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AUTOMORPHISMS OF CUSPS AND INOUE-HIRZEBRUCH SURFACES

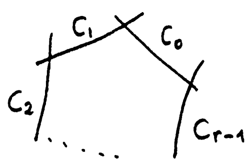
H.C. Pinkham *

Introduction

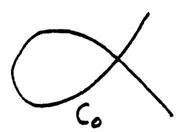
In [8] Inoue constructs a family of minimal compact complex surfaces S with no meromorphic functions, first Betti number equal to 1 and second Betti number positive. These are surfaces of type VII_0 in Kodaira's classification [9], and are now generally called Inoue-Hirzebruch surfaces. The construction which is explained in Section 1 but need not concern us yet, depends on modules in real quadratic fields.

The main purpose of this paper is to determine the full group of complex automorphisms of S. By the same method we obtain an interesting group of automorphisms of "cusp" singularities, which we now describe.

The only complex curves on S are arranged in 2 connected components consisting of rational curves $C_0,\dots,C_{r-1}$, and $D_0,\dots,D_{s-1}$, both arranged in a cycle, i.e. if $r>1$:



and if $r=1$:



The intersection matrices $|C_i\cdot C_j|$ and $|D_i\cdot D_j|$ are negative definite, so that each connected component can be contracted to a normal singular point,

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called a cusp (Hirzebruch [5], §2). Call them p and q, and the contracted surface S'. Since each cusp occurs on precisely one Inoue-Hirzebruch surface, the second cusp appearing on S' is well-defined, and called the dual of the first. This duality has been studied by Looijenga [10] and Nakamura [11] and is a generalization of the strange duality of Arnold [1]. A somewhat incidental purpose of this paper is to show that there is a lattice theoretic explanation for the duality, along the same lines as the explanation of the strange duality given in [14], at least for those cusps that have an interesting deformation theory. This is done in the last section. (The unicity statement conjectured in [14] can be verified using the work of Nikulin [13].) The only new idea here is to embed two apparently unrelated lattices as orthogonal complements in a unimodular lattice, using results of [13]. This construction is the lattice theoretic analog of a geometric construction of Looijenga ([10] Prop. 2.8). The main result of this section is used to study the deformation theory of certain cusps in [18].

This explanation of the duality, and indeed the rest of this paper, depends on a certain finite abelian group T, which we now define. Remove the curves $\cup C_i$ and $\cup D_j$ from S (or p and q from S', if you prefer). Call the remaining (smooth) variety S''. Inoue ([8], §4) shows that S'' is homeomorphic to $\mathbb{R} \times \tau$, where τ is an $S^1 \times S^1$ bundle over S^1. Then T is the torsion subgroup of $H_1(\tau, \mathbb{Z})$.

In the course of the determination of the automorphism group of S we will see that it contains a subgroup isomorphic to T which acts trivially on $H_2(S, \mathbb{Z})$. The quotient by any subgroup of T is again an Inoue-Hirzebruch surface (Section 2). Let me conclude by the following concrete corollary of my results, which was actually the starting point of this investigation:

COROLLARY: *Consider the hypersurface cusp*:

$$x^p + y^q + z^r + xyz = 0, \quad 1/p + 1/q + 1/r < 1$$

(*the resolution of the cusp is given, for example, in* [11], *Lemma* 2.5: *the dual cusp has three components with self intersections* $-(p-1)$, $-(q-1)$, $-(r-1)$.) *Then T is isomorphic to the group*
$G \simeq \{\lambda, \mu, \nu \in \mathbb{C} \mid \lambda^p = \mu^q = \nu^r = \lambda\mu\nu\}$ *acting by* $x \to \lambda x$, $y \to \mu y$, $z \to \nu z$, *and the quotient of the cusp by G is the dual cusp.*

Note that J. Wahl has considered examples of quotients of cusps in [21], 5.9.4.

1. Review of known results

The results in this section come from [2], pp. 39–53, [5], §2, and [8]. See also [17].

Throughout this paper M denotes a complete (or full, in the terminology of [3]) module in the real quadratic field K over $\mathbb{Q}$. An excellent account of the results we need from number theory is given in Chapter 2 of [3]. As usual if $x \in K$, x' denotes the conjugate of x. We will consider two equivalence relations on modules M: M_1 and M_2 are equivalent (resp. strictly equivalent) if there exists an element $\gamma \in K$, (resp. $\gamma > 0$, $\gamma' > 0$), such that $\gamma M_1 = M_2$. For any strict equivalence class represented by M there is a unique strict equivalence class represented by δM, $\delta > 0$ and $\delta' < 0$. This is denoted M^*. I will often abuse language and use M to denote the (strict) equivalence class of M as well as M itself.

Let U_M and U_M^+ be the groups of positive and totally positive units of M, respectively; e.g.

$$U_M^+ = \{\gamma \in K, \gamma > 0, \gamma' > 0 | \gamma M = M\}.$$

Both groups are infinite cyclic and $[U_M : U_M^+]$ is either 1 or 2.

We consider a subgroup V of finite index in U_M^+ and let α denote a generator of V.

The semi-direct product $G(M, V)$ acts on $\mathbb{H} \times \mathbb{C}$ (and $\mathbb{H} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half plane) by

$$m \in M, \gamma \in V: (z_1, z_2) \to (\gamma z_1 + m, \gamma' z_2 + m'). \tag{1}$$

The group structure on $G(M, V)$ is given by:

$$(\gamma_1, m_1) \circ (\gamma_2, m_2) = (\gamma_1\gamma_2, \gamma_1 m_2 + m_1).$$

The action is free and properly discontinuous in both cases. Let $S''(M, V)$ be the quotient of $\mathbb{H} \times \mathbb{C}$ by this action, and $X''(M, V)$ that of $\mathbb{H} \times \mathbb{H}$. Both spaces are smooth complex spaces, but of course not compact. We first need to understand the topology of $S''(M, V)$ or equivalently $X''(M, V)$.

$$\text{Let } \begin{Bmatrix} z_1 = x_1 + \mathrm{i} y_1 \\ z_2 = x_2 + \mathrm{i} y_2 \end{Bmatrix}$$

where x_1, x_2, y_1, y_2 are real coordinates. $\mathbb{H} \times \mathbb{C}$ is fibered by subvarieties $W_d = \{(z_1, z_2) \in \mathbb{H} \times \mathbb{C} | y_1 \cdot y_2 = d\}$, invariant under $G(M, V)$. The quotient τ_d fibers over $\{(y_1, y_2) \in \mathbb{R} \times \mathbb{R} | y_1 y_2 = d\}/V$ which is just S^1, and the fiber is $\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}\}/M$ which is just $S^1 \times S^1$. Therefore τ_d is an $S^1 \times S^1$ bundle over S^1. Since it clearly does not depend on d, we now drop the subscript d. The monodromy of τ, which is its only invariant, is given by the action of α, the generator of V, on M. Therefore from the

Wang sequence one easily deduces, as in [8], §4 Eqn. (24) that the torsion subgroup of $H_1(\tau, \mathbb{Z})$ is $M/(\alpha-1)M$. We call this group $T(M, V)$. Another more direct way of computing $T(M, V)$ will be useful later: $G(M, V)$ is the fundamental group of τ so that if $G'(M, V)$ denotes the commutator subgroup of $G(M, V)$ then $H_1(\tau, \mathbb{Z}) = G(M, V)/G'(M, V)$. Thus the following lemma gives $T(M, V)$:

LEMMA: $G'(M, V) = (1, (\alpha-1)M)$.

$T(M, V)$ is fundamental for the rest of our investigation. It is of course isomorphic to $\mathbb{Z}/e \oplus \mathbb{Z}/f$ where e and f are the elementary divisors of $\alpha - 1$ viewed as an endomorphism of M.

We have seen that $S''(M, V)$ is homeomorphic to $\mathbb{R} \times \tau$.

We will now compactify $S''(M, V)$ by adding two cycles of rational curves, one at $+\infty$, the other at $-\infty$ in the representation $S''(M, V) \simeq \mathbb{R} \times \tau$. We can of course do each cycle separately, so let's do the cycle at $+\infty$ first. We can therefore restrict the action to $\mathbb{H} \times \mathbb{H}$. From this point on *I* will mainly use the notation from Hirzebruch [5], Section 2, which differs from that of Inoue.

The module M is strictly equivalent to a module $M(w)$ generated by 1 and w, where $w \in K$ is *reduced*, i.e. $w > 1 > w' > 0$. Since the construction that follows is easily seen to depend only on the strict equivalence class of M, we may as well replace M by $M(w)$. w has a purely periodic modified continued fraction expansion

$w = [[\overline{b_0, b_1, \ldots b_{r-1}}]]$ where all the b_i are $\geqslant 2$ and some b_i is $\geqslant 3$. (For basic information on modified continued fraction expansions see Hirzebruch [5], 2.3.) Here r does not denote the length of the primitive period of w, but that of the degree $[U_M^+ : V]$ period: cf. the theorem p. 216 of Hirzebruch.

Take an infinite number of copies of $\mathbb{C}^2$, indexed by $k \in \mathbb{Z}$, with coordinates u_k and v_k. Glue them together by

$$\begin{aligned} u_{k+1} &= u_k^{b_k} v_k \\ v_{k+1} &= 1/u_k. \end{aligned} \tag{2}$$

Call this space Y. The curve C_k given by $u_{k+1} = v_k = 0$ is a $\mathbb{P}^1$ with self-intersection $-b_k$. The group V acts freely on Y, with α sending the point with coordinates (u_k, v_k) in the k-th chart to the point with the same coordinates in the $k+r$-th chart.

We have an isomorphism

$$\Phi : Y - \bigcup_{k \in \mathbb{Z}} C_k \to \mathbb{C}^2/M(w)$$

defined on the 0-th coordinate chart of Y by

$$\begin{aligned} 2\pi i z_1 &= w \log u_0 + \log v_0 \\ 2\pi i z_2 &= w' \log u_0 + \log v_0. \end{aligned} \tag{3}$$

We now restrict Φ to $\Phi^{-1}(\mathbb{H} \times \mathbb{H}/M(w))$. Φ is compatible with the action of V on both sides, so that we can patch Y/V to $X''(M(w), V) \approx \mathbb{H} \times \mathbb{H}/G(M(w), V)$. Call the resulting space $X(M(w), V)$. We have added to $X''(M(w), V)$ a cycle of r rational curves C_i, $0 \leqslant i < r$ with self-intersections $-b_0, -b_1, \ldots, -b_{r-1}$, or $-b_0 + 2$ if $r = 1$. The condition on the b_i is equivalent to the intersection matrix $(C_i \cdot C_j)$ being negative definite, so we can contract the curves C_i to a normal singular point p, which is a cusp (in dimension 2, a cusp is any isolated normal singularity having a resolution with exceptional divisor consisting of a cycle of smooth rational curves, or with an irreducible exceptional divisor that is a rational curve with a node). By suitable choice of w, hence M, one can obtain all cusps in this fashion. We call the contracted space $X'(M(w), V)$. For later purposes we need to be able to write down a set of generators for the ring of holomorphic functions at the singular point of $X'(M(w), V)$.

Let M^* be the dual lattice of M under trace, i.e. $M^* = \{v \in \mathbb{Q}(w) | vm + v'm' \in \mathbb{Z},\ m \in M\}$. This use of * does not really conflict with our previous definition since one easily sees (Nakamura [12], Section 6) that the dual under trace of M is strictly equivalent to our old M^*.

For $m \in M^*$, m totally positive, let

$$f_m(z_1, z_2) = \sum_{n \in \mathbb{Z}} \exp\left(2\pi i\left(\alpha^n m z_1 + (\alpha')^n m' z_2\right)\right). \tag{4}$$

According to Hirzebruch [5], Section 2, Eqn. (4), these Fourier series generate the convergent power series ring of the cusp (in [12], §6, Nakamura shows how to find a minimal set of generators, but we will not need his result.)

We now return to the surface $S''(M, V)$; we want to add a cycle of rational curves at $-\infty$. For this we must look at the action of $G(M, V)$ on $\mathbb{H} \times \mathbb{L}$ ($\mathbb{L}$ = lower half plane). This amounts to looking at the action of $G(M^*, V)$ on $\mathbb{H} \times \mathbb{H}$, so that if $M^* = M(w^*)$, w^* reduced with modified continued fraction expansion $[[\overline{c_0, \ldots c_{s-1}}]]$, then we must add on a cycle of rational curves $D_0, \ldots D_{s-1}$ with self-intersections $-c_0, -c_1, \ldots, -c_{s-1}$ or $-c_0 + 2$ if $s = 1$. There is a simple algorithm for getting from the b_i to the c_i, which first occurs in a paper of Hirzebruch-Zagier ([6], p. 50) and later much studied by Looijenga [10], Wahl and Nakamura [12].

The surface obtained by glueing on both cycles of rational curves to $S''(M, V)$ is called the Inoue-Hirzebruch surface of type (M, V) and is denoted $S(M, V)$. If both cycles of curves are contracted to points p and

q we get the singular Inoue-Hirzebruch surface, denoted $S'(M, V)$. As mentioned in the introduction the cusp q is called the dual of the cusp p.

2. The automorphism group of Inoue-Hirzebruch surfaces

Let K, M, V, etc. be as in Section 1. The module $(\alpha - 1)^{-1}M$, which we write $\overline{M}$ from now on, is equivalent to M so we can form the semi-direct product $G(\overline{M}, U_M)$. Recall (Section 1) that U_M is the group of positive units of M. The following lemma is a trivial computation.

LEMMA: *$G(M, V)$ is a normal subgroup of $G(\overline{M}, U_M)$.*

We can now state the main theorem of this paper.

THEOREM: *$G(\overline{M}, U_M)/G(M, V)$ is the full (complex) automorphism group of $S(M, V)$, $S'(M, V)$ and $S''(M, V)$. Any subgroup of $G(\overline{M}, U_M)/G(M, V)$ is of the form $G(N, W)/G(M, V)$ and the quotient of $S(M, V)$ (resp. S', S'') by such a subgroup is "bimeromorphically" $S(N, W)$ (resp. exactly S', S'').*

Note: if $W = U_M \supsetneqq U_M^+$, then $S(N, W)$ denotes the quotient of $S(N, U_M^+)$ by the involution associated to the non-trivial element of U_M/U_M^+ (cf. [8], Section 6).

The proof is in two steps. First we show the group in question is the automorphism group of $S''(M, V)$ and second we show it extends to S and S'. The second step is easy by general considerations but we give a more computational proof in order to exhibit the form of the automorphisms explicitly.

Step I

It is clear that any automorphism of S or S' restricts to an automorphism of S'', since $S \setminus S''$ consists of all the complex curves of S, and $S' \setminus S''$ of the only singular points of S'.

So consider any automorphism f: $S'' \to S''$. S'' has $\mathbb{H} \times \mathbb{C}$ as universal cover, and f lifts to an automorphism F: $\mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C}$ which satisfies the following equivariance property with respect to $G(M, V)$:

$$F(g \cdot z) = h(g) \cdot F(z)$$

where $z \in \mathbb{H} \times \mathbb{C}$, $g \in G(M, V)$, h a group automorphism of $G(M, V)$ and $\cdot$ the action of $G(M, V)$ on $\mathbb{H} \times \mathbb{C}$ described in Section 1. More explicitly write $z = (z_1, z_2)$, $z_1 \in \mathbb{H}$, $z_2 \in \mathbb{C}$; $g = (v, m)$, $v \in V$, $m \in M$; $h(v, m) = (h_1(v, m), h_2(v, m))$ where h_1 is a group homomorphism

$G(M, V) \to V$ and h_2 a map $G(M, V) \to M$ satisfying the cocycle rule $h_2(g_1 g_2) = h_1(g_1) h_2(g_2) + h_2(g_1)$.

Now V is abelian so h_1 factors through the quotient by the commutator subgroup G' of $G(M, V)$ which by Section 1 consists of $(1, m)$, $m \in (\alpha - 1)M$. We will often identify G' with $(\alpha - 1)M$. Restricting to G' we see that $F(z) = (f_1(z), f_2(z))$ satisfies

$$F(z_1 + m, z_2 + m') = (f_1(z) + h_2(m), f_2(z) + h_2'(m))$$

for $m \in G'$, where h_2 is now a homomorphism of G'.

Thus $\partial f_j / \partial z_i$ is periodic with respect to G', so that it has a Fourier series expansion

$$\sum_{m \in G'^*} a_m \exp(2\pi i(m z_1 + m' z_2))$$

where * denotes dual under trace, as in Section 1. Now use the equivariance of F with respect to V. Since $h_1(\alpha, 0) = \alpha^k$ for some integer k,

$$F(\alpha z_1, \alpha' z_2) = \left(\alpha^k f_1(z) + h_2(\alpha, 0), \alpha'^k f_2(z) + h_2'(\alpha, 0)\right)$$

so for example taking the f_1 term:

$$\frac{\partial f_1}{\partial z_1}(\alpha z_1, \alpha' z_2) = \alpha^k \frac{\partial f_1}{\partial z_1}(z_1, z_2)$$

so that the coefficients of the Fourier series of $\partial f_1 / \partial z_1$ satisfy

$$a_{\alpha^{-1} m} = \alpha^k a_m.$$

This series must converge in $\mathbb{H} \times \mathbb{C}$, in particular when $z_1 \to i\infty$, and $z_2 \to \pm i\infty$. A standard argument (see for example [17], Section 1) then implies that all the coefficients of the series are 0 except perhaps the constant term.

Therefore $f_1(z) = a z_1 + b z_2 + c$, where $a, b, c \in \mathbb{C}$. Since $z_2 \in \mathbb{C}$ and f_1 takes values in $\mathbb{H}$, $b = 0$, so f_1 is a function of z_1 alone. Recall that

$$\begin{array}{ccc} f_1(v z_1 + m) & = & h_1(v, m) f_1(z_1) + h_2(v, m) \\ \| & & \| \\ a v z_1 + a m + c & & h_1(v, m)(a z_1 + c) + h_2(v, m) \end{array}$$

so that

$$h_1(v, m) = v \qquad \text{for all} \quad m \in M.$$

Next set $v = 1$. Then $am = h_2(1, m) \in M$, so that a is a unit of M. Since f_1

is a map $\mathbb{H} \to \mathbb{H}$, a must be positive. Finally we have

$$h_2(v, m) = (1 - v)c + am$$

so $(1 - v)c \in M$ for all $v \in V$. Thus $c \in (\alpha - 1)^{-1}M = \overline{M}$.

A similar analysis for f_2, using the expressions for h_1 and h_2 given above, shows that $f_2(z) = a'z_2 + c'$. Thus the group $G(\overline{M}, U_M)$ acts on $\mathbb{H} \times \mathbb{C}$, and its action descends to $S''(M, V)$ with kernel $G(M, V)$. This establishes Step I.

Step II

It is trivial to see how the U_M/V part of the automorphism group extends to S and S', so we will only concern ourselves with the $\overline{M}/M$ part. (In fact the U_M^+/V part of the automorphism group is described in [11], 2.2, and as already mentioned the U_M/U_M^+ part in [8], Section 6.)

It suffices of course to show the action extends to $X(M, V)$ and $X'(M, V)$. We will do $X(M, V)$ first. We replace M by a $M(w)$ in the same strict equivalence class, where w is reduced.

By Eqn. (1) of the previous section $m \in \overline{M}$ acts on $(z_1, z_2) \in \mathbb{H} \times \mathbb{H}$ by $m: (z_1, z_2) \to (z_1 + m, z_2 + m')$. Therefore by Eqn. (3), m acts on Y, in the 0-th coordinate chart, by $m: (u_0, v_0) \to (\exp(2\pi i m_1)u_0, \exp(2\pi i m_2)v_0)$ where $m = m_1 w + m_2$; $m_1, m_2 \in \mathbb{Q}$.

We want this action to descend to an action on Y/V. By iterating Eqn. (2) we see that

$$u_k = u_0^{p_k} v_0^{q_k}$$

$$v_k = u_0^{-p_{k-1}} v_0^{-q_{k-1}}$$

where the p's and q's are defined by

$$\begin{pmatrix} p_k & q_k \\ -p_{k-1} & -q_{k-1} \end{pmatrix} = \begin{pmatrix} b_{k-1} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\text{Set } N = \begin{pmatrix} p_r & q_r \\ -p_{r-1} & -q_{r-1} \end{pmatrix}.$$

By Inoue [8], Eqn. (4) and Proposition 1.2 we have

$$(w, 1)N = \alpha \cdot (w, 1).$$

So m acts on the r-th coordinate chart by

$$(u_r, v_r) \to$$

$$(\exp(2\pi i(p_r m_1 + q_r m_2)u_r, \quad \exp(2\pi i(-p_{r-1}m_1 - q_{r-1}m_2))v_r).$$

Therefore for the action of $\overline{M}$ to descend to Y/V, since u_r is identified to u_0 by α, and v_r to v_0, we must have

$$\exp(2\pi i m_1) = \exp(2\pi i(p_r m_1 + q_r m_2))$$

$$\exp(2\pi i m_2) = \exp(2\pi i(-p_{r-1} m_1 - q_{r-1} m_2))$$

or

$$(m_1, m_2) \equiv (m_1, m_2)\cdot N \mod \mathbb{Z}.$$

In other words, we must have $\alpha m \equiv m \bmod M$ or $m \in [1/(\alpha-1)]M$, which is precisely our hypothesis.

NOTE: Here is another way of expressing this computation. The space Y of Section 1 is the toroidal compactification ([19], Section 15) of a subspace of $\mathbb{C}^* \times \mathbb{C}^*$, so it admits a $\mathbb{C}^* \times \mathbb{C}^*$ action given in the coordinates by

$$\begin{matrix} u_0 \mapsto s u_0 \\ v_0 \mapsto v_0 \end{matrix} \quad \text{and} \quad \begin{matrix} u_0 \mapsto u_0 \\ v_0 \mapsto t v_0 \end{matrix}$$

What we have done is simply to compute what part of the action descends to the quotient by V.

Note that $\overline{M}/M$ is isomorphic to the group $T(M, V)$ of the previous section.

Let's now turn to $X'(M, V)$ (f_m is defined by Eqn. (4) in §1).

PROPOSITION: *$\overline{M}/M$ acts as a group of "monomial" automorphisms, on the local ring of the singular point of $X'(M, V)$, with $\overline{m} \in \overline{M}$ acting on f_m, $m \in M^* = \{v \in K \mid \mathrm{tr}(vn) \in \mathbb{Z}, \forall n \in M\}$, by*

$$\overline{m}: f_m(z_1, z_2) \to \exp(2\pi i\, \mathrm{tr}(m\overline{m})) f_m(z_1, z_2).$$

REMARKS:

(1) as pointed out by the referee, since X' is the Stein completion of X'', all automorphisms of X'' extend to X'. It is useful to know explicitly how they extend, however.

(2) if M is isomorphic to its conjugate module M' then $X'(M, V)$ has an extra automorphism of order 2 given on $\mathbb{H} \times \mathbb{H}$ by $(z_1, z_2) \to (z_2, z_1)$. As we have seen in Step I, this involution does not extend to $S''(M, V)$. I have not determined what the quotient of X' by this involution is. †

The proof of Step II is now complete. The remaining assertions in the statement of the theorem, concerning the type of the quotients, are obvious.

† See note added in proof.

We have the following easy corollary of the proof of the theorem:

COROLLARY:

(1) *The kernel of the induced action of* $G(\overline{M}, U_M)/G(M, V)$ *on* $H_2(S(M, V), \mathbb{Z})$ *is* $G(\overline{M}, V)/G(M, V) = \overline{M}/M$.

(2) *The induced action of* $G(\overline{M}, U_M)/G(M, V)$ *on* $H_1(\tau, \mathbb{Z}) = G(M, V)/G'(M, V)$ *is given by* $(u, \overline{m}) \in G(\overline{M}, U_M)$ *sends* $(v, m) \in G(M, V)$ *to* $(v, um + (1 - v)\overline{m})$. (*It is easy to see this action descends to the quotients*).

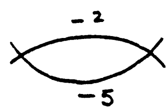
Part (1) follows from Step II of the proof, or alternatively from (2) of the corollary, and Part (2) is a triviality from topology (we have just written down explicitly the action of $G(\overline{M}, U_M)$ on $G(M, V)$ by conjugation) which we have given simply to express intrinsically what was done in the last part of Step I.

EXAMPLE: Take the hypersurface cusp $x^3 + y^3 + z^5 + xyz = 0$. Its minimal resolution has 2 curves with self intersection -2 and -5. So one possible choice for w is $w = [[2,5]] = 1 + \sqrt{15}/5$, $\alpha = 4 + \sqrt{15}$, the norm of $\alpha - 1$ is -6, so that $\overline{M}/M$ is cyclic of order 6. The action of $\overline{M}/M$ on the cusp is given explicitly by

$$x \to \zeta x$$

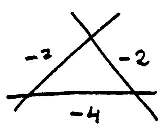
$$y \to \zeta^5 y$$

$$z \to \zeta^3 z$$

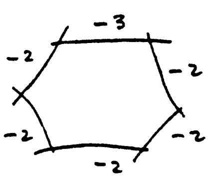


where ζ is a sixth root of unity.

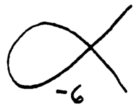
The quotient by the full group gives the dual cusp $x^2 + y^4 + z^7 + xyz = 0$ with resolution



Quotienting by the subgroup of order 3 we get the (hypersurface) cusp:



and quotienting by the subgroup of order 2 we get its dual:



The involution $x \leftrightarrow y$ is the automorphism described in Remark (2) above.

Much more complicated examples can be computed using the tables in [7].

3. Quadratic forms on T(M, V)

In this section we fix a cusp $D = X'(M, V)$ and its dual $D' = X'(M^*, V)$: $T(M, V)$ we now just call T, and the smooth Inoue-Hirzebruch surface on which both cusps have resolutions we call S.

The exceptional divisor of the resolution of D consists of curves $C_0, \ldots, C_{r-1}$; that of D', $D_0, \ldots, D_{s-1}$. For the relationship between the numerical invariants of D and D' see [10], [11], or [4].

Let L be the lattice spanned by $C_0, \ldots, C_{r-1}$ in $H_2(S, \mathbb{Z})$ and L' that spanned by $D_0, \ldots, D_{s-1}$. It follows from [6], §4 (for more details see [11], §4) that we have two exact sequences: ($L^* = \mathrm{Hom}(L, \mathbb{Z})$.)

$$0 \to L \to L^* \to T \to 0 \quad \text{and} \quad 0 \to L' \to L'^* \to T \to 0$$

so that T inherits 2 finite bilinear forms with values in $\mathbb{Q}/\mathbb{Z}$ that differ by a sign, since one easily sees that L is the orthogonal of L' in $H_2(S, \mathbb{Z})$. My general reference for quadratic forms will be [13].

Let us now assume that the configuration of curves $C_0, \ldots, C_{r-1}$ together with the appropriate intersection matrix (resp. the curves $D_0, \ldots, D_{s-1}$) lies in a smooth proper rational surface V (resp. V') as an anti-canonical divisor. The importance of this notion is explained in [10] and [4]. [4] uses the awkward terminology "rational" cusp to describe this notion, but I prefer the less confusing periphrase "sits on a rational surface". This will always imply that exceptional locus of resolution of the cusp is an anticanonical divisor of the rational surface. Denote by R (resp. R') the orthogonal complement in $H_2(V, \mathbb{Z})$ (resp. $H_2(V', \mathbb{Z})$) of the lattice generated by the C_i (resp. D_i), which of course is isomorphic to L (resp. L').

NOTE: R and R' are in general not unique: [4], §6. The non-unicity is related to the number of smoothing components of D' and D, respectively. See [4], 6.1.

For simplicity let us assume both embeddings $L \to H_2(V, \mathbb{Z})$ and $L' \subset H_2(V', \mathbb{Z})$ are primitive (examples in [4], §6, show this is not always the case). Then

THEOREM 1:

(1) *R and R′ are even lattices of rank* $10 - r + s$ *and* $10 + r - s$, *respectively, and signature* $(1, -)$.

(2) $R^*/R \approx R'^*/R' \approx T$, *and the associated finite quadratic forms* q_R, $q_{R'}: T \to \mathbb{Q}/2\,\mathbb{Z}$ *differ by a sign.*

(3) *There exists a primitive embedding of R into* $-E_8 \oplus -E_8 \oplus H \oplus H$ *with orthogonal complement R′. H is the hyperbolic plane and* $-E_8$ *is the unique negative definite even unmodular lattice of rank* 8 [16].

PROOF:

(1) follows from the adjunction formula on V, since classes in R do not intersect the canonical divisor. The formula for the rank of R follows from the fact that dim. Pic$(V) = 10 - K^2$, where $-K^2 =$ multiplicity of $D =$ length of $D' = s$ (see [11] or [4]). That the signature is of the form $(1, -)$ follows from the Hodge index theorem.

(2) $R^*/R \approx T$ since L was assumed to be primitively embedded. Now R^*/R and R'^*/R' have bilinear forms into $\mathbb{Q}/\mathbb{Z}$ which differ by a sign, since L^*/L and L'^*/L' do, so to check that their associated quadratic form into $\mathbb{Q}/2\mathbb{Z}$ differ by a sign it suffices ([13], theorem 1.11.3) to check that their signatures are opposite mod 8 (this is the Arf-invariant).

$$\begin{aligned}&\text{Signature } (R^*/R) = 1 - 9 - s + r \equiv -s + r \quad (8)\\ &\text{Signature } (R'^*/R') = \qquad\qquad \equiv s - r \quad (8)\end{aligned}$$

so we are done.

(3) follows from [13], Prop. 1.6.1, plus the standard fact that the only even unmodular lattice of rank 20 and signature (2, 18) is $-E_8 \oplus -E_8 \oplus H \oplus H$.

For applications we need a slight improvement of this result.

THEOREM 2: *Suppose the cusp D sits on a rational surface V and that the embedding of L in* $H_2(V, \mathbb{Z})$ *is primitive. Let R be the orthogonal of L in* $H_2(V, \mathbb{Z})$. *Then a necessary condition for the dual cusp D′ to sit on a rational surface V′ is that R admit an embedding into* $-E_8 \oplus -E_8 \oplus H \oplus H$.

PROOF: Same as that of the previous proposition, except that we need the correspondence between overlattices of a lattice and isotropic subgroups of the finite discriminant form ([13], §1.4).

The hypothesis that L embed primitively in $H_2(V, \mathbb{Z})$ is obviously satisfied any time every component D_i, $0 \leqslant i \leqslant r - 1$ meets an exceptional curve of the first kind of V. This is the case when r is $\leqslant 5$ as can be seen from [10], Theorem 1.1.

We conclude by an example which ties together the methods of

Sections 2 and 3. We will need an explicit description of the map $L^* \to T$ used above.

Consider $X(M, V)$, where we assume M is in the form $M(w)$ with w reduced (Section 1). Of course $L = H_2(X(M, V), \mathbb{Z})$ and on L^* we will use the dual basis C_k^*, $0 \leqslant k \leqslant r-1$, of the curves C_k. On M we use the basis w, 1, and we write the elements of M as row vectors. It is easy to see (the computation is essentially done in [20], Example 2.3) that the isomorphism $L^*/L \approx T \approx M/(\alpha - 1)M$ is induced from the $\mathbb{Z}$-linear map g: $L^* \to M$ such that (notation as in Section 2)

$$g(C_k^*) = (0 \quad 1)\begin{pmatrix} p_k & q_k \\ -p_{k-1} & -p_{k-1} \end{pmatrix}.$$

EXAMPLE: Let D be the cusp (5, 11, 2), i.e. $r = 3$, $b_0 = 5$, $b_1 = 11$, $b_2 = 2$. Then D' is $(2, 2, 3, \underbrace{2\ldots,2}_{8}, 4)$, so $s = 12$, and by [4], Prop. 4.8, D' sites on a rational surface V'. The group T is isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/30$ ([15], Lemma 1), so by [13] the embedding of L' in $H_2(V', \mathbb{Z})$ cannot be primitive, as the orthogonal complement of L' in $H_2(V', \mathbb{Z})$ is one-dimensional. In fact, if $\overline{L}'$ is the primitive lattice generated by L', then $\overline{L}'/L' \approx \mathbb{Z}/3$ and a lift of a generator of $\overline{L}'/L'$ to $\overline{L}'$ can be written:

$$B = \tfrac{1}{3}(D_0 + 2D_2 + D_4 + 2D_5 + D_7 + 2D_8 + D_{10} + 2D_{11}).$$

Using B one can construct a degree 3 cover of V', such that if the resolution of the cusp D' is contracted to a point on $\overline{V}'$, then the cover $Z \to \overline{V}'$ is unramified outside of the cusp, and totally ramified there. By the results of Section 2 it is clear that Z has a cusp above D'; using the explicit description of the map $L'^* \to T$ given above we see it is $X'(g(\overline{L}'), V)$. In the case at hand we get the cusp (4, 3, 2, 3, 2, 2, 2) (the dual of (2, 3, 4, 6)) which has multiplicity 4 and hence is a complete intersection. By [11], Lemma 2.5 its equations can be written

$$x^2 + w^4 = yz$$

$$y^3 + z^6 = xw$$

and a direct computation shows that the $\mathbb{Z}/3$ quotient is given by $(x, y, z, w) \mapsto (x, \omega y, \omega^2 z, w)$, where ω is a cube root of unity. Z is a rational surface, as follows directly from computation or from a general result of Mérindol, with trivial canonical divisor, a cusp (4, 3, 2, 3, 2, 2, 2) and a $\mathbb{Z}/3$ group of automorphisms. Furthermore we have a one-parameter family of such, since V' itself depends on one modulus.

Deformation theoretic consequences of this construction, which applies to all cusps D such that D' sits non-primitively on a rational surface, will be dealt with elsewhere.

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References

[1] V.I. ARNOLD: Critical points of smooth functions. *Proc. Intern. Cong. Math. Vancouver* (1974) 19–39.

[2] A. ASH, D. MUMFORD, M. RAPOPORT and Y. TAI: *Smooth compactification of locally symmetric varieties.* Brookline, Mass: Math. Sci. Press (1975).

[3] Z.I. BOREVICH and I.R. SHAFAREVICH: *Number Theory.* New York: Academic Press (1966).

[4] R. FRIEDMAN and R. MIRANDA: Smoothing cusp singularities of small length. *Math. Annalen 263* (1983) 185–212.

[5] F. HIRZEBRUCH: Hilbert modular surfaces. *Enseignement Math. 19* (1973) 183–282.

[6] F. HIRZEBRUCH and D. ZAGIER: Classification of Hilbert modular surfaces. In: W.L. Baily and T. Shioda (eds.), *Complex Analysis and Algebraic Geometry.* Cambridge University Press (1977).

[7] E.L. INCE: Cycles of reduced ideals in quadratic fields. *British Assoc. Advancement science, Math. tables, Volume IV,* London (1934).

[8] M. INOUE: New surfaces with no meromorphic functions II. In: W.L. Baily and T. Shioda (eds.), *Complex Analysis and Algebraic Geometry.* Cambridge University Press (1977).

[9] K. KODAIRA: On the structure of compact complex analytic surfaces, I and II. *Amer. J. Math. 86* (1964) 751–798 and *88* (1966) 682–721.

[10] E. LOOIJENGA: Rational surfaces with an anticanonical cycle. *Ann. Math. 114* (1981) 267–322.

[11] I. NAKAMURA: Inoue-Hirzebruch surfaces and a duality of hyperbolic unimodular singularities. *Math. Ann. 252* (1980) 221–235.

[12] I. NAKAMURA: Duality of cusp singularities in: *Complex Analysis of Singularities* (R.I.M.S. Symposium, K. Aomoto, organizer) Kyoto (1981).

[13] V.V. NIKULIN: Integral symmetric bilinear forms and some of their applications. *Math. USSR Izvestija 14* (1980) 103–167.

[14] H. PINKHAM: Singularités exceptionnelles, la dualité etrange d'Arnold et les surfaces K-3. *C.R. Acad. Sc. Paris, Série A, 284* (1977) 615–618.

[15] H. PINKHAM: Smoothings of the D_{pqr} singularities, $p+q+r=22$. Appendix to Looijenga's paper: the smoothing components of a triangle singularity. Proceedings of the Arcata conference on singularities (1981). *Proc. Symp. Pure Math. 40* (1983) Part 2, 373–377.

[16] J.-P. SERRE: *Cours d'arithmétique.* Paris: P.U.F. (1970).

[17] K. BEHNKE: Infinitesimal deformations of cusp singularities. *Math. Ann. 265* (1983) 407–422.

[18] R. FRIEDMAN and H. PINKHAM: Smoothings of cusp singularities via triangle singularities. To appear in *Comp. Math.*

[19] T. ODA: On torus embeddings and its applications. *TATA Institute Lecture Notes*, Bombay (1978).

[20] P. WAGREICH: Singularities of complex surfaces with solvable local fundamental group. *Topology 11* (1972) 51–72.
[21] J. WAHL: Smoothings of normal surface singularities. *Topology 20* (1981) 219–246.

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Note added in proof

(1) As J. Wahl has pointed out, the "extra automorphism" considered in Section 2, Remark 2 has already been studied by U. Karras, *Math. Ann. 215* (1975) 117–129, Section 3.

(2) Lattice theoretic considerations, somewhat similar to those in Section 3, have been used by Wahl and Looijenga to study smoothings of all Gorenstein surface singularities (to appear).