$$A(x_i) = \sum_j a_{ji} x_j, \quad B(a_{\alpha}) = \sum_{\beta} b_{\beta\alpha} a_{\beta}.$$

Then we have

THEOREM 4.1: *The Killing vector fields ξ of $(N, \langle,\rangle)$ are given by*

$$\xi = \sum_i \xi_i \frac{\partial}{\partial v_i} + \sum_{\alpha} \xi_{\alpha} \frac{\partial}{\partial u_{\alpha}}$$

where

$$\xi_i = \sum_j a_{ij} v_j + \lambda_i, \quad \lambda_i = \text{const.}, \tag{4.5}$$

$$\xi_{\alpha} = \sum_{\beta} b_{\alpha\beta} u_{\beta} + \frac{1}{2} \sum_{i,j} A_{ij}^{\alpha} v_i \lambda_j + \mu_{\alpha}, \quad \mu_{\alpha} = \text{const.} \tag{4.6}$$

PROOF: The Killing equations can be written as follows:

$$\begin{cases} g(\nabla_{x_i}\xi, x_j) + g(\nabla_{x_j}\xi, x_i) = 0, \\ g(\nabla_{a_{\alpha}}\xi, a_{\beta}) + g(\nabla_{a_{\beta}}\xi, a_{\alpha}) = 0, \\ g(\nabla_{a_{\alpha}}\xi, x_i) + g(\nabla_{x_i}\xi, a_{\alpha}) = 0. \end{cases} \tag{4.7}$$

Let ρ be the Ricci tensor of N. Then the Lie derivative $L_{\xi}\rho$ vanishes. More specifically we have

$$\rho([\xi, a_{\alpha}], x_i) + \rho([\xi, x_i], a_{\alpha}) = 0. \tag{4.8}$$

Using (3.7) and (3.8) we derive the following conditions which are equivalent to (4.7) and (4.8):

$$\begin{cases} x_i(\xi_j) + x_j(\xi_i) = 0, \\ a_\alpha(\tilde{\xi}_\beta) + a_\beta(\tilde{\xi}_\alpha) = 0, \\ a_\alpha(\xi_i) = 0, \\ x_i(\tilde{\xi}_\alpha) + \sum\limits_h \xi_h \langle [x_i, x_h], a_\alpha \rangle = 0, \end{cases} \tag{4.9}$$

where

$$\tilde{\xi}_\alpha = \xi_\alpha - \frac{1}{2} \sum_{i,j} A^\alpha_{ji} v_j \xi_i. \tag{4.10}$$

From the first and third condition we derive

$$\frac{\partial \xi_i}{\partial u_\alpha} = 0, \quad \frac{\partial \xi_i}{\partial v_j} + \frac{\partial \xi_j}{\partial v_i} = 0$$

and hence

$$\xi_i = \sum_j a_{ij} v_j + \lambda_i, \tag{4.11}$$

where $a_{ij} + a_{ji} = 0$ and a_{ij}, λ_i are constants. Similarly, the second condition gives

$$\tilde{\xi}_\alpha = \sum_\beta b_{\alpha\beta}(v) u_\beta + \eta_\alpha(v) \tag{4.12}$$

with $b_{\alpha\beta} + b_{\beta\alpha} = 0$. Next we determine the functions $b_{\alpha\beta}$ and η_α using the last equation in (4.9). Therefore we substitute (4.12) in the equation. Differentiation with respect to u_β gives that $b_{\alpha\beta}$ are constant. Moreover

$$\frac{\partial \eta_\alpha}{\partial v_i} + \tfrac{1}{2} \sum_\beta \left(\sum_j A^\beta_{ji} v_j \right) b_{\alpha\beta} + \sum_{h,j} A^\alpha_{ih} a_{hj} v_j = 0. \tag{4.13}$$

The integrability conditions of this system are

$$\sum_\beta A^\beta_{ij} b_{\alpha\beta} - \sum_h A^\alpha_{ih} a_{hj} + \sum_h A^\alpha_{jh} a_{hi} = 0. \tag{4.14}$$

Taking into account (4.3) and (3.4), (4.14) is equivalent to (4.4).

Conversely, suppose that we have (4.4). Then we have

$$\eta_\alpha = -\frac{1}{2}\sum_{i,j,h} A^\alpha_{ih} a_{hj} v_i v_j + \mu_a, \quad \mu_\alpha = \text{const.} \tag{4.15}$$

So the required formula (4.6) follows from (4.15), (4.12) and (4.10).

The geodesics of the manifold $(N, \langle,\rangle)$ have been calculated explicitly in [8], [9]. Let $\gamma(t) = \exp(x(t) + a(t))$ be the geodesic tangent at 0 to the vector $\dot\gamma(0) = \lambda + \mu$, $\lambda \in V$, $\mu \in Z$. Then we have

$$\begin{cases} x(t) = \dfrac{1-\cos|\mu|t}{|\mu|^2} j(\mu)\lambda + \dfrac{\sin|\mu|t}{|\mu|}\lambda, \\ a(t) = \left\{ t + \dfrac{|\lambda|^2}{2|\mu|^2}\left(t - \dfrac{\sin|\mu|t}{|\mu|}\right)\right\}\mu \end{cases} \tag{4.16}$$

for $\mu \neq 0$ and

$$\begin{cases} x(t) = t\lambda, \\ a(t) = 0 \end{cases} \tag{4.17}$$

for $\mu = 0$.

From this it is clear that when $\mu = 0$, $\gamma(t) = (\exp t\lambda)O$ is the orbit of a one-parameter subgroup of isometries of $(N, \langle,\rangle)$ but in general we have

THEOREM 4.2: *The geodesic $\gamma(t)$ with $\dot\gamma(0) = \lambda + \mu$ is an orbit of a one-parameter subgroup of isometries of $(N, \langle,\rangle)$ if and only if there exist skew-symmetric endomorphisms A and B of V and Z such that*

$$\begin{cases} A(\lambda) = j(\mu)\lambda, \\ B(\mu) = 0, \\ Aj(a) - j(a)A = j(B(a)), \end{cases} \tag{4.18}$$

for all $a \in Z$.

PROOF. The conditions (4.18) are the necessary and sufficient conditions for the existence of a Killing vector field ξ such that $\xi(\gamma(t)) = \dot\gamma(t)$ for all t.

5. The geometry of the 6-dimensional group of type H

It is not difficult to see that there is only one group of type H of dimension 6 (see below). This manifold has been discussed by Kaplan [9].

In this section we discuss this manifold and its geometry in detail because of its remarkable properties.

This example can be described as follows. Let $V = \mathbb{H}$, the space of the quaternions, and let Z be a 2-dimensional subspace of purely imaginary quaternions. Let $j: Z \to \mathrm{End}(V)$ be the linear map defined by

$$j(a)x = ax \qquad , a \in Z, \quad x \in V, \tag{5.1}$$

i.e. $j(a)x$ is the ordinary multiplication of x by a. It is clear that $\mathfrak{n} = V \oplus Z$ is a Lie algebra of type H. Further it follows from the results in section 4 that the corresponding Lie group N of type H is a homogeneous space which is not naturally reductive.

Kaplan proved in [9] that the geodesics of this group N are still orbits of one parameter subgroups of isometries. Now we shall give a new proof of this result using Theorem 4.2.

THEOREM 5.1: *Let $(N, \langle,\rangle)$ denote the 6-dimensional group of type H. Then the geodesics of $(N, \langle,\rangle)$ are orbits of one-parameter subgroups of isometries of $(N, \langle,\rangle)$.*

PROOF: Let $\gamma(t)$ be the geodesic of N through $\gamma(0) = 0$ and such that $\dot\gamma(0) = \lambda + \mu$, where $\lambda \in V$, $\mu \in Z$. First we suppose $\lambda \neq 0$ and $\mu \neq 0$. Let (a_1, a_2) be an orthonormal frame of Z such that $\mu = \mu_1 a_1$ and let x_1 be a unit vector of V such that $\lambda = \lambda_1 x_1$. Then $\{x_1, j(a)_1)x_1, j(a_2)x_1, j(a_1)j(a_2)x_1\}$ is an orthonormal basis of V.

It follows from Theorem 4.2 that there exists a Killing vector field ξ such that $\xi(\gamma(t)) = \dot\gamma(t)$ if and only if there exist skew-symmetric endomorphisms A and B of V and Z satisfying (4.18). Now it is clear that in our case $B = 0$ and further A is uniquely determined by

$$\begin{cases} A(x_1) = j(a_1)x_1, & A(j(a_1)x_1) = -x_1, \\ A(j(a_2)x_1) = -j(a_1)j(a_2)x_1, & A(j(a_1)j(a_2)x_1) = j(a_2)x_1. \end{cases} \tag{5.2}$$

Hence these geodesics are orbits of *unique* one-parameter subgroups of isometries.

If $\lambda = 0$ or $\mu = 0$ we have $\gamma(t) = \exp t\mu$ or $\gamma(t) = \exp t\lambda$. Hence we put $A = B = 0$. In this case the geodesics are again orbits of one-parameter subgroups but these subgroups are *not uniquely* determined.

REMARK: The property proved in Theorem 5.1 cannot be extended to all groups of type H. Indeed, it can be proved by further research that there exist groups of type H such that not all the geodesics are orbits of one-parameter subgroups (see [9] for a proof in the case $m \equiv 0 \pmod 4$).

Next we want to concentrate on another property which holds at least partly for all groups of type H. Before doing this we need some preliminaries.

Let (M, g) be an n-dimensional Riemannian manifold and m a point of M. Further let $(x_1, \ldots, x_n)$ be a system of normal coordinates centered at m and p a point of M such that $r = \mathrm{d}(m, p) < i(m)$ where $i(m)$ is the injectivity radius at m. Then p can be joined to m by a unique geodesic γ. Put $\gamma(0) = m$, $\gamma(r) = p = \exp_m(r\xi)$ where ξ is the unit velocity vector. The *geodesic symmetry* γ (about m) is defined by

$$\psi: M \to M, \quad p \mapsto \psi(p) = \exp_m(-r\xi) = -p$$

and this is an involutive local diffeomorphism.

Riemannian manifolds with volume-preserving or, equivalently, divergence-preserving geodesic symmetries were studied in [3], [4], [5] and such manifolds are called *D'Atri spaces* in [15], [16]. They can be characterized as follows. Let

$$\theta = \left(\det g_{ij}\right)^{1/2} = \left(\det g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\right)^{1/2}.$$

Then it is easily seen that (M, g) is a D'Atri space if and only if

$$\theta(-p) = \theta(p)$$

for all $m \in M$ and all p near m. So θ has *antipodal symmetry*.

Examples of D'Atri spaces are the so-called *commutative spaces* (see for example [15]). This class includes the harmonic spaces, their products and also all symmetric spaces. Also the naturally reductive homogeneous Riemannian manifolds have this property [5] and these were, among the homogeneous manifolds, the only known examples. It is the search for non-naturally reductive examples which gave rise to our work [14], [15], [16] and to part of the work of Kaplan [9]. Note that up to now no examples of non-homogeneous D'Atri spaces are known.

In [9] Kaplan proved the following remarkable result

THEOREM 5.2: *All the groups of type H are D'Atri spaces.*

Hence the volume-preserving geodesic symmetry property does not characterize the naturally reductive spaces among the homogeneous spaces. But D'Atri proved in [5] a stronger result. Let $G = (g_{ij})$ be the matrix of g with respect to a normal coordinate system at m. Then the eigenvalues of G are independent of the choice of the normal coordinate system at m. D'Atri proved that *all* these eigenvalues have the antipodal symmetry. A different proof for this property can be given using the special form of the Jacobi equation in terms of the canonical connection and the associated tensor T or the torsion tensor of this connection. In what follows we shall prove that the 6-dimensional group of type H has also this property so that again this cannot be characteristic for the class of naturally reductive homogeneous spaces.

THEOREM 5.3: *Let $(N, \langle,\rangle)$ be the 6-dimensional group of type H. Then all the eigenvalues of the matrix of the metric tensor $\langle,\rangle$ with respect to any normal coordinate system have the antipodal symmetry.*

PROOF: Let G be the matrix of $\langle,\rangle$ with respect to a normal coordinate system centered at 0 and let γ be a geodesic through 0 and $q \in \gamma$. Then we have to prove that

$$\det(G - \lambda I)(-q) = \det(G - \lambda I)(q).$$

First we have to compute $G(q)$. Therefore we put $t = 1$ in (4.16) and (4.17). This gives the relation between the global coordinates (v_i, u_α) of $p = \exp(x(1) + a(1))$ and the normal coordinates (λ_i, μ_α) with respect to the basis $(x_i(0), a_\alpha(0))$ of T_0N. Further, from (4.2) we obtain that in general the dual frame of (x_i, a_α) is determined by the left invariant 1-forms θ_i, ψ_α, where

$$\theta_i = \mathrm{d}v_i, \tag{5.3}$$

$$\psi_\alpha = \mathrm{d}u_\alpha - \frac{1}{2}\sum_{j,i} A^\alpha_{ji} v_j \mathrm{d}v_i \tag{5.4}$$

for $i, j = 1,\dots,n$ and $\alpha = 1,\dots,m$. Hence

$$\langle,\rangle = \sum_i \theta_i^2 + \sum_\alpha \psi_\alpha^2. \tag{5.5}$$

To obtain the components of $G(p)$ with respect to normal coordinates we have to express $\mathrm{d}v_i$ and $\mathrm{d}u_\alpha$ as functions of $\mathrm{d}\lambda_i$, $\mathrm{d}\mu_\alpha$. Therefore we use the fact that the eigenvalues of G are independent of the normal coordinate system choosen at 0 and hence we put

$$\lambda = \lambda_1 x_1, \quad \mu = \mu_1 a_1.$$

On the 6-dimensional manifold we then choose a basis (a_1, a_2) for Z and the basis (x_1, x_2, x_3, x_4) of V with

$$x_2 = j(a_1)x_1, \quad x_3 = j(a_2)x_1, \quad x_4 = j(a_1)j(a_2)x_1. \tag{5.6}$$

An easy calculation now shows that

$$\begin{cases} \psi_1 = \mathrm{d}u_1 - \frac{1}{2}(v_1\mathrm{d}v_2 - v_2\mathrm{d}v_1) - \frac{1}{2}(v_3\mathrm{d}v_4 - v_4\mathrm{d}v_3), \\ \psi_2 = \mathrm{d}u_2 - \frac{1}{2}(v_1\mathrm{d}v_3 - v_3\mathrm{d}v_1) + \frac{1}{2}(v_2\mathrm{d}v_4 - v_4\mathrm{d}v_2). \end{cases} \tag{5.7}$$

Moreover from (4.16) and (4.17) we derive

$$\begin{cases} v_1 = -\beta(\mu_1\lambda_2 + \mu_2\lambda_3) + \alpha\lambda_1, \\ v_2 = \beta(\mu_1\lambda_1 + \mu_2\lambda_4) + \alpha\lambda_2, \\ v_3 = -\beta(\mu_1\lambda_4 - \mu_2\lambda_1) + \alpha\lambda_3, \\ v_4 = \beta(\mu_1\lambda_3 - \mu_2\lambda_2) + \alpha\lambda_4, \\ u_1 = \gamma\mu_1, \\ u_2 = \gamma\mu_2, \end{cases} \tag{5.8}$$

where

$$\begin{cases} \alpha = \dfrac{\sin|\mu|}{|\mu|}, \\ \beta = \dfrac{1-\cos|\mu|}{|\mu|}, \\ \gamma = 1 + \dfrac{|\lambda|^2}{2|\mu|^2}\left(1 - \dfrac{\sin|\mu|}{|\mu|}\right) \end{cases} \tag{5.9}$$

if $|\mu| \neq 0$. The case $|\mu| = 0$ can be obtained by continuity.

Note that

$$\alpha(-p) = \alpha(p), \quad \beta(-p) = \beta(p), \quad \gamma(-p) = \gamma(p)$$

and

$$\begin{cases} \mathrm{d}\alpha_{|p} = A(p)\mathrm{d}\mu_{1|p}, \\ \mathrm{d}\beta_{|p} = B(p)\mathrm{d}\mu_{1|p}, \\ \mathrm{d}\gamma_{|p} = C(p)\mathrm{d}\mu_{1|p} + D(p)d\lambda_{1|p} \end{cases} \tag{5.10}$$

where

$$A(-p) = -A(p), \quad B(-p) = -B(p),$$

$$C(-p) = -C(p), \quad D(-p) = -D(p).$$

Now from (5.8) we compute $\mathrm{d}v_i$, $\mathrm{d}u_\alpha$; then use (5.3) and (5.4) and substitute in (5.5) after evaluating at p. So we obtain the following form for the characteristic polynomial:

$$P_p(\lambda) = \det(G - \lambda I)(p)$$

$$= \begin{vmatrix} e_1-\lambda & O_1 & 0 & 0 & e_7 & 0 \\ O_1 & e_2-\lambda & 0 & 0 & O_3 & 0 \\ 0 & 0 & e_3-\lambda & O_2 & 0 & O_4 \\ 0 & 0 & O_2 & e_4-\lambda & 0 & e_8 \\ e_7 & O_3 & 0 & 0 & e_5-\lambda & 0 \\ 0 & 0 & O_4 & e_8 & 0 & e_6-\lambda \end{vmatrix} \tag{5.11}$$

$$e_j(-p) = e_j(p) \quad \text{and} \quad O_\beta(-p) = -O_\beta(p) \tag{5.12}$$

for $j = 1,\ldots,8$ and $\beta = 1,\ldots,4$.

From this it is clear that

$$P_{-p}(\lambda) = P_p(\lambda)$$

when $O_\beta = 0$ for all β. When at least one $O_\beta \neq 0$, the same result is obtained as can be seen easily by multiplying the first, third and fifth row by O_β. This finishes the proof since p can be any arbitrary point q near 0.

REMARKS:

A. As we already mentioned we do not know of any nonhomogeneous manifold having the properties given in Theorem 5.2 or Theorem 5.3. It would be nice to know if a Riemannian manifold with one of these properties is (locally) homogeneous. This would imply that a harmonic space is (locally) homogeneous.

Further it would also be of some interest to know if the property of Theorem 5.3 has something to do with the fact that all geodesics are orbits of one-parameter subgroups. Is an extension of that property to the class of manifolds such that all geodesics are orbits of one-parameter subgroups, possible?

B. In [1] the authors consider also the class of homogeneous manifolds such that $\nabla_X T_X = 0$ for all $X \in \mathfrak{X}(M)$. It is straightforward but tedious to prove that such a (1,2)-tensor T cannot exist on the 6-dimensional group of type H.

Note that $T_X X = 0$ implies $\nabla_X T_X = 0$ as can be seen from (AS(iii)).

Finally we derive another property for the 6-dimensional example in relation with the theory of k-symmetric spaces. We refer to [7],[11] for a detailed theory about these spaces and for further references.

First we write down explicitly the brackets for the Lie algebra $\mathfrak{n}$ of the 6-dimensional group $(N, \langle,\rangle)$. We obtain easily

$$\begin{cases} [x_1, x_2] = a_1, \quad [x_1, x_3] = a_2, \\ [x_2, x_4] = -a_2, \quad [x_3, x_4] = a_1, \\ \text{all the other brackets being zero.} \end{cases} \tag{5.13}$$

Next put

$$\begin{cases} U_1 = x_1 + \mathrm{i}x_4, \\ U_2 = x_2 + \mathrm{i}x_3 \quad , \\ U_3 = -a_1 + \mathrm{i}a_2 \end{cases} \tag{5.14}$$

and define the linear map S of $\mathfrak{n}$ by

$$SU_j = \mathrm{e}^{2\pi\mathrm{i}/3}U_j, \quad j = 1, 2, 3.$$

It follows at once from (5.13) and (5.14) that S is an isometric automorphism of the Lie algebra $(\mathfrak{n}, \langle,\rangle)$ and moreover $S^3 = I$. Hence N is a 3-*symmetric space*. Note that the canonical almost complex structure J associated with S, i.e.

$$S = -\frac{1}{2}I + \frac{\sqrt{3}}{2}J,$$

is neither nearly Kähler nor almost Kähler (see [14], [7]). The fact that J is not nearly Kähler agrees with the fact that $(N, \langle,\rangle)$ is not naturally reductive.

Next consider the linear map S defined by

$$SU_1 = \mathrm{i}U_1, \quad SU_2 = \mathrm{i}U_2, \quad SU_3 = -U_3.$$

It is easily seen that S is again an isometric automorphism of $(\mathfrak{n}, \langle,\rangle)$ but now $S^4 = \mathrm{I}$. Hence $(N, \langle,\rangle)$ is also a 4-*symmetric space*. So we proved

THEOREM 5.4: *The* 6-*dimensional group of type H is* 3- *and* 4-*symmetric.*

We note that these two facts are implicitly included in [12]. See also [14]. Further a more detailed research about the relation between k-symmetric spaces and general groups of type H would be of some interest. We hope to come back on this in another paper.

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Istituto Matematico
Politecnico di Torino
Corso Duca degli Abruzzi 24
10129 Torino
Italy

Current Address
Dipartimento di Matematica
Università di Torino
via Principe Amedeo 8
10126 Torino
Italy

Departement Wiskunde
Katholieke Universiteit Leuven
Celestijnenlaan 200 B
B-3030 Leuven
Belgium